Weierstrass Institute for

Applied Analysis and Stochastics

Asymptotic analysis of an isothermal model for nematic liquid crystal flow

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ERC Group "Entropy Formulation of Evolutionary Phase Transitions"

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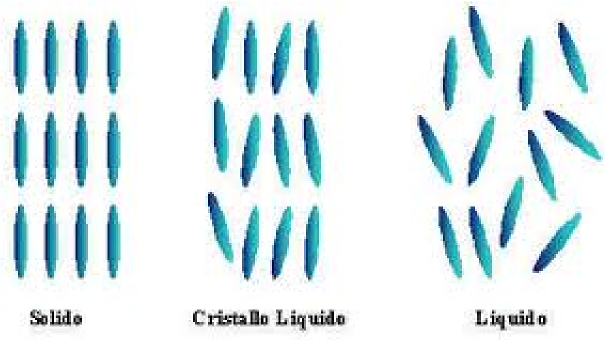
- Materials consisting of molecules with **elongated shape**
- Fluid has anisotropic properties over a limited temperature range: molecules lined up in a specific direction (uniaxial), but no positional order
- **the director** $\mathbf{d}$ is the average, over a small volume element, of unit vectors representing the long axis of each molecule



Semplification of the original Ericksen-Leslie system with termal and e.m. effects neglected

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\pi = -\lambda\mathrm{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}) + \mathbf{h}$$

$$\mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \eta(\Delta\mathbf{d} - \nabla_{\mathbf{d}}W(\mathbf{d}))$$

$$\mathrm{div}\mathbf{u} = 0$$

- $W(\mathbf{d})$ **double-well regular potential**, e.g. $W(\mathbf{d}) = (|\mathbf{d}|^2 - 1)^2$
- W relaxation of the constraint $|\mathbf{d}| = 1$

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Reasonable b.c. are

- no-slip for $\mathbf{u}$, Dirichlet for $\mathbf{d}$ (strong anchoring)
- free-slip for $\mathbf{u}$, i.e.

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \mathbb{T}\mathbf{n}\wedge\mathbf{n} = 0 \qquad \text{on } \partial\Omega$$

where

$$\mathbb{T} = \mathbb{S} - \rho\lambda(\nabla\mathbf{d}\odot\nabla\mathbf{d}) - p\mathbb{I} \qquad \mathbb{S} = \nu(\nabla\mathbf{u} + \nabla^T\mathbf{u})$$

and homogeneous Neumann for $\mathbf{d}$

In $\Omega \times (0,\infty)$, $\Omega \subset \mathbb{R}^3$

$$\mathbf{u}_t + \text{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p = \text{div}(\nu(\nabla \mathbf{u} + \nabla^T \mathbf{u})) - \text{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d})$$
$$- \text{div}\big(\alpha(\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \otimes \mathbf{d} - (1-\alpha)\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d}))\big) + \mathbf{h}$$
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$$\text{div}(\mathbf{u}) = 0$$

- $\alpha \in [0,1]$ related to the shape of liquid crystal molecules
  $\alpha = 1$ rod-like, $\alpha = 1/2$ spherical, $\alpha = 0$ disc-like

- b.c. considered: periodic, or no-slip for $\mathbf{u}$+homogeneous Neumann or non homogeneous Dirichlet for $\mathbf{d}$

$$\partial_{\mathbf{n}} \mathbf{d} = 0 \qquad \text{or} \qquad \mathbf{d} = \mathbf{g} \qquad \text{on } \partial\Omega$$

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Remarks

- Lin-Liu model neglects kinematic transport: liquid crystal molecules assumed small
- Strechting term $\mathbf{d} \cdot \nabla \mathbf{u}$ included and a new component $\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d}))$ added in the stress tensor

$$\mathbb{T} = \mathbb{S} - \nabla \mathbf{d} \odot \nabla \mathbf{d} - \alpha(\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \otimes \mathbf{d} + (1-\alpha)\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d}))$$

to ensure **energy balance**

$$\frac{d}{dt}\mathcal{E}(\mathbf{u}, \mathbf{d}) + \| - \Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d})\|^2 + \nu \|\nabla \mathbf{u}\|^2 = 0$$

where

$$\mathcal{E}(\mathbf{u}, \mathbf{d}) := \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}\|\nabla \mathbf{d}\|^2 + \int_\Omega W(\mathbf{d})$$

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- Mathematical difficulty: **lack of maximum principle** and of $L^\infty-$estimate for $\mathbf{d}$

