Nonlocal Cahn-Hilliard-Navier-Stokes systems

Sergio Frigeri-Università di Milano
DIMO-2013 Diffuse Interface Models, Levico Terme, September 10-13 2013

Supported by the FP7-IDEAS-ERC-StG Grant "EntroPhase"

The nonlocal Cahn-Hilliard-Navier-Stokes system

A well-known model which describes the evolution of an incompressible isothermal mixture of two immiscible fluids is the so-called model H (see [10,9]). This is a diffuse-interface model (cf. [1]) in which the sharp interface separating the two fluids (e.g., oil and water) is replaced by a diffuse one by introducing an order parameter φ. The dynamics of φ, which represents the (relative) concentration of one of the fluids (or the difference of the two concentrations), is governed by a Cahn-Hilliard type equation with a transport term. This parameter influences the (average) fluid velocity $\mathbf{u}$ through a capillarity force (called Korteweg force) proportional to $\mu\nabla\varphi$, where μ is the chemical potential (see, e.g., [11,Appendix]). Note that this force is concentrated close to the diffuse interface.
Assuming constant density and viscosity, the model reduces to the following system in $\Omega\times(0,\infty)$, $\Omega\subset\mathbb{R}^d$, $d=2,3$

$$\begin{aligned}&\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla\pi=\mu\nabla\varphi+\mathbf{h}\\&\mathrm{div}(\mathbf{u})=0\\&\varphi_t+\mathbf{u}\cdot\nabla\varphi=\mathrm{div}\,(m(\varphi)\nabla\mu)\end{aligned}$$

μ chemical potential, first variation of the (total Helmholtz) nonlocal free energy

- **Nonlocal free energy** (van der Waals) rigorously justified by Giacomin and Lebowitz (see [7,8]) as macroscopic limit of microscopic phase segregation models

$$\mathcal{E}(\varphi)=\frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2dxdy+\int_\Omega F(\varphi(x))dx$$

where $J:\mathbb{R}^d\to\mathbb{R}$ s.t. $J(x)=J(-x)$
Local free energy (having $\int_\Omega|\nabla\varphi|^2$ in place of the interaction integral) is an approximation of the nonlocal one

- **Nonlocal chemical potential**

$$\mu=a\varphi-J*\varphi+F'(\varphi)$$

where

$$(J*\varphi)(x):=\int_\Omega J(x-y)\varphi(y)dy\quad a(x):=\int_\Omega J(x-y)dy$$

F double-well potential: Helmholtz free energy density of uniform mixture

- Singular

$$F(s)=-\frac{\theta_c}{2}s^2+\frac{\theta}{2}((1+s)\log(1+s)+(1-s)\log(1-s))$$

for all $s\in(-1,1)$, with $0<\theta<\theta_c$

- Regular

$$F(s)=(1-s^2)^2\qquad\forall s\in\mathbb{R}$$

Weak solutions-Regular potentials, constant mobility [2]

Theorem 1 (Colli, F. & Grasselli '11)
Assume that $J\in W^{1,1}(\mathbb{R}^d)$, $a(x)\ge 0$ and that $h\in L^2_{loc}(\mathbb{R}^+;H^1_{div}(\Omega)')$.
Let $u_0\in L^2_{div}(\Omega)^d$, $\varphi_0\in L^2(\Omega)$ with $F(\varphi_0)\in L^1(\Omega)$. Then, $\forall T>0$ $\exists$ a weak solution $[u,\varphi]$ on $[0,T]$ s.t.

