Recent results on nonlocal diffuse-interface models for binary fluids

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Local Cahn-Hilliard-Navier-Stokes systems

Flow of viscous incompressible Newtonian macroscopically immiscible two-phase fluids (diffuse-interface model).

In $\Omega \times (0, \infty)$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$

$$
\begin{aligned}
&\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{h} \\
&\mathrm{div}(\mathbf{u}) = 0 \\
&\varphi_t + \mathbf{u} \cdot \nabla \varphi = \mathrm{div}\, (m(\varphi) \nabla \mu) \\
&\mu = -\epsilon \Delta \varphi + \epsilon^{-1} F'(\varphi)
\end{aligned}
$$

- μ chemical potential, first variation of the (total Helmholtz) free energy

$$E(\varphi) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} F(\varphi) \right) dx$$

Local Cahn-Hilliard-Navier-Stokes systems

- $(\epsilon/2)|\nabla\varphi|^2$ free energy increase due to presence of two components
- F double-well potential: Helmholtz free energy density
    - Singular

$$F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\left((1+s)\log(1+s) + (1-s)\log(1-s)\right)$$

    for all $s \in (-1,1)$, with $0 < \theta < \theta_c$
    - Regular

$$F(s) = (1-s^2)^2 \qquad \forall s \in \mathbb{R}$$

- Some literature: Starovoitov '97, Boyer '99, Abels '09, Abels & Feireisl '08; Abels '09, Gal & Grasselli '09 , Zhao, Wu & Huang '09; Abels '09, Gal & Grasselli '09, '10 and '11

Nonlocal model for binary fluid motion

- **Nonlocal free energy** rigorously justified by Giacomin and Lebowitz ('97 & '98) as macroscopic limit of microscopic phase segregation models

$$E(\varphi) = \frac{1}{4}\int_\Omega \int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \int_\Omega F(\varphi(x))dx$$

$J : \mathbb{R}^d \to \mathbb{R}$ is an interaction kernel s.t. $J(x) = J(-x)$ (usually nonnegative and radial)

- **Nonlocal chemical potential**

$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

$$(J * \varphi)(x) := \int_\Omega J(x-y)\varphi(y)dy \quad a(x) := \int_\Omega J(x-y)dy$$

Nonlocal Cahn-Hilliard-Navier-Stokes systems

Consider in $\Omega \times (0, \infty)$

$$\varphi_t + \mathbf{u} \cdot \nabla\varphi = \text{div}\,(m(\varphi)\nabla\mu)$$
$$\mu = a\varphi - J * \varphi + F'(\varphi)$$
$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla\pi = \mu\nabla\varphi + \mathbf{h}$$
$$\text{div}(\mathbf{u}) = 0$$

subject to

$$\frac{\partial\mu}{\partial n} = 0 \qquad \mathbf{u} = 0 \qquad \text{on} \quad \partial\Omega \times (0, \infty)$$
$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \varphi(0) = \varphi_0 \qquad \text{in} \quad \Omega$$

- Mass is conserved

$$\overline{\varphi(t)} := |\Omega|^{-1} \int_\Omega \varphi(x, t)dx = \overline{\varphi}_0$$

First mathematical results on nonlocal CHNS

- Constant mobility+ regular potential
  - $\exists$ global weak sols in 2D-3D (Colli, F. & Grasselli, J. Math. Anal. Appl. '12)
  - global attractor in 2D and trajectory attractor in 3D (F. & Grasselli, J. Dynam Differential Equations '12)
- Constant mobility+singular potential
  - $\exists$ global weak sols in 2D-3D; global attractor in 2D and trajectory attractor in 3D (F. & Grasselli, Dyn. Partial Differ. Equ. '12)

- **Assumptions on kernel and external force**

$$J \in W^{1,1}(\mathbb{R}^d) \qquad a(x) = \int_\Omega J(x-y)dy \geq 0$$

$$\mathbf{h} \in L^2_{loc}(\mathbb{R}^+; H^1_{div}(\Omega)') \qquad \mathbb{R}^+ := [0, \infty)$$

