Equazioni di evoluzione

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PREREQUISITI

Banach Spaces

- X vector space on $\mathbb{R}$
- A *norm* on X is a function $\|\cdot\| : X \to [0, +\infty)$ s.t.
  - $\|x\| = 0$ if and only if $x = 0$
  - $\|\lambda x\| = |\lambda| \|x\|, \ \forall x \in X, \ \forall \lambda \in \mathbb{R}$
  - $\|x + y\| \leq \|x\| + \|y\|, \ \forall x, \ y \in X$
- $(X, \|\cdot\|)$ is a *normed* space
- $(X, \|\cdot\|)$ is a metric space (X, d) w.r.t. d induced by $d(x, y) = \|x - y\|, \ \forall x, \ y \in X$
- $x_n \to x^*$ in X if $\|x_n - x^*\| \to 0$ as $n \to +\infty$ (*strong convergence*)
- A *Banach* space is a complete normed space (any Cauchy sequence is convergent in X)

Banach Spaces: separability and compactness

- $Y \subset X$ is *dense* if $\forall x \in X \; \exists \; \{y_n\} \subset Y \; : \; y_n \to x$ $(\overline{Y} = X)$
- A Banach space is *separable* if there exists a countable $Y \subset X$ such that $\overline{Y} = X$
- $E \subset X$ is *compact* if every open cover of E contains a finite subcover
- $E \subset X$ is compact if and only if every bounded sequence in E contains a convergent subsequence in E.

Some remarks

Remark

We want to solve in X a problem P we cannot treat directly.
We formulate easier problems P_n, approximating P.
We find a solution x_n in a compact $E \subset X$.
We construct a subsequence $x_{n_j} \to x^* \in E$.
We show that x^* is solution to problem P.

Remark

The unit ball in an infinite-dimensional Banach space X
IS NOT compact.
The compact sets of X are THIN.
Introduction of weak convergence.

Hilbert Spaces

- X vector space on $\mathbb{R}$
- An *inner product* on X is a function $(\cdot,\cdot) : X \times X \to \mathbb{R}$ s.t.
  - $(x,x) \geq 0$, $\forall x \in X$, $(x,x) = 0$ iff $x = 0$
  - $(y,x) = (x,y)$, $\forall x, y \in X$
  - $(\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z)$, $\forall x, y, z \in X$, $\forall\, \lambda, \mu \in \mathbb{R}$
- $(X, (\cdot,\cdot))$ is an *inner product* space
- $(X, (\cdot,\cdot))$ is a normed space $(X, \|\cdot\|)$ w.r.t. $\|\cdot\|$ induced by $\|x\| = (x,x)^{1/2}$, $\forall x \in X$
- $|(x,y)| \leq \|x\|\|y\|$, $\forall x, y \in X$ Cauchy-Schwarz inequality
- A *Hilbert* space is a complete inner product space

Orthogonal projections and bases in Hilbert Spaces

- $M^{\perp} = \{u \in H : (u, v) = 0, \forall\, v \in M\}$
  orthogonal complement of $M \subset H$, H Hilbert space
- If M is a closed subspace of H then $\exists!$ decomposition
  $x = u + v,\ u \in M,\ v \in M^{\perp}, \quad \forall\, x \in H$
- $P_M x = u$ *orthogonal projection* of x onto M
  $\|x\|^2 = \|P_M x\|^2 + \|x - P_M x\|^2, \quad \|P_M x\| \leq \|x\|$
- $\{e_j\} : (e_i, e_j) = \delta_{ij}$ *orthonormal* (countable) set in H
- $\{e_j\}$, orthonormal set, is a (countable) *basis* for H if
  $$x = \sum_{j=1}^{\infty} (x, e_j) e_j, \ \forall\, x \in H$$
- H is separable iff H has a countable basis
- If $\{e_j\}$ is a basis for H then $\|x\|^2 = \sum_{j=1}^{\infty} (x, e_j)^2, \ \forall\, x \in H$