$$\begin{aligned}&u\in L^\infty(0,T;L^2_{div}(\Omega)^d)\cap L^2(0,T;H^1_{div}(\Omega)^d)\qquad u_t\in L^{4/d}(0,T;H^1_{div}(\Omega)')\\&\varphi\in L^\infty(0,T;L^4(\Omega))\cap L^2(0,T;H^1(\Omega))\qquad\varphi_t\in L^2(0,T;H^1(\Omega)')\\&\mu\in L^2(0,T;H^1(\Omega))\end{aligned}$$

and which satisfies the energy inequality (identity if $d=2$)

$$\mathcal{E}(u(t),\varphi(t))+\int_0^t(\nu\|\nabla u(\tau)\|^2+\|\nabla\mu(\tau)\|^2)d\tau\le\mathcal{E}(u_0,\varphi_0)+\int_0^t\langle h,u(\tau)\rangle d\tau\qquad\forall t>0$$

where we have set

$$\mathcal{E}(u(t),\varphi(t))=\frac{1}{2}\|u(t)\|^2+\frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t)-\varphi(y,t))^2dxdy+\int_\Omega F(\varphi(t))$$

Theorem 1 stills holds if the double-well potential satisfies the following
More general assumptions

(A1) $F\in C^2(\mathbb{R})$ and $\exists c_0>0$ s.t.

$$F''(s)+a(x)\ge c_0\quad\forall s\in\mathbb{R}\quad\text{a.e. }x\in\Omega$$

(A2) $\exists c_1>0$, $c_2>0$ and $p>2$ s.t.

$$F''(s)+a(x)\ge c_1|s|^{p-2}-c_2\quad\forall s\in\mathbb{R}\quad\text{a.e. }x\in\Omega$$

(A3) $\exists c_3>0$, $c_4\ge 0$ and $r\in(1,2]$ s.t.

$$|F'(s)|^r\le c_3|F(s)|+c_4\quad\forall s\in\mathbb{R}$$

Strong solutions in 2D-Regular potentials and constant mobility [5]

Theorem 2 (F., Grasselli & Krejčí '13)
Let $h\in L^2_{loc}(\mathbb{R}^+;L^2_{div}(\Omega)^2)$ and in addition $J\in W^{2,1}(\mathbb{R}^2)$. If

$$u_0\in H^1_{div}(\Omega)^2\qquad\varphi_0\in H^2(\Omega)$$

then, $\forall T>0$ $\exists$ unique strong solution $z:=[u,\varphi]$ s.t.

$$\begin{aligned}&u\in L^\infty(0,T;H^1_{div}(\Omega)^2)\cap L^2(0,T;H^2(\Omega)^2)\qquad u_t\in L^2(0,T;L^2_{div}(\Omega)^2)\\&\varphi\in L^\infty(0,T;H^2(\Omega))\qquad\varphi_t\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\end{aligned}$$

Moreover, the following continuous dependence estimate holds

$$\begin{aligned}&\|u_2(t)-u_1(t)\|^2+\|\varphi_2(t)-\varphi_1(t)\|^2_{H^1(\Omega)'}+\int_0^t\Big(\|\nabla u_2(\tau)-\nabla u_1(\tau)\|^2+\|\varphi_2(\tau)-\varphi_1(\tau)\|^2\Big)d\tau\\&\le\Lambda\Big(\|u_{02}-u_{01}\|^2+\|\varphi_{02}-\varphi_{01}\|^2_{H^1(\Omega)'}+\|h_2-h_1\|^2_{L^2(0,T;L^2_{div}(\Omega)^2)}\Big)\end{aligned}$$

Instantaneous regularization of weak solutions
For $\eta\ge 0$ given, introduce

$$\begin{aligned}&\mathcal{X}_\eta=L^2_{div}(\Omega)^2\times\mathcal{Y}_\eta\quad\mathcal{Y}_\eta=\{\varphi\in L^2(\Omega):F(\varphi)\in L^1(\Omega),|\bar\varphi|\le\eta\}\qquad(\textit{phase space of weak sols})\\&\mathcal{X}^1_\eta:=H^1_{div}(\Omega)^2\times\mathcal{Y}^1_\eta\qquad\mathcal{Y}^1_\eta:=\{\psi\in H^2(\Omega):|\overline{\psi}|\le\eta\}\qquad(\textit{phase space of strong sols})\end{aligned}$$