- **Notion of weak sol**
  Let $\mathbf{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $0 < T < +\infty$ be given.
  $[\mathbf{u}, \varphi]$ is a weak sol to nonlocal CHNS system on $[0, T]$ if

$$\mathbf{u} \in L^\infty(0, T; L^2_{div}(\Omega)^d) \cap L^2(0, T; H^1_{div}(\Omega)^d)$$

$$\mathbf{u}_t \in L^{4/d}(0, T; H^1_{div}(\Omega)'),$$

$$\varphi \in L^\infty(0, T; L^4(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

$$\varphi_t \in L^2(0, T; H^1(\Omega)')$$

$$\mu \in L^2(0, T; H^1(\Omega))$$

$\exists$ weak sols (regular potential, constant mobility)

and $\forall\psi \in H^1(\Omega)$, $\forall\mathbf{v} \in H^1_{div}(\Omega)^d$ and for a.e. $t \in (0,T)$

$$\langle \varphi_t, \psi \rangle + (\nabla\mu, \nabla\psi) = (\mathbf{u}, \varphi\nabla\psi)$$
$$\langle \mathbf{u}_t, \mathbf{v} \rangle + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) = -(\mathbf{v}, \varphi\nabla\mu) + \langle \mathbf{h}, \mathbf{v} \rangle$$

with

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \varphi(0) = \varphi_0$$

where

$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

and

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) := \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} \qquad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in H^1_{div}(\Omega)^d$$

$\exists$ weak sols (regular potential, constant mobility)

Theorem (Colli, F. & Grasselli '11)

Assume $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$. *Then,* $\forall T > 0\ \exists$ *a weak sol* $[\boldsymbol{u}, \varphi]$ *on* $[0, T]$ *which satisfies the energy inequality (identity if* $d = 2$*)*

$$\mathcal{E}(\boldsymbol{u}(t), \varphi(t)) + \int_0^t (\nu \|\nabla \boldsymbol{u}(\tau)\|^2 + \|\nabla \mu(\tau)\|^2) d\tau$$
$$\leq \mathcal{E}(\boldsymbol{u}_0, \varphi_0) + \int_0^t \langle \boldsymbol{h}, \boldsymbol{u}(\tau) \rangle d\tau \qquad \forall t > 0$$

where we have set

$$\mathcal{E}(\boldsymbol{u}(t), \varphi(t)) = \frac{1}{2}\|\boldsymbol{u}(t)\|^2$$
$$+ \frac{1}{4}\int_\Omega \int_\Omega J(x-y)(\varphi(x,t) - \varphi(y,t))^2 dxdy + \int_\Omega F(\varphi(t))$$

Remarks (regular potential, constant mobility)

- All results hold for more general double-well regular potentials F, i.e., for F **with polynomial growth of arbitrary order**
- **Main difficulty**: the nonlocal term implies that φ is not as regular as for the standard (local) CHNS system

$$\varphi \in L^2(H^1) \;\; (\textit{nonlocal}), \; \text{instead of } \varphi \in L^\infty(H^1) \;\; (\textit{local})$$

- **Consequence**: regularity results (higher order estimates in 2D and 3D) and uniqueness of weak sols in 2D difficult issues

Strong sols in 2D (reg. pot., const. mob.)

Theorem (F., Grasselli & Krejčí '13)

Let $\boldsymbol{h} \in L^2_{loc}(\mathbb{R}^+; L^2_{div}(\Omega)^2)$ *and in addition* $J \in W^{2,1}(\mathbb{R}^2)$. *If*

$$\boldsymbol{u}_0 \in H^1_{div}(\Omega)^2 \qquad \varphi_0 \in H^2(\Omega)$$

then, $\forall T > 0$, $\exists$ *unique strong sol* $z := [\boldsymbol{u}, \varphi]$ *s.t.*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T; H^1_{div}(\Omega)^2) \cap L^2(0,T; H^2(\Omega)^2)\\
&\boldsymbol{u}_t \in L^2(0,T; L^2_{div}(\Omega)^2)\\
&\varphi \in L^\infty(0,T; H^2(\Omega))\\
&\varphi_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))
\end{aligned}$$

Moreover, a continuous dependence estimate w.r.t. data $(\boldsymbol{u}_0, \varphi_0, \boldsymbol{h}) \in L^2_{div}(\Omega)^2 \times H^1(\Omega)' \times L^2_{loc}(\mathbb{R}^+; L^2_{div}(\Omega)^2)$ *holds*

