Phase transitions and hysteresis: new perspectives and results

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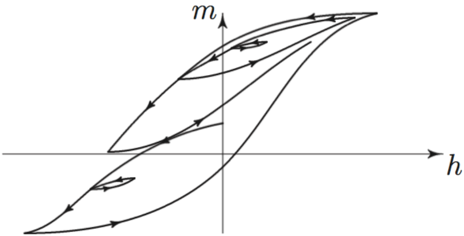
Hysteresis: a rate-independent memory effect

- **Hysteresis:** a rate-independent memory effect (multidisciplinary character)



Tipical hysteresis diagram in ferromagnetism (h magnetic field, m magnetization).

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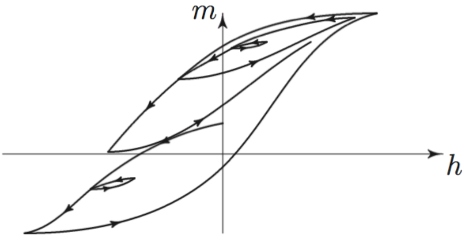

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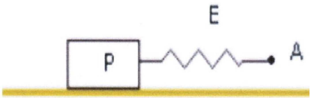
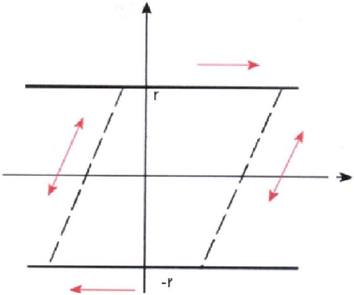
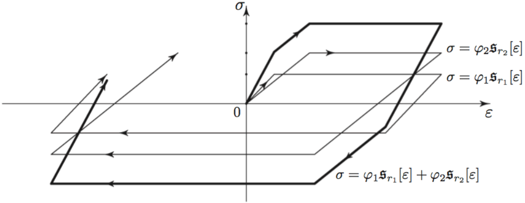
The stop and the Prandtl-Ishlinskii operators







A classical hysteresis-type model for 1D elastoplasticity

- Introduced by L. Prandtl and A. Yu. Ishlinskii (extensions to the multidimensional case are possible)
- The relation between (one-dimensional) strain ε and stress σ is given in the form of the so-called Prandtl-Ishlinskii operator

$$\sigma = \mathscr{P}[\varepsilon](t) = \int_0^\infty \mathfrak{s}_r[\varepsilon](t)\,\varphi(r)\,\mathrm{d}r$$

for all $\varepsilon \in W^{1,1}(0,T)$. Here $\varphi > 0$ is a nonnegative weight function not known a priori and $\mathfrak{s}_r$ represents the **one-dimensional elastic-ideally plastic element or stop operator**, with the threshold $r > 0$

- Prandtl-Ishlinskii description of elastoplasticity: a superposition of infinitely many stop operators having different thresholds (very imaginative and easily understood) BUT engineers very often prefer classical engineering approaches like the three-dimensional von Mises or Tresca models
- Motivation: the disadvantage that the weight function φ **is not known a priori** and must be identified

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New theory of oscillating elastoplastic beams and plates

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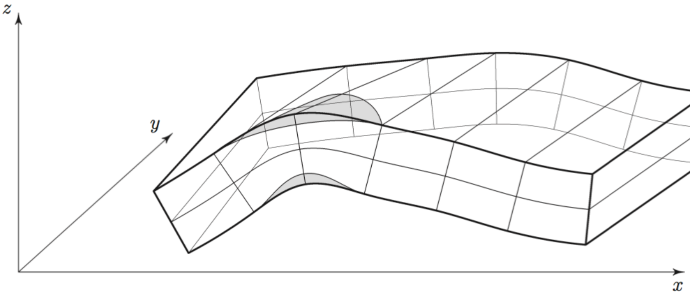

A plate section with grey plasticized zone.