If $z_0=[u_0,\varphi_0]\in\mathcal{X}_\eta$, then $\forall\tau>0$ $\exists s_\tau\in(0,\tau]$ s.t. $z(s_\tau)\in\mathcal{X}^1_\eta$. Starting from s_τ the weak solution corresponding to z_0 becomes a (unique) strong solution $z\in C([s_\tau,\infty);\mathcal{X}^1_\eta)$. The regularization is also uniform w.r.t. bdd in $\mathcal{X}_\eta$ sets of initial data
Convergence to equilibria of weak solutions
Set of stationary solutions

$$\begin{aligned}\mathcal{E}_\eta:=\Big\{&z_\infty=[0,\varphi_\infty]:\ \varphi_\infty\in L^2(\Omega),\ F(\varphi_\infty)\in L^1(\Omega),\ |\overline{\varphi}_\infty|\le\eta,\\&a\varphi_\infty-J*\varphi_\infty+F'(\varphi_\infty)=\mu_\infty,\ \ \mu_\infty=\overline{F'(\varphi_\infty)}\quad\text{a.e. in }\Omega\Big\}\end{aligned}$$

Theorem 3 (F., Grasselli & Krejčí '13)
Take $z_0\in\mathcal{X}_\eta$ and let $z\in C(\mathbb{R}^+;\mathcal{X}_\eta)$ be a corresponding weak solution. Then

$$\emptyset\ne\omega(z)\subset\mathcal{E}_\eta$$

and $\exists t^*=t^*(z_0)$ s.t. the trajectory $\cup_{t\ge t^*}\{z(t)\}$ is precompact in $\mathcal{X}_\eta$. Moreover $\exists z_\infty\in\mathcal{E}_\eta$ s.t.

$$z(t)\to z_\infty\qquad\text{in }\mathcal{X}_\eta\qquad\text{as }\ t\to\infty$$

Weak solutions-Singular potential, degenerate mobility [6]

Relevant situation: mobility m degenerates at ±1 and singular double-well potential F on $(-1,1)$ (e.g. logarithmic like). A $\varphi-$dependent mobility appears in the original derivation of CH eq. (J.W. Cahn & J.E. Hilliard, 1971). Thermodynamically reasonable choice

$$m(\varphi)=k(1-\varphi^2)$$

- Key assumption (cf. [Elliot & Garcke '96], [Gajewski & Zacharias '03], [Giacomin & Lebowitz '97,'98])

$$mF''\in C([-1,1])$$

We are not able to control $\nabla\mu$ in some L^p space; hence we need to reformulate the definition of weak solution in such a way that μ does not appear any more.
Theorem 4 (F., Grasselli & Rocca '13)
Let $M\in C^2(-1,1)$ s.t. $m(s)M''(s)=1$, $M(0)=M'(0)=0$. Let

$$u_0\in L^2_{div}(\Omega)^d,\quad\varphi_0\in L^\infty(\Omega),\quad F(\varphi_0)\in L^1(\Omega),\quad M(\varphi_0)\in L^1(\Omega)$$

Then $\exists$ a weak solution $z:=[u,\varphi]$ on $[0,T]$ s.t. $\overline{\varphi(t)}=\overline{\varphi}_0$ and $|\varphi(x,t)|\le 1$ a.e $(x,t)\in\Omega\times(0,T)$
In addition, z satisfies the energetic inequality (identity if $d=2$)

$$\begin{aligned}&\frac{1}{2}(\|u(t)\|^2+\|\varphi(t)\|^2)+\int_0^t\int_\Omega(m(\varphi)F''(\varphi)+am(\varphi))|\nabla\varphi|^2+\nu\int_0^t\|\nabla u\|^2\le\frac{1}{2}(\|u_0\|^2+\|\varphi_0\|^2)\\&+\int_0^t\int_\Omega(a\varphi-J*\varphi)u\cdot\nabla\varphi+\int_0^t\int_\Omega m(\varphi)(\nabla J*\varphi-\varphi\nabla a)\cdot\nabla\varphi+\int_0^t\langle h,u\rangle\qquad\forall t>0\end{aligned}$$