- Well-posedness
  - H. Sun & C. Liu 09': $\exists$ strong sols, periodic b.c., in 2D (3D, ν large)
  - H. Wu, X. Xu & C. Liu '10: uniqueness and continuous dependence on in. data of strong sols, periodic b.c., in 2D
  - C. Cavaterra & E. Rocca '12: $\exists$ weak sols in 3D, Neumann or non-hom. Dirichlet b.c. for $\mathbf{d}$ and no-slip b.c. for $\mathbf{u}$
  - E. Feireisl, M Frémond, E. Rocca & G. Schimperna '11: $\exists$ weak sols for non-isothermal system, Neumann b.c. for $\mathbf{d}$, free-slip b.c. for $\mathbf{u}$
- Asymptotic behavior
  - H. Wu, X. Xu & C. Liu '10: conv. to eq. strong sols, conv. rate, periodic bc, in 2D (3D, ν large)
  - H. Petzeltová, E. Rocca & G. Schimperna '12: conv. to eq. weak sols, conv. rate, Neumann bc for $\mathbf{d}$ in 2D and 3D
  - M. Grasselli & H. Wu '11: smooth global attractor in 2D for strong sols, with finite fractal dimension, periodic b.c.

Consider, e.g., homogeneous Neumann b.c. for $\mathbf{d}$ (no-slip for $\mathbf{u}$)

(A1) $W = W_1 + W_2 \qquad W \in C^2(\mathbb{R}^3), \quad W_1$ convex, $W_2 \in C^{1,1}$

(A2) $\mathbf{h} \in L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div}) \qquad \mathbb{R}^+ := [0, \infty)$

Theorem (C. Cavaterra & E. Rocca '12)

Assume (A1), (A2) and that

$$\mathbf{u}_0 \in \boldsymbol{H}_{div}, \qquad \mathbf{d}_0 \in H^1(\Omega)^3, \qquad W(\mathbf{d}_0) \in L^1(\Omega)$$

Then, ∃ a weak sol $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ *corresponding to* $\mathbf{u}_0, \mathbf{d}_0$ *s.t.*

$$\mathbf{u} \in L^\infty_{loc}(\mathbb{R}^+; \boldsymbol{H}_{div}) \cap L^2_{loc}(\mathbb{R}^+; \boldsymbol{V}_{div})$$
$$\mathbf{u}_t \in L^2_{loc}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3)$$
$$\mathbf{d} \in L^\infty_{loc}(\mathbb{R}^+; H^1(\Omega)^3) \cap L^2_{loc}(\mathbb{R}^+; H^2(\Omega)^3), W(\mathbf{d}) \in L^\infty_{loc}(\mathbb{R}^+; L^1(\Omega))$$
$$\mathbf{d}_t \in L^2_{loc}(\mathbb{R}^+; L^{3/2}(\Omega)^3)$$

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and satisfying the energy inequality

$$\mathcal{E}(\mathbf{w}(t)) + \int_s^t \Big(\| -\Delta\mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d})\|^2 + \nu \|\nabla\mathbf{u}\|^2 \Big) d\tau$$
$$\leq \mathcal{E}(\mathbf{w}(s)) + \int_s^t \langle \mathbf{h}, \mathbf{u} \rangle d\tau$$

for all $t \geq s$*, for a.e.* $s \in (0, \infty)$*, including* $s = 0$*. We have set*

$$\mathcal{E}(\mathbf{w}(t)) := \frac{1}{2}\|\mathbf{u}(t)\|^2 + \frac{1}{2}\|\nabla\mathbf{d}(t)\|^2 + \int_\Omega W\big(\mathbf{d}(t)\big), \qquad \mathbf{w} = [\mathbf{u}, \mathbf{d}]$$

Remark

The space of test functions for weak sols is $W^{1,3}_{0,div}(\Omega)$ *(uniqueness or strong-weak uniqueness in 2D not known)*

Abstract evolution equation in a Banach space E

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- **Trajectory space**: for each $\sigma \in \Sigma$, $\mathcal{K}^+_\sigma$ is the set of *some sols* from $\mathcal{W}^+_{loc}$ and $\mathcal{K}^+_\Sigma := \cup_{\sigma\in\Sigma}\mathcal{K}^+_\sigma$ $\big($united trajectory space of the family $\{\mathcal{K}^+_\sigma\}_{\sigma\in\Sigma}\big)$