Spaces of continuous functions

- $C^0(\Omega) = \{u : \Omega \subset \mathbb{R}^m \to \mathbb{R} \text{ continuous on } \Omega\}$
- If Ω is bounded ($\Rightarrow \overline{\Omega}$ is compact):
  $\|u\|_{C^0(\overline{\Omega})} = \|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$
- $(C^0(\overline{\Omega}), \|\cdot\|_\infty)$ is a separable Banach space
- $C^r(\Omega) = \{u : \Omega \to \mathbb{R} \ : \ D^\alpha u \in C^0(\Omega)\}, \ r \in \mathbb{N}, \ |\alpha| \leq r$
- $\|u\|_{C^r(\overline{\Omega})} = \sum_{|\alpha| \leq r} \sup_{x \in \overline{\Omega}} |D^\alpha u(x)| \quad (\Omega \text{ bounded})$
- $(C^r(\overline{\Omega}), \|\cdot\|_{C^r(\overline{\Omega})})$ is a separable Banach space
- $\text{supp } u = \overline{\{x \in \Omega \ : \ u(x) \neq 0\}}$
- $C^r_c(\Omega) = \{u \in C^r(\Omega) \text{ with compact support in } \Omega\}$
- $C^\infty(\Omega) = \cap_{r=0}^\infty C^r(\Omega) \quad C^\infty_c(\Omega) = \cap_{r=0}^\infty C^r_c(\Omega)$

L^p spaces, $1 \leq p < \infty$

- $\Omega \subset \mathbb{R}^m$ L-measurable
- u: L-measurable on Ω ($\Rightarrow u^p$: L-measurable on Ω)
- $L^p(\Omega) = \{u \,:\, \left(\int_\Omega |u(x)|^p dx\right)^{1/p} < +\infty\}, \quad 1 \leq p < +\infty$
- $u = 0$ in $L^p(\Omega) \Leftrightarrow u = 0$ a.e. in Ω
- $\|u\|_p = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$
- $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space

- $C_c^0(\Omega)$ is dense in $L^p(\Omega)$, $\quad 1 \leq p < +\infty$
- $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, $\quad 1 \leq p < +\infty$
- $L^p(\Omega)$ is separable, $\quad 1 \leq p < +\infty$
- $L^2(\Omega)$ is a Hilbert space, $\quad (u, v) = \int_\Omega u(x)v(x)dx$

L^∞ space

- $\Omega \subset \mathbb{R}^m$ L-measurable
- u: L-measurable on Ω
- $L^\infty(\Omega) = \{u : \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < +\infty\}$
- $\operatorname*{ess\,sup}_{x\in\Omega} |u(x)| = \inf\{M : |u(x)| \leq M \text{ a.e. in } \Omega\}$
- $u = 0$ in $L^\infty(\Omega) \Leftrightarrow u = 0$ a.e. in Ω
- $\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$
- $(L^\infty(\Omega), \|\cdot\|_\infty)$ is a Banach space (not separable!)

ℓ^2 space

- $\{a_k\} = (a_1, a_2, \dots), \quad a_k \in \mathbb{R}$
- $\ell^2 = \left\{ \mathbf{a} = \{a_k\} : \sum_{k=1}^{\infty} a_k^2 < +\infty \right\}$
- $(\mathbf{a}, \mathbf{b}) = \sum_{k=1}^{\infty} a_k b_k, \quad \|u\|_{\ell^2} = \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2}$
- ℓ^2 is a Hilbert space
- $\mathbf{a}_1 = (1, 0, \dots), \quad \mathbf{a}_2 = (0, 1, \dots)$
- $\mathbf{a}_k$ is an orthonormal basis in ℓ^2
- ℓ^2 is separable

Results on L^p spaces

Theorem

If $u_j \to u$ in $L^p(\Omega)$, $1 \leq p \leq +\infty$, then there exists a subsequence that converges pointwise to u, a.e. in Ω.