An idea of the proof

1) The fact that $\varphi \in L^\infty(\Omega \times (0, T))$ and NS regularity in 2D $\Rightarrow$ regularity for $\mathbf{u}$
2) (Nonlocal CH)$\times \mu_t$ in $L^2(\Omega)$ and use the above regularity to get

$$\|\nabla \mu\|^2 + \int_0^t \|\varphi_t\|^2 d\tau \leq \|\nabla \mu_0\|^2 + C + \int_0^t \alpha(\tau)\|\nabla \mu(\tau)\|^2 d\tau$$

where $\alpha \in L^1(0, T)$ and C depend on $\|\nabla \mathbf{u}_0\|$, $\|\varphi_0\|_{H^2}$, T. Hence

$$\varphi \in L^\infty(0, T; H^1(\Omega)) \qquad \varphi_t \in L^2(0, T; L^2(\Omega))$$

3) (Nonlocal CH)$_t \times \mu_t$ in $L^2(\Omega)$ and use regularity at point 1). By means of *some technical arguments* (Gagliardo-Nirenberg in 2D) we deduce

$$\frac{d}{dt}\int_\Omega (a+F''(\varphi))\varphi_t^2 + \frac{1}{4}\|\nabla\mu_t\|^2 \leq \beta(t)\|\varphi_t\|^2 + C\|\varphi_t\|^4 + \gamma(t)$$

with $\beta, \gamma \in L^1(0,T)$. Then, use a nonlinear Gronwall lemma

$$\left.\begin{array}{r} w'(t) \leq C_1\left(1+w^2(t)\right) \\ \int_0^T w(\tau)d\tau \leq C_2 \end{array}\right\} \Rightarrow w(t) \leq C_3 = C_3(w(0), C_1, C_2, T)$$

and the improved regularity at point 2) to get

$$\varphi_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

4) By comparison in the nonlocal CH we get $\mu \in L^\infty(0,T;H^2(\Omega))$ and finally, using assumption $J \in W^{2,1}(\mathbb{R}^2)$, we get

$$\varphi \in L^\infty(0,T;H^2(\Omega))$$

Strong sols in 2D (reg. pot., const. mob.)

- **Regularization in finite time of weak sols**

For $\eta \geq 0$ given, introduce the *phase spaces*

$$\mathcal{X}_\eta = L^2_{div}(\Omega)^2 \times \mathcal{Y}_\eta \quad \mathcal{Y}_\eta = \{\varphi \in L^2(\Omega) : F(\varphi) \in L^1(\Omega), |\bar{\varphi}| \leq \eta\}$$

$$\mathcal{X}^1_\eta := H^1_{div}(\Omega)^2 \times \mathcal{Y}^1_\eta \qquad \mathcal{Y}^1_\eta := \{\psi \in H^2(\Omega) : |\overline{\psi}| \leq \eta\}$$

If $z_0 = [\mathbf{u}_0, \varphi_0] \in \mathcal{X}_\eta$, then $\forall \tau > 0\ \exists s_\tau \in (0, \tau]$ s.t. $z(s_\tau) \in \mathcal{X}^1_\eta$. Starting from s_τ the weak sol corresponding to z_0 becomes a (unique) strong sol $z \in C([s_\tau, \infty); \mathcal{X}^1_\eta)$.

The regularization is also uniform w.r.t. bdd in $\mathcal{X}_\eta$ sets of initial data, i.e.

Theorem (F., Grasselli & Krejčí '13)

$\exists \Lambda(\eta) > 0$ *s.t.* $\forall z_0 \in H^1_{div}(\Omega)^2 \times H^2(\Omega)$ *with* $|\overline{\varphi}_0| \leq \eta\ \exists t^* = t^*(\mathcal{E}(z_0))$ *s.t. the strong sol corresponding to* z_0 *satisfies*

$$\|\nabla \boldsymbol{u}(t)\| + \|\varphi(t)\|_{H^2(\Omega)} + \int_t^{t+1} \|\boldsymbol{u}(s)\|_{H^2(\Omega)^2} \leq \Lambda(\eta) \qquad \forall t \geq t^*$$

- **The global attractor** (autonomous case)

Let $\mathcal{G}_\eta$ be the set of all weak sols corresponding to all initial data $z_0 = [\mathbf{u}_0, \varphi_0] \in \mathcal{X}_\eta$

Theorem (F. & Grasselli '11)

Let $\boldsymbol{h} \in H^1_{div}(\Omega)'$. Then $\mathcal{G}_\eta$ is a generalized semiflow on $\mathcal{X}_\eta$ which possesses the global attractor $\mathcal{A}_\eta$

Take $z_0 \in \mathcal{B}$ bdd subset of $\mathcal{X}_\eta$ and $\tau = 1$. Then $\exists t^* = t^*(\mathcal{B})$ s.t.