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- The **translation semigroup** $\{T(t)\}_{t\geq 0}$ is defined on $\mathcal{W}^+_{loc}$. For all $w \in \mathcal{W}^+_{loc}$

$$T(t)w := w(\cdot + t), \qquad \forall t \geq 0$$

$\{T(t)\}$ is continuous in $(\mathcal{W}^+_{loc}, \Theta^+_{loc})$ and acts on $\mathcal{K}^+_\Sigma$ (if the family $\{\mathcal{K}^+_\sigma\}_{\sigma\in\Sigma}$ is translation coordinated, i.e. $\sigma \in \Sigma$, $w \in \mathcal{K}^+_\sigma \Rightarrow T(t)w \in \mathcal{K}^+_{T(t)\sigma}$, $\forall t \geq 0$. We assume also that $T(t)\Sigma \subset \Sigma$, $\forall t \geq 0$)

- Introduce a **subspace** $\mathcal{W}_b^+$ of $\mathcal{W}_{loc}^+$: usually Banach, but also metric space with metric $\rho_{\mathcal{W}_b^+}$. We assume that $\mathcal{K}_\sigma^+ \subset \mathcal{W}_b^+$, for all $\sigma \in \Sigma$. The subspace $\mathcal{W}_b^+$ is **used to define bounded subsets** of the trajectory space $\mathcal{K}_\Sigma^+$

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- $\mathcal{A}_\Sigma \subset \mathcal{W}_{loc}^+$ **uniform (w.r.t. $\sigma \in \Sigma$) trajectory attractor** if

  1) $\mathcal{A}_\Sigma$ is compact in Θ_{loc}^+
  2) $\mathcal{A}_\Sigma$ is a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for $\{\mathcal{K}_\sigma^+\}_{\sigma\in\Sigma}$ in the topology Θ_{loc}^+: $\forall B \subset \mathcal{K}_\Sigma^+$ bdd in $\mathcal{W}_b^+$ and $\forall \mathcal{O}(\mathcal{A}_\Sigma)$, neighbourhood of $\mathcal{A}_\Sigma$ in Θ_{loc}^+, $\exists t_1 \geq 0$ s.t. $T(t)B \subset \mathcal{O}(\mathcal{A}_\Sigma)$, $\forall t \geq t_1$.
  3) $\mathcal{A}_\Sigma$ is the minimal compact and uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for the family $\{\mathcal{K}_\sigma^+\}_{\sigma\in\Sigma}$ in Θ_{loc}^+

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- If $\mathcal{A}_\Sigma$ exists, it is **unique**

- Since $T(t)$ is continuous in Θ_{loc}^+, then $\mathcal{A}_\Sigma$ is **strictly invariant**

$$T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma, \qquad \forall t \in \mathbb{R}^+$$

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To this aim we need

- A **dissipative estimate** of the form

$$\rho_{\mathcal{W}_b^+}(T(t)w, w_0) \leq \Lambda_0\Big(\rho_{\mathcal{W}_b^+}(w, w_0)\Big)e^{-kt} + \Lambda_1, \quad \forall t \geq t_0$$

  for every $w \in \mathcal{K}^+_\Sigma$, where $\Lambda_0 : \mathbb{R}^+ \to \mathbb{R}^+$ locally bdd, k, Λ_0, Λ_1 independent of w

- That the ball

$$B_{\mathcal{W}_b^+}(w_0, 2\Lambda_1) := \{w \in \mathcal{W}_b^+ : \rho_{\mathcal{W}_b^+}(w, w_0) \leq 2\Lambda_1\}$$

  is **compact** in Θ^+_{loc}

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If, in addition, $\{\mathcal{K}^+_\sigma\}_{\sigma\in\Sigma}$ is $(\Theta^+_{loc}, \Sigma)-$**closed**, Σ compact, then $\mathcal{A}_\Sigma \subset \mathcal{K}^+_\Sigma$ and

$$\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$$

+further properties of the trajectory attractor

GENERAL SMOOTH POTENTIALS

Banach-metric setting

Set $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ and introduce the **Banach space**

$$\mathcal{W}^{+}_{loc} := \Big\{ \mathbf{w} \in L^{\infty}_{loc}(\mathbb{R}^{+}; \mathbf{H}_{div} \times H^{1}(\Omega)^{3}) \cap L^{2}_{loc}(\mathbb{R}^{+}; \mathbf{V}_{div} \times H^{2}(\Omega)^{3}) : \\ \mathbf{u}_t \in L^{2}_{loc}(\mathbb{R}^{+}; W^{-1,3/2}(\Omega)^{3}), \mathbf{d}_t \in L^{2}_{loc}(\mathbb{R}^{+}; L^{3/2}(\Omega)^{3}) \Big\}$$