Theorem

(Hölder inequality)
If $p, q \in [1, +\infty] : \frac{1}{p} + \frac{1}{q} = 1$ (p, q conjugate indices) and $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ then

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$$

Young inequalities

Theorem

If $a, b \geq 0,\ \varepsilon > 0,\ p, q \in (1, +\infty)\ :\ \frac{1}{p} + \frac{1}{q} = 1,$ *then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

$$ab \leq \varepsilon\frac{a^p}{p} + \varepsilon^{-q/p}\frac{b^q}{q} \qquad ab \leq \varepsilon\frac{a^2}{2} + \frac{1}{\varepsilon}\frac{b^2}{2}$$

Gronwall inequality

Theorem

Let $y \in C^1([0,\infty))$, g, $h \in C([0,\infty))$ *be such that*

$$y' \leq g(t)y + h(t), \quad \forall\, t \geq 0$$

then

$$y(t) \leq y(0)e^{\int_0^t g(\sigma)d\sigma} + \int_0^t e^{\int_s^t g(\sigma)d\sigma} h(s)ds, \quad \forall\, t \geq 0$$

Linear operators on normed spaces

- $A : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$
- A is *linear* if $A(\lambda x + \mu z) = \lambda Ax + \mu Az, \quad \forall\, x, z \in X$
- A is *bounded* if $\exists\, M > 0 \;:\; \|Ax\|_Y \le M\|x\|_X, \;\forall\, x \in X$
- A is *continuous* if $x_n \to x$ in $X \;\Rightarrow\; Ax_n \to Ax$ in Y
- $\mathcal{L}(X, Y)$: set of all bounded linear operator from X to Y

  $\mathcal{L}(X, Y)$ vector space
- $$\|A\|_{\mathcal{L}(X,Y)} = \sup_{x \ne 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Ax\|_Y$$
- $\|Ax\|_Y \le \|A\|_{\mathcal{L}(X,Y)} \|x\|_X, \;\forall\, x \in X$

Results on linear operators

Theorem

If Y is a Banach space then $\mathcal{L}(X, Y)$ is a Banach space

Theorem

$A : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$, *A linear.*
A is continuous $\Leftrightarrow$ *A is bounded.*

Theorem

Uniform Boundedness Principle
Let X be a Banach space and $S \subset \mathcal{L}(X, Y)$. Then:

$$\sup_{T \in S} \|Tx\|_Y \leq M_x, \forall x \in X \quad \Rightarrow \quad \sup_{T \in S} \|T\|_{\mathcal{L}(X,Y)} \leq M$$

Linear functionals and dual spaces

- $f : (X, \|\cdot\|_X) \to \mathbb{R}$ bounded and linear: *functional* on X
- $\mathcal{L}(X, \mathbb{R})$ is the *dual space* of X: X^* or X'
- $X^* = \mathcal{L}(X, \mathbb{R}), \quad \|f\|_{X^*} = \|f\|_{\mathcal{L}(X,\mathbb{R})} = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_X}$
- $(X^*, \|\cdot\|_{X^*})$ is a Banach space ($\mathbb{R}$ is a Banach space)
- ${}_{X^*}\langle \cdot, \cdot \rangle_X : X^* \times X \to \mathbb{R} \quad {}_{X^*}\langle f, x \rangle_X = f(x)$ is a bilinear form
- ${}_{X^*}\langle f, x \rangle_X$ or $\langle f, x \rangle$ (duality)
- $|\langle f, x \rangle| \leq \|f\|_{X^*} \|x\|_X$

Theorem

Let X be a Banach space.
If $x, z \in X$ and $\langle f, x \rangle = \langle f, z \rangle$ for any $f \in X^*$, then $x = z$.