$$z(t) \in B_{\mathcal{X}^1_\eta}(0, \Lambda(\eta)) \qquad \forall t \geq t^*$$

$\Rightarrow$ **regularity of the global attractor**

$$\mathcal{A}_\eta \subset B_{\mathcal{X}^1_\eta}(0, \Lambda(\eta))$$

- **Convergence to equilibria of weak sols**

  Set of stationary sols

$$\mathcal{E}_\eta := \Big\{ z_\infty = [\mathbf{0}, \varphi_\infty] : \ \varphi_\infty \in L^2(\Omega),\ F(\varphi_\infty) \in L^1(\Omega),\ |\overline{\varphi}_\infty| \le \eta,$$
$$a\varphi_\infty - J * \varphi_\infty + F'(\varphi_\infty) = \mu_\infty,\ \ \mu_\infty = \overline{F'(\varphi_\infty)} \quad \text{a.e. in } \Omega \Big\}$$

Theorem (F., Grasselli & Krejčí '13)

Take $z_0 \in \mathcal{X}_\eta$ and let $z \in C(\mathbb{R}^+; \mathcal{X}_\eta)$ be a corresponding weak sol. Then

$$\emptyset \neq \omega(z) \subset \mathcal{E}_\eta$$

and $\exists t^* = t^*(z_0)$ s.t. the trajectory $\cup_{t \ge t^*} \{z(t)\}$ is precompact in $\mathcal{X}_\eta$. Moreover $\exists z_\infty \in \mathcal{E}_\eta$ s.t.

$$z(t) \to z_\infty \qquad in\ \ \mathcal{X}_\eta \qquad as \quad t \to \infty$$

Regular potentials, constant mobility

Theorem (F., Gal & Grasselli '13)

Let $\mathbf{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$. Then, $\exists$ a unique weak sol $[\mathbf{u}, \varphi]$ corresponding to $[\mathbf{u}_0, \varphi_0]$.

- **Idea of the proof.** By redefining the pressure π, the Korteweg force $\mu\nabla\varphi$ can be rewritten as

$$-(\nabla a/2)\varphi^2 - (J * \varphi)\nabla\varphi$$

Consider two weak sols corresponding to the same initial data $[\mathbf{u}_0, \varphi_0]$. Then, setting $\mathbf{u} := \mathbf{u}_2 - \mathbf{u}_1$ and $\varphi := \varphi_2 - \varphi_1$

$$\varphi_t = \Delta\widetilde{\mu} - \mathbf{u} \cdot \nabla\varphi_2 - \mathbf{u}_1 \cdot \nabla\varphi$$
$$\widetilde{\mu} = a\varphi - J * \varphi + F'(\varphi_2) - F'(\varphi_1)$$
$$\mathbf{u}_t - \nu\Delta\mathbf{u} + ((\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2 - (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1)$$
$$= -\varphi(\varphi_1 + \varphi_2)(\nabla a/2) - (J * \varphi)\nabla\varphi_2 - (J * \varphi_1)\nabla\varphi$$

Uniqueness of weak sol in 2D

Multiply NS by $\mathbf{u}$, nonlocal CH by $(-\Delta_N)^{-1}\varphi$ and sum. By means of some technical arguments (Gagliardo-Nirenberg in 2D) we are led to

$$\frac{d}{dt}\Big(\|\mathbf{u}\|^2 + \|(-\Delta_N)^{-1/2}\varphi\|^2\Big) + c_0\|\varphi\|^2 + \frac{\nu}{2}\|\nabla\mathbf{u}\|^2$$
$$\leq \beta\Big(\|\mathbf{u}\|^2 + \|(-\Delta_N)^{-1/2}\varphi\|^2\Big)$$
$$\beta := c(\|\varphi_1\|_{L^4}^4 + \|\varphi_2\|_{L^4}^4 + \|\mathbf{u}_1\|_{L^4}^4 + \|\nabla\mathbf{u}_2\|^2 + 1) \in L^1(0,T)$$

- A continuous dependence estimate in $L^2_{div} \times (H^1)'$ also holds

$$\|\mathbf{u}_2(t) - \mathbf{u}_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2_{(H^1)'}$$
$$+ \int_0^t \Big(c_0\|\varphi_2(\tau) - \varphi_1(\tau)\|^2 + \frac{\nu}{2}\|\nabla(\mathbf{u}_2(\tau) - \mathbf{u}_1(\tau))\|^2\Big)d\tau$$
$$\leq \Gamma_1(t)\big(\|\mathbf{u}_{02} - \mathbf{u}_{01}\|^2 + \|\varphi_{02} - \varphi_{01}\|^2_{(H^1)'}\big) + C_\eta\Gamma_2(t)|\overline{\varphi}_{02} - \overline{\varphi}_{01}|$$