The condition $|\overline{\varphi}_0|<1$ not required (only less strict condition $|\overline{\varphi}_0|\le 1$). This is due to the different weak solution formulation w.r.t. the case of constant mobility.
Theorem 5 (F., Grasselli & Rocca '13)
Let φ_0 be such tha $F'(\varphi_0)\in L^2(\Omega)$. Then, $\exists$ weak solution $z=[u,\varphi]$ that also satisfies

$$\mu\in L^\infty(0,T;L^2(\Omega))\qquad\nabla\mu\in L^2(0,T;L^2(\Omega)^d)$$

As a consequence, $z=[u,\varphi]$ also satisfies the weak formulation and the energy inequality (identity for $d=2$) of the non degenerate mobility case.

Asymptotic behavior [6,3,4]

Nonlocal CHNS IN 2D-Singular potential, degenerate mobility
Let $\mathcal{G}_\eta$ be the set of all weak solutions corresponding to all initial data $z_0=[\mathrm{u}_0,\varphi_0]\in\mathcal{X}_\eta$, where

$$\mathcal{X}_\eta=L^2_{div}(\Omega)^2\times\mathcal{Y}_\eta\qquad\mathcal{Y}_\eta:=\{\varphi\in L^\infty(\Omega):|\varphi|\le 1,\ \ F(\varphi),M(\varphi)\in L^1(\Omega),|\overline{\varphi}|\le\eta\},$$

and $\eta\in[0,1]$ is fixed. The metric on $\mathcal{X}_\eta$ is

$$\mathrm{d}(z_1,z_2)=\|\mathrm{u}_1-\mathrm{u}_2\|+\|\varphi_1-\varphi_2\|\quad\forall z_i=[\mathrm{u}_i,\varphi_i]\in\mathcal{X}_\eta,\quad i=1,2$$

Theorem 6 (F., Grasselli & Rocca '13)
Let $h\in H^1_{div}(\Omega)'$. Then $\mathcal{G}_\eta$ is a generalized semiflow on $\mathcal{X}_\eta$ which possesses the global attractor $\mathcal{A}_\eta$.
Existence of the global attractor in 2D (autonomous case) and of the trajectory attractor in 3D (non-autonomous case) also for nonlocal CHNS system with regular or singular potentials and constant mobility. In particular, for the case of regular potential and constant mobility we have

$$\mathcal{A}_\eta\subset B_{\mathcal{X}^1_\eta}(0,\Lambda(\eta)),$$

where $\Lambda(\eta)$ is a positive constant and $\mathcal{X}^1_\eta$ is the phase space of strong solutions

The convective nonlocal CH with degenerate mobility [6]

Given $\mathbf{u}\in L^2_{loc}(\mathbb{R}^+;H^1_{div}(\Omega)^d\cap L^\infty(\Omega)^d)$, consider in $\Omega\times(0,\infty)$, $\Omega\subset\mathbb{R}^d$, $d=2,3$

$$\begin{aligned}&\varphi_t+\mathbf{u}\cdot\nabla\varphi=\ \mathrm{div}\,(m(\varphi)\nabla\mu)\\&\mu=a\varphi-J*\varphi+F'(\varphi)\end{aligned}$$

As by-product of the previous analysis we obtain

- $\exists$ and uniqueness of a weak solution $\Longrightarrow$ we can define a semiflow $S(t)$ on $\mathcal{Y}_\eta$, $\eta\in[0,1]$
- $\exists$ of a connected global attractor ($\mathbf{u}$ independent of time)

Remark: uniqueness of solution and $\exists$ of the global attractor for the local CH with degenerate mobility are open issues