endowed with the **topology Θ^{+}_{loc} of local weak convergence**

$$\mathbf{w}_m := [\mathbf{u}_m, \mathbf{d}_m] \to \mathbf{w} := [\mathbf{u}, \mathbf{d}] \qquad \text{in } \Theta^{+}_{loc}$$

iff for all $M > 0$

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{weakly}^* \text{ in } L^{\infty}(0, M; \mathbf{H}_{div}) \text{ and weakly in } L^{2}(0, M; \mathbf{V}_{div})$$

$$(\mathbf{u}_m)_t \rightharpoonup \mathbf{u}_t \quad \text{weakly in } L^{2}(0, M; W^{-1,3/2}(\Omega)^{3})$$

$$\mathbf{d}_m \rightharpoonup \mathbf{d} \quad \text{weakly}^* \text{ in } L^{\infty}(0, M; H^{1}(\Omega)^{3}) \text{ and weakly in } L^{2}(0, M; H^{2}(\Omega)^{3})$$

$$(\mathbf{d}_m)_t \rightharpoonup \mathbf{d}_t \quad \text{weakly in } L^{2}(0, M; L^{3/2}(\Omega)^{3})$$

In $\mathcal{W}_{loc}^{+}$ we consider the following **metric subspace**

$$\mathcal{W}_b^+ := \Big\{ \mathbf{w} \in L^\infty(\mathbb{R}^+; \mathbf{H}_{div} \times H^1(\Omega)^3) \cap L^2_{tb}(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \\ \mathbf{u}_t \in L^2_{tb}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L^2_{tb}(\mathbb{R}^+; L^{3/2}(\Omega)^3), \\ W(\mathbf{d}) \in L^\infty(\mathbb{R}^+; L^1(\Omega)) \Big\}$$

used to define bounded subsets of the space of trajectories $\mathcal{K}_\Sigma^+$. The metric on $\mathcal{W}_b^+$ is

$$\rho_{\mathcal{W}_b^+}(\mathbf{w}_1, \mathbf{w}_2) := \|\mathbf{w}_1 - \mathbf{w}_2\|_{L^\infty(0,\infty;\mathbf{H}_{div}\times H^1(\Omega)^3)} + \|\mathbf{w}_1 - \mathbf{w}_2\|_{L^2_{tb}(0,\infty;\mathbf{V}_{div}\times H^2(\Omega)^3)} \\ + \|(\mathbf{u}_1)_t - (\mathbf{u}_2)_t\|_{L^2_{tb}(0,\infty;W^{-1,3/2}(\Omega)^3)} + \|(\mathbf{d}_1)_t - (\mathbf{d}_2)_t\|_{L^2_{tb}(0,\infty;L^{3/2}(\Omega)^3)} \\ + \Big\| \int_\Omega W(\mathbf{d}_1) - \int_\Omega W(\mathbf{d}_2) \Big\|^{1/2}_{L^\infty(0,\infty)}$$

for every $\mathbf{w}_1 = [\mathbf{u}_1, \mathbf{d}_1], \mathbf{w}_2 = [\mathbf{u}_2, \mathbf{d}_2] \in \mathcal{W}_b^+$

Definition

For every $\mathbf{h} \in L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div})$ the **trajectory space** $\mathcal{K}^+_{\mathbf{h}}$ with external force $\mathbf{h}$ is the set of all weak sols $\mathbf{w} = [\mathbf{u}, \mathbf{d}]$ satisfying the energy inequality for all $t \geq s$ and for a.a. $s \in (0, \infty)$

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- Set $\mathcal{K}^+_\Sigma := \cup_{\mathbf{h}\in\Sigma}\mathcal{K}^+_{\mathbf{h}}$. We have $\mathcal{K}^+_\Sigma \subset \mathcal{W}^+_b$
- For the **symbol space** Σ we choose the **hull** of $\mathbf{h}_0$ in $L^2_{loc,w}(\mathbb{R}^+; \mathbf{V}'_{div})$

$$\Sigma = \mathcal{H}_+(\mathbf{h}_0) := \Big[\{T(t)\mathbf{h}_0, t \geq 0\}\Big]_{L^2_{loc,w}(\mathbb{R}^+;\mathbf{V}'_{div})}$$

and take $\mathbf{h}_0$ translation bounded in $L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div})$, i.e.