$|\overline{\varphi}_{01}|, |\overline{\varphi}_{02}| \leq \eta$, with $\Gamma_i \in C(\mathbb{R}^+)$ depending on weak sols norms

Consequence: the nonlocal CHNS system generates a *semigroup* $S(t)$ of *closed* operators on $\mathcal{X}_\eta$

$$z(t) := [\mathbf{u}(t), \varphi(t)] = S(t)z_0 := S(t)[\mathbf{u}_0, \varphi_0]$$

Remark: by similar arguments uniqueness of the weak sol in 2D holds for the nonlocal CHNS system also for the following cases

- constant mobility+singular potential
- degenerate mobility+singular potential

Exponential attractors in 2D (reg. pot., const. mob.)

Definition

A compact set $\mathcal{M} \subset \mathcal{X}_\eta$ is an *exponential attractor* for the semigroup $S(t)$ if the following properties are satisfied

(i) Positively invariance: $S(t)\mathcal{M} \subset \mathcal{M}\ \forall t \geq 0$

(ii) Finite dimensionality: $\dim_F \mathcal{M} < \infty$

(iii) Exponential attraction: $\exists J : \mathbb{R}^+ \to \mathbb{R}^+$ increasing and $\kappa > 0$ s.t., $\forall R > 0$ and $\forall \mathcal{B} \subset \mathcal{X}_\eta$ with $\sup_{z \in \mathcal{B}} \mathbf{d}_{\mathcal{X}_\eta}(z, 0) \leq R$ there holds

$$\mathrm{dist}(S(t)\mathcal{B}, \mathcal{M}) \leq J(R)e^{-\kappa t}$$

Exponential attractors in 2D (reg. pot., const. mob.)

Theorem (Efendiev & Zelik '09)

Let $\mathcal{H}$ be a metric space and $\mathcal{V}, \mathcal{V}_1$ Banach spaces s.t. $\mathcal{V}_1 \hookrightarrow\hookrightarrow \mathcal{V}$. Let B be a bdd subset of $\mathcal{H}$ and $\mathbb{S} : B \to B$ a map s.t.

$$d_{\mathcal{H}}(\mathbb{S}z_{02}, \mathbb{S}z_{01}) \leq \gamma d_{\mathcal{H}}(z_{02}, z_{01}) + K\|\mathbb{T}z_{02} - \mathbb{T}z_{01}\|_{\mathcal{V}}$$

$\forall z_{01}, z_{02} \in B$, where $\gamma < 1/2$, $K \geq 0$ and $\mathbb{T} : B \to \mathcal{V}_1$ is a globally Lipschitz continuous map, i.e.,

$$\|\mathbb{T}z_{02} - \mathbb{T}z_{01}\|_{\mathcal{V}_1} \leq L d_{\mathcal{H}}(z_{02}, z_{01}), \quad \forall z_{01}, z_{02} \in B,$$

for some $L \geq 0$. Then, $\exists$ a (discrete) exponential attractor $\mathcal{M}_d \subset B$ for the (time discrete) semigroup $\{\mathbb{S}^n\}_{n=0,1,2,\ldots}$ on B (with the topology of $\mathcal{H}$ induced on B).

Exponential attractors in 2D (reg. pot., const. mob.)

Theorem (F., Gal & Grasselli '13)

For every $\eta \geq 0$ the dynamical system $(\mathcal{X}_\eta, S(t))$ possesses an exponential attractor $\mathcal{M}_\eta$

Main steps of the proof:

- using the results on existence of strong sols, we need estimates for the difference of two sols in the $L^2_{div} \times L^2$ -norms with data in $H^1_{div} \times H^2$ (also for time derivatives)
- introduce $B_1 := \bigcup_{t \geq t_0} S(t)\mathcal{B}_0$ ($\mathcal{B}_0$ a bdd absorbing set in $\mathcal{X}_\eta$) and by means of eventual regularization result, construct $\mathbb{B} = S(t^*)B_1$ bdd in $H^1_{div} \times H^2$, positively invariant and absorbing in $\mathcal{X}_\eta$
- uniform Hölder-continuity of $(t, z_0) \mapsto S(t)z_0$ on $[0, T] \times \mathbb{B}$ to get an exponential attractor for the continuous $S(t)$