L^p and dual spaces

- $\forall\, u \in L^p(\Omega)$, fix $v \in L^q(\Omega)\ :\ p, q \in (1, +\infty)\ :\ \frac{1}{p} + \frac{1}{q} = 1$
- $\langle f, u \rangle = \int_\Omega u(x)v(x)dx$
- $\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$
- $f \in (L^p(\Omega))^*$ and $\|f\|_{(L^p(\Omega))^*} = \|v\|_{L^q(\Omega)}$

- $(L^p(\Omega))^* \simeq L^q(\Omega)$ with $p, q \in (1, +\infty)\ :\ \frac{1}{p} + \frac{1}{q} = 1$
- $(L^1(\Omega))^* \simeq L^\infty(\Omega)$
- ATTENTION! $(L^\infty(\Omega))^* \supset Y \simeq L^1(\Omega)$

Reflexive spaces

- $X^{**} = (X^*)^* \ = \ \{g : X^* \to \mathbb{R} \text{ bounded linear functional } \}$
- $G : X \to X^{**} \ : \ {}_{X^{**}}\langle Gx, f\rangle_{X^*} = {}_{X^*}\langle f, x\rangle_X, \quad \forall f \in X^*$
- $Gx \in X^{**}$ and $\|Gx\|_{X^{**}} = \|x\|_X \Rightarrow X \simeq Y \subset X^{**}$
- If $X^{**} \simeq X$ then X is *reflexive* (G is surjective)
- X is reflexive iff X^* is reflexive
- $L^p(\Omega)$ is reflexive for $p \in (1, +\infty)$
- $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive

Dual spaces of Hilbert spaces

Theorem

Riesz representation Theorem

Let H be a Hilbert space. Then, $\forall f \in H^* \Rightarrow \exists!\, y = y_f \in H$:
$\langle f, x\rangle = (x, y), \forall x \in H$ *and* $\|f\|_{H^*} = \|y\|_H$

- If H is a Hilbert space then $H^* \simeq H$
- $(L^2(\Omega))^* \simeq L^2(\Omega) \quad (\ell^2)^* \simeq \ell^2$
- Any Hilbert space is reflexive

Remark

Riesz Theorem $\Rightarrow$ *we identify* H *and* H^*.

ATTENTION!

If V *and* H *are Hilbert spaces* : $V \subset H$, *then* $H^* \subset V^*$
We identify only H *and* H^*, *that is* $V \subset H \equiv H^* \subset V^*$

Weak convergence

Definition

Let X be a Banach space, $x_n, x \in X$.
$x_n \rightharpoonup x$ (*weak* convergence) if $\langle f, x_n \rangle \to \langle f, x \rangle$ for any $f \in X^*$.

Example

$H = \ell^2 \qquad f \in (\ell^2)^* \rightleftarrows \mathbf{b} \in \ell^2$

$\langle f, \mathbf{a} \rangle = (\mathbf{a}, \mathbf{b}),\ \forall\, \mathbf{a} \in \ell^2$ and $\|\mathbf{b}\|_{\ell^2} = \|f\|_{(\ell^2)^*}$

N.B. $\mathbf{b} \in \ell^2 \quad \Rightarrow \quad \sum_{k=1}^{\infty} b_k^2 < +\infty \quad \Rightarrow \quad b_k \to 0$

$\mathbf{e}_k$ orthonormal basis

$\forall\, f \in (\ell^2)^* \quad \Rightarrow \quad \langle f, \mathbf{e}_k \rangle = (\mathbf{e}_k, \mathbf{b}) = b_k \to 0 \quad \Rightarrow \quad \mathbf{e}_k \rightharpoonup 0$

$\mathbf{e}_k \not\to 0$ strongly in H
($\mathbf{e}_k$ is not a Cauchy sequence: $\|\mathbf{e}_k - \mathbf{e}_j\|_{\ell^2} = \sqrt{2}, \quad k \neq j$)

Strong and weak convergence

Theorem

X Banach space, $x_n, x \in X$.
$x_n \to x$ (strong convergence) $\Rightarrow$ $x_n \rightharpoonup x$ (weak convergence)

Proof.