$$\|\mathbf{h}_0\|^2_{L^2_{tb}(\mathbb{R}^+;\mathbf{V}'_{div})} := \sup_{t\geq 0}\int_t^{t+1} \|\mathbf{h}_0(\tau)\|^2_{\mathbf{V}'_{div}}\, d\tau < \infty$$

$\Leftrightarrow$ $\mathbf{h}_0$ translation compact (tr.-c.) in $L^2_{loc,w}(\mathbb{R}^+; \mathbf{V}'_{div})$, i.e., Σ compact in $L^2_{loc,w}(\mathbb{R}^+; \mathbf{V}'_{div})$

(A3) W satisfies (A1) and $\exists\, c_0 \geq 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$ and $\delta > 0$ s.t. for all $\mathbf{d} \in \mathbb{R}^3$

$$W_1(\mathbf{d}) \leq c_0(1 + |\nabla_{\mathbf{d}} W_1(\mathbf{d})|^2) \qquad W_1(\mathbf{d}) \geq c_1|\mathbf{d}|^{2+\delta} - c_2$$

(A4) $W(\mathbf{d}) \leq b(1 + |\mathbf{d}|^6) \qquad \forall \mathbf{d} \in \mathbb{R}^3 \qquad b > 0$

Theorem (S.F. & E. Rocca '12)

Assume (A3) and $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \boldsymbol{V}'_{div})$. *Then, the semigroup* $\{T(t)\}$ *acting on* $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ *possesses the uniform (w.r.t.* $\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)$*) trajectory attractor* $\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)}$. *This set is strictly invariant, bounded in* $\mathcal{W}^+_b$ *and compact in* Θ^+_{loc}.

In addition, if (A4) holds and $\mathbf{h}_0$ *is tr.-c. in* $L^2_{loc}(\mathbb{R}^+; \boldsymbol{V}'_{div})$ *or* $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \boldsymbol{H}_{div})$, *then* $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ *is closed in* Θ^+_{loc}, $\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ *and*

$$\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\mathbf{h}_0))}$$

Main steps of the proof

- Assume (A3). Then $\exists\, \kappa, \eta, l > 0$ (independent of $\mathbf{d}$) s.t.

$$\| - \Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d})\|^2 \geq \kappa \|\nabla \mathbf{d}\|^2 + \eta \int_\Omega W(\mathbf{d}) - l$$

for all $\mathbf{d} \in H^2(\Omega)^3$, with $\partial_{\boldsymbol{n}} \mathbf{d} = 0$ on $\partial\Omega$

Main steps of the proof

- Assume (A3). Then $\exists\, k, l > 0$ (independent of $\mathbf{d}$) s.t.

$$\| -\Delta\mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d})\|^2 + \frac{\nu}{2}\|\nabla\mathbf{u}\|^2 \geq k\mathcal{E}(\mathbf{w}) - l \qquad \mathbf{w} = [\mathbf{u}, \mathbf{d}]$$

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- From the energy inequality we get

$$\mathcal{E}(\mathbf{w}(t)) + k\int_0^t \mathcal{E}(\mathbf{w}(\tau))d\tau \leq l(t-s) + \frac{1}{2\nu}\int_s^t \|\mathbf{h}(\tau)\|^2_{\mathbf{V}'_{div}} d\tau$$
$$+ \mathcal{E}(\mathbf{w}(s)) + k\int_0^s \mathcal{E}(\mathbf{w}(\tau))d\tau \qquad \forall t \geq s \quad \text{for a.e. } s \in (0, \infty)$$

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- By using a generalization of an integral Gronwall lemma, some technical arguments and comparison in the system we get $\mathcal{K}^+_{\mathbf{h}} \subset \mathcal{W}^+_b$, $\forall \mathbf{h} \in \mathcal{H}(\mathbf{h}_0)$, $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \mathbf{V}'_{div})$ and the dissipative estimate

$$\rho_{\mathcal{W}^+_b}(T(t)\mathbf{w}, \mathbf{0}) \leq c\rho^2_{\mathcal{W}^+_b}(\mathbf{w}, \mathbf{0})e^{-\frac{k}{2}t} + \Lambda_1, \qquad \forall t \geq 1,$$

  for all $\mathbf{w} \in \mathcal{K}^+_{\mathbf{h}}$, with Λ_1 depending on $\|\mathbf{h}_0\|_{L^2_{tb}(\mathbb{R}^+; \mathbf{V}'_{div})}$