Uniqueness of weak solution and exponential attractors in 2D

Regular potentials, constant mobility
By redefining the pressure π, the Korteweg force $\mu\nabla\varphi$ can be rewritten as

$$-(\nabla a/2)\varphi^2-(J*\varphi)\nabla\varphi$$

This allows, by some technical arguments (Gagliardo-Nirenberg in 2D) to prove
Theorem 7 (F., Gal & Grasselli '13)
Let $u_0\in L^2_{div}(\Omega)^2$, $\varphi_0\in L^2(\Omega)$ with $F(\varphi_0)\in L^1(\Omega)$. Then, the weak solution $[u,\varphi]$ corresponding to $[u_0,\varphi_0]$ is unique . Furthermore, a continuous dependence estimate in $L^2_{div}\times(H^1)'$ also holds.
$\Longrightarrow$ the nonlocal CHNS system generates a *semigroup $S(t)$* of *closed* operators on $\mathcal{X}_\eta$

$$z(t):=[\mathrm{u}(t),\varphi(t)]=S(t)z_0:=S(t)[\mathrm{u}_0,\varphi_0]$$

Theorem 8 (F., Gal & Grasselli '13)
For every $\eta\ge 0$ the dynamical system $(\mathcal{X}_\eta,S(t))$ possesses an exponential attractor $\mathcal{M}_\eta$.
We recall that a set $\mathcal{M}\subset\mathcal{X}_\eta$ is an *exponential attractor* for the semigroup $S(t)$ if $\mathcal{M}$ is compact, positively invariant, with finite fractal dimension and such that $\exists J:\mathbb{R}^+\to\mathbb{R}^+$ increasing and $\kappa>0$ s.t., $\forall R>0$ and $\forall\mathcal{B}\subset\mathcal{X}_\eta$ with $\sup_{z\in\mathcal{B}}d_{\mathcal{X}_\eta}(z,0)\le R$ there holds $\mathrm{dist}(S(t)\mathcal{B},\mathcal{M})\le J(R)e^{-\kappa t}$
Remark: by similar arguments uniqueness of the weak sol in 2D holds for the nonlocal CHNS system with constant mobility+singular potential and with degenerate mobility+singular potential

References

1. D.M. Anderson, G.B. McFadden, A.A. Wheeler, *Diffuse-interface methods in fluid mechanics*, Annu. Rev. Fluid Mech. **30**, Annual Reviews, Palo Alto, CA, 1998, 139-165.
2. P. Colli, S. Frigeri, M. Grasselli, *Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Math. Anal. Appl. **386** (2012), 428-444.
3. S. Frigeri, M. Grasselli, *Global and trajectory attractors for a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Dynam Differential Equations **24** (2012), 827-856.
4. S. Frigeri, M. Grasselli, *Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials*, Dyn. Partial Differ. Equ. **9** (2012), 273-304.
5. S. Frigeri, M. Grasselli, P. Krejčí, *Strong solutions for two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems*, J. Differential Equations **255** (2013), 2597-2614.
6. S. Frigeri, M. Grasselli and E. Rocca, *A diffuse interface model for two-phase incompressible flows with nonlocal interactions and nonconstant mobility*, preprint arXiv 1303.6446 (2013).
7. G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits*, J. Statist. Phys. **87** (1997), 37-61.
8. G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. II. Phase motion*, SIAM J. Appl. Math. **58** (1998), 1707-1729.
9. M.E. Gurtin, D. Polignone, J. Viñals, *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Meth. Appl. Sci. **6** (1996), 8-15.
10. P.C. Hohenberg, B.I. Halperin, *Theory of dynamical critical phenomena*, Rev. Mod. Phys. **49** (1977), 435-479.
11. D. Jasnow, J. Viñals, *Coarse-grained description of thermo-capillary flow*, Phys. Fluids **8** (1996),

email: sergio.frigeri@unimi.it