If $f \in X^*$ (f bounded linear functional) then f is continuous.
If $x_n \to x$ then $\langle f, x_n \rangle \to \langle f, x \rangle$, $\forall f \in X^*$ $\Rightarrow$ $x_n \rightharpoonup x$ □

Theorem

X finite dimensional Banach space, $x_n, x \in X$.
$x_n \to x$ (strong convergence) $\Leftrightarrow$ $x_n \rightharpoonup x$ (weak convergence)

Uniqueness of weak limit

Theorem

X Banach space, $x_n \in X$.
If $x_n \rightharpoonup x \in X$ then x is unique

Proof.

If $x_n \rightharpoonup x$, $x_n \rightharpoonup z$ then $\langle f, x_n \rangle \to \langle f, x \rangle$, $\langle f, x_n \rangle \to \langle f, z \rangle \Rightarrow$
$\langle f, x \rangle = \langle f, z \rangle$, $\forall f \in X^* \Rightarrow x = z$ (cf. previous Theorem) □

Boundedness of weak convergent sequences

Theorem

X Banach space, $x_n \in X$.
If $x_n \rightharpoonup x \in X$ then x_n is bounded

Proof.

- $\forall f \in X^*$, $\langle f, x_n \rangle$ is convergent (to $\langle f, x \rangle$ in $\mathbb{R}$) $\Rightarrow$ $|\langle f, x_n \rangle| \leq C_f, \forall n$
- $Gx_n \in X^{**} : \langle Gx_n, f \rangle = \langle f, x_n \rangle, \forall f \in X^*$, $\|Gx_n\|_{X^{**}} = \|x_n\|_X$
- $|\langle Gx_n, f \rangle| \leq C_f, \forall n$, and X^* is complete $\Rightarrow$ (Uniform Boundedness Principle) $\|Gx_n\|_{X^{**}}$ is bounded
- $\|x_n\|_X$ is bounded

$\square$

An estimate of the norm of the weak limit

Theorem

X Banach space, $x_n \in X$.
If $x_n \rightharpoonup x \in X$ then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$

Proof.

($X = H$ Hilbert space case)

$x_n \rightharpoonup x \quad \Rightarrow \quad$ (Riesz representation Theorem)

$\|x\|^2 = (x, x) = \lim_{n\to\infty}(x_n, x) = \liminf_{n\to\infty}(x_n, x)$

$\leq \liminf_{n\to\infty} \|x_n\| \|x\| = \|x\| \liminf_{n\to\infty} \|x_n\|$

□

Compact operators and weak convergence

Definition

$K : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is *compact* if

$\overline{K(W)}$ is compact in Y, for any bounded set $W \subset X$.

Remark: A compact operator is bounded.

Theorem

$A : X \to Y$, A compact, and $x_n \rightharpoonup x$ in X.
Then $Ax_n \to Ax$ in Y (strong convergence).

Weak-$*$ convergence

Definition

Let X be a Banach space, f_n and $f \in X^*$.
$f_n \overset{*}{\rightharpoonup} f$ (*weak*-$*$ convergence) if $\langle f_n, x\rangle \to \langle f, x\rangle \;\; \forall\, x \in X$.

Theorem

Weak-$*$ limits are unique.
Weak-$*$ convergent sequences are bounded.
Weak convergence implies weak-$*$ convergence.
If X reflexive, weak-$*$ convergence implies weak convergence.

Weak and weak-$*$ compactness Theorems

Theorem

Banach-Alaoglu Theorem

Let X be a Banach space.
Let f_n be a bounded sequence in X^*.
Then f_n has a weak-$*$ convergent subsequence in X^*.

Theorem

Let X be a reflexive Banach space.
Let x_n be a bounded sequence in X.
Then x_n has a weak convergent subsequence in X.

Corollary

Any bounded sequence in a Hilbert space H has a weak convergent subsequence in H.