POLYNOMIAL POTENTIALS

Banach-Banach setting

Set $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ and, for $p \geq 2$, introduce the **Banach space**

$$\mathcal{W}^+_{p,loc} := \Big\{ \mathbf{w} \in L^\infty_{loc}(\mathbb{R}^+; \mathbf{H}_{div} \times (H^1(\Omega)^3 \cap L^p(\Omega)^3)) \\ \mathbf{w} \in L^2_{loc}(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \\ \mathbf{u}_t \in L^2_{loc}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L^2_{loc}(\mathbb{R}^+; L^{3/2}(\Omega)^3) \Big\}$$

endowed with its inductive limit topology $\Theta^+_{p,loc}$.

Bdd subsets of $\mathcal{K}^+_\Sigma$ defined w.r.t the **Banach subspace** of $\mathcal{W}^+_{p,loc}$

$$\mathcal{W}^+_{p,b} := \Big\{ \mathbf{w} \in L^\infty(\mathbb{R}^+; \mathbf{H}_{div} \times (H^1(\Omega)^3 \cap L^p(\Omega)^3)) \\ \mathbf{w} \in L^2_{tb}(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \\ \mathbf{u}_t \in L^2_{tb}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L^2_{tb}(\mathbb{R}^+; L^{3/2}(\Omega)^3) \Big\}$$

(A5) $\exists\, C_1, C_2 > 0$ and $p \in (2, +\infty)$ s.t.

$$C_1(|\mathbf{d}|^p - 1) \leq W(\mathbf{d}) \leq C_2(1 + |\mathbf{d}|^p), \qquad \forall \mathbf{d} \in \mathbb{R}^3$$

$\mathcal{K}^+_{p,\mathbf{h}}$ and $\mathcal{K}^+_{p,\mathcal{H}_+(\mathbf{h}_0)} := \bigcup_{\mathbf{h}\in\mathcal{H}_+(\mathbf{h}_0)} \mathcal{K}^+_{p,\mathbf{h}}$ are the trajectory spaces: defined exactly as before, with the additional requirement that

$$\mathbf{d} \in L^\infty_{loc}(\mathbb{R}^+; L^p(\Omega)^3)$$

Theorem (S.F. & E. Rocca '12)

Assume (A3), (A5) and $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \boldsymbol{V}'_{div})$. *Then,* $\{T(t)\}$ *acting on* $\mathcal{K}^+_{p,\mathcal{H}_+(\mathbf{h}_0)}$ *possesses the uniform (w.r.t.* $\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)$*) trajectory attractor* $\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)}$. *This set is strictly invariant, bounded in* $\mathcal{W}^+_{p,b}$, *compact in* $\Theta^+_{p,loc}$.
In addition, if $\mathbf{h}_0$ *is tr.-c. in* $L^2_{loc}(\mathbb{R}^+; \boldsymbol{V}'_{div})$ *or* $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \boldsymbol{H}_{div})$, *then* $\mathcal{K}^+_{p,\mathcal{H}_+(\mathbf{h}_0)}$ *is closed in* $\Theta^+_{p,loc}$, $\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}^+_{p,\mathcal{H}_+(\mathbf{h}_0)}$ *and*

$$\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{p,\omega(\mathcal{H}_+(\mathbf{h}_0))}$$

- If $\mathbf{g} \in H^1_{loc}(\mathbb{R}^+; H^{-1/2}(\Gamma)^3) \cap L^2_{loc}(\mathbb{R}^+; H^{3/2}(\Gamma)^3)$ (same assumptions for $\mathbf{u}_0$, $\mathbf{d}_0$ and $\mathbf{h}$), then $\exists$ a weak sol $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ satisfying

$$\mathcal{E}(\mathbf{w}(t)) + \int_s^t \Big(\| -\Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d}) \|^2 + \nu \|\nabla \mathbf{u}\|^2 \Big) d\tau$$
$$\leq \mathcal{E}(\mathbf{w}(s)) + \int_s^t \langle \mathbf{g}_t, \partial_{\boldsymbol{n}} \mathbf{d} \rangle_{H^{-1/2}(\Gamma)^3 \times H^{1/2}(\Gamma)^3} d\tau + \int_s^t \langle \mathbf{h}, \mathbf{u} \rangle d\tau$$

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for all $t \geq s$, for a.e. $s \in (0, \infty)$, including $s = 0$

- Symbol space for the Dirichlet datum $\mathbf{g}$

$$\mathcal{H}_+(\mathbf{g}_0) := [\{T(t)\mathbf{g}_0, t \geq 0\}]_{\Xi^+_{loc,w}}$$

$$\Xi^+_{loc,w} := \{\mathbf{g} \in C(\mathbb{R}^+; H^{3/2}(\Gamma)^3) : \mathbf{g}_t \in L^2_{loc,w}(\mathbb{R}^+; H^{-1/2}(\Gamma)^3)\}$$

Consider, e.g., general smooth potentials. Trajectory spaces

$$\mathcal{K}^+_{\mathbf{g},\mathbf{h}} \quad \text{and} \quad \mathcal{K}^+_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)} := \bigcup_{\mathbf{g}\in\mathcal{H}_+(\mathbf{g}_0),\mathbf{h}\in\mathcal{H}_+(\mathbf{h}_0)} \mathcal{K}^+_{\mathbf{g},\mathbf{h}}$$

Theorem (S.F. & E. Rocca '12)

Assume (A3) and that $\mathbf{g}_0$ *is tr.-c. in* $C(\mathbb{R}^+; H^{3/2}(\Gamma)^3)$ *with* $\partial_t \mathbf{g}_0 \in L^2_{tb}(\mathbb{R}^+; H^{-1/2}(\Gamma)^3)$*, and* $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \boldsymbol{V}'_{div})$*. Then,* $\{T(t)\}$ *acting on* $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)}$ *possesses the uniform (w.r.t.* $[\mathbf{g},\mathbf{h}] \in \mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)$*) trajectory attractor* $\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)}$*. This set is strictly invariant, bounded in* $\mathcal{W}^+_b$ *and compact in* Θ^+_{loc}*.*
In addition, if (A4) holds, if $\mathbf{g}_0$ *is tr.-c. in* Ξ^+_{loc} *and if* $\mathbf{h}_0$ *is tr.-c. in* $L^2_{loc}(\mathbb{R}^+; \boldsymbol{V}'_{div})$ *or* $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \boldsymbol{H}_{div})$*, then* $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)}$ *is closed in* Θ^+_{loc}*,* $\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}^+_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)}$ *and*

$$\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0))}$$

Open issue: eventual regularization and energy identity for weak sols in 2D $\Rightarrow$ existence of trajectory attractor in 2D for the *strong* topology of $\mathcal{W}_{loc}^{+}$ not known

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Alternative analysis: for $q > 3$ consider **Problem P$_\epsilon$**

$$\mathbf{u}_t + \text{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p = 2\nu \text{div}(D\mathbf{u}) + \epsilon \text{div}(|Du|^{q-2} Du) - \text{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d})$$

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$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} - \alpha \mathbf{d} \cdot \nabla \mathbf{u} + (1-\alpha)\mathbf{d} \cdot \nabla^T \mathbf{u} = \Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})$$

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Banach space of sols

$$\mathcal{Z}^{+}_{q,loc} := \Big\{\mathbf{w} := [\mathbf{u},\mathbf{d}] \in L^{\infty}_{loc}(\mathbb{R}^+;\mathbf{H}_{div}\times H^1(\Omega)^3)$$
$$\mathbf{u}\in L^{q}_{loc}(\mathbb{R}^+;\mathbf{W}_{q,div}), \mathbf{d}\in L^2_{loc}(\mathbb{R}^+;\times H^2(\Omega)^3):$$
$$\mathbf{u}_t \in L^{q'}_{loc}(\mathbb{R}^+;W^{-1,q'}(\Omega)^3), \mathbf{d}_t\in L^2_{loc}(\mathbb{R}^+;L^2(\Omega)^3)\Big\}$$

where $\mathbf{W}_{q,div} := \{\mathbf{v}\in W^{1,q}_0(\Omega)^3 : \text{div}(\mathbf{v}) = 0\}$.

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where $\mathbf{W}_{q,div} := \{\mathbf{v} \in W^{1,q}_0(\Omega)^3 : \text{div}(\mathbf{v}) = 0\}$.

$\mathcal{Z}^+_{q,loc}$ with **topologies $\Theta^+_{q,loc,w}$, $\Theta^+_{q,loc,s}$ of local weak and strong convergence**, resp.

In $\mathcal{Z}^{+}_{q,loc}$ we consider the following **metric subspace** (with $\epsilon-$dependent metric $\rho_{\mathcal{Z}^{+}_{q,b,\epsilon}}$)

$$\mathcal{Z}^{+}_{q,b,\epsilon} := \Big\{ \mathbf{w} := [\mathbf{u},\mathbf{d}] \in L^{\infty}(\mathbb{R}^{+};\mathbf{H}_{div} \times H^{1}(\Omega)^{3}), \mathbf{u} \in L^{q}_{tb}(\mathbb{R}^{+};\mathbf{W}_{q,div}),$$
$$\mathbf{d} \in L^{2}_{tb}(\mathbb{R}^{+};\times H^{2}(\Omega)^{3}), \mathbf{u}_{t} \in L^{q'}_{tb}(\mathbb{R}^{+};W^{-1,q'}(\Omega)^{3}),$$
$$\mathbf{d}_{t} \in L^{2}_{tb}(\mathbb{R}^{+};L^{2}(\Omega)^{3}), W(\mathbf{d}) \in L^{\infty}(\mathbb{R}^{+};L^{1}(\Omega)) \Big\}$$

with respect to which bdd subsets of the trajectory space $\mathcal{K}^{+}_{\mathcal{H}_{+}(\mathbf{h}_{0})}$ are defined.

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$$\mathbf{d} \in L^2_{tb}(\mathbb{R}^+; \times H^2(\Omega)^3), \mathbf{u}_t \in L^{q'}_{tb}(\mathbb{R}^+; W^{-1,q'}(\Omega)^3),$$

$$\mathbf{d}_t \in L^2_{tb}(\mathbb{R}^+; L^2(\Omega)^3), W(\mathbf{d}) \in L^\infty(\mathbb{R}^+; L^1(\Omega)) \Big\}$$

with respect to which bdd subsets of the trajectory space $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ are defined.

Preliminary results

- **Strong trajectory attractor** for P$_\epsilon$

  Every weak sol to P$_\epsilon$ satisfies the **energy identity**

$$\frac{d}{dt}\mathcal{E}(\mathbf{w}(t)) + \| -\Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d})\|^2 + \nu \|\nabla \mathbf{u}\|^2 + \epsilon \|D\mathbf{u}\|^q_{L^q} = \langle \mathbf{h}, \mathbf{u} \rangle$$

  Then, under assumption (A1) and

$$|\partial^2_{d_i d_j} W(\mathbf{d})| \leq c(1 + |\mathbf{d}|^{4-\sigma}) \qquad \forall \mathbf{d} \in \mathbb{R}^3 \quad \forall i, j \qquad \sigma > 0$$

  Problem P$_\epsilon$ admits a (unique) strong trajectory attractor $\mathcal{A}^\epsilon_{\mathcal{H}_+(\mathbf{h}_0)}$ (uniform w.r.t. $\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)$) in the space of weak sols $\mathcal{Z}^+_{q,loc}$ with its strong topology $\Theta^+_{q,loc,s}$

Convergence of the family of strong trajectory attractors

Introducing the space

$$
\begin{aligned}
\widetilde{\mathcal{Z}}^{+}_{q,loc} := \Big\{ & \mathbf{w} := [\mathbf{u}, \mathbf{d}] \in L^{\infty}_{loc}(\mathbb{R}^{+}; \mathbf{H}_{div} \times H^{1}(\Omega)^{3}) \\
& \mathbf{u} \in L^{2}_{loc}(\mathbb{R}^{+}; \mathbf{V}_{div}), \mathbf{d} \in L^{2}_{loc}(\mathbb{R}^{+}; \times H^{2}(\Omega)^{3}) : \\
& \mathbf{u}_t \in L^{q'}_{loc}(\mathbb{R}^{+}; W^{-1,q'}(\Omega)^{3}), \mathbf{d}_t \in L^{2}_{loc}(\mathbb{R}^{+}; L^{3/2}(\Omega)^{3}) \Big\}
\end{aligned}
$$

of sols *of both P_ϵ and P_0*, endowed with its weak topology $\widetilde{\Theta}^{+}_{q,loc}$, so that $\mathcal{A}^{\epsilon}_{\mathcal{H}_+(\mathbf{h}_0)}, \mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} \subset \widetilde{\mathcal{Z}}^{+}_{q,loc}$, we have

$$
\mathcal{A}^{\epsilon}_{\mathcal{H}_+(\mathbf{h}_0)} \to \mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} \qquad \text{in } \widetilde{\Theta}^{+}_{q,loc} \qquad \text{as } \epsilon \to 0
$$