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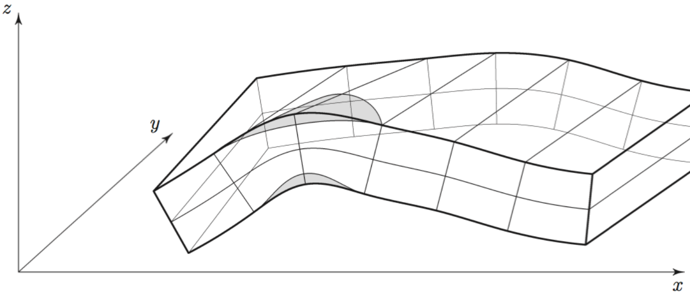

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Motivation for material fatigue

- Plastic deformations lead to energy dissipation and material fatigue, manifested by material softening, heat release, material failure in finite time
- Very important: take into account the effects of energy exchange and estimating the lifetime of oscillating thermoelastoplastic structures under material fatigue
- **Aim:** develop a thermodynamically consistent theory of oscillating thermoelastoplastic plates under material fatigue (dynamic approach - different from literature)
- The resulting system from the theory developed by Krejčí & al:

$$\partial_{tt} w - \partial_{tt}\Delta w + \mathbf{D}_2^* \sigma = g\,,$$
$$\sigma = \mathbf{B}\varepsilon + \int_0^\infty \mathfrak{s}_{rZ}[\varepsilon](t)\,\varphi(r)\,\mathrm{d}r$$
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- We introduce $\theta > 0$ (absolute temperature) and $m(x,t) \geq 0$ (material fatigue); **aim:** get an $\boxed{\text{evolution equation for } m}$ consistent from the thermodynamic point of view

Evolution equation for the fatigue

- **Main assumption:** proportionality between rate of fatigue $\partial_t m$ and

$$\begin{aligned}\mathscr{D} &= \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathscr{S}[\theta,\varepsilon] - \partial_t \mathscr{F}[\theta,\varepsilon] \\ &= -\frac{1}{2}\left\langle \mathbf{B}'(m)\varepsilon, \varepsilon \right\rangle \partial_t m + \int_0^\infty \langle \partial_t(\varepsilon - \mathfrak{s}_{rZ}[\varepsilon]), \mathfrak{s}_{rZ}[\varepsilon] \rangle \, \varphi(\theta, r)\, \mathrm{d}r\end{aligned}$$

  where $\mathscr{F}$ is the **specific free energy** and $\mathscr{S}$ is the **specific entropy**
- Justified by the so-called **rainflow method for cyclic fatigue accumulation** in uniaxial processes (counts closed hysteresis loops in the loading hystory - mechanism of energy dissipation)
- In multiaxial loading processes? Experimental measurements at the point of material failure: strong temperature increase, manifested by energy dissipation peak (temperature tests are in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry))

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The model with phase transition

- **Motivation:**
  - possibility to account also for decreasing fatigue rate (in view of engineering applications)
  - the material can be partially repaired by local melting
- **How to achieve this goal:**
- Phase transition equation in the form of melting-solidification law

$$\gamma \chi_t \in -\partial_\chi \mathscr{F}[\varepsilon, \theta, \chi] \qquad \chi \in [0,1]$$

$\chi_0 \in [0,1]$ some initial condition, $A(x,t) := \int_0^t \frac{1}{\gamma} \left(\frac{L}{\theta_c}(\theta - \theta_c) \right)(x,\tau)\, d\tau$

$\boxed{(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0,1]}$

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    w material fatigue and χ degree of melting
    the time of failure of the material can be shifted
    possibly considering a sufficiently large time interval of observation (useful engineering viewpoint) a global solution of the corresponding PDEs system can be found
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Thermodynamical consistency

- If we introduce $\mathscr{F}[\varepsilon,\theta,\chi]$ **specific free energy**, $\mathscr{S}[\varepsilon,\theta,\chi]$ **specific entropy** and $\mathscr{U}[\varepsilon,\theta,\chi]$ **internal energy** we are able to show that the first and second principles of thermodynamics are satisfied

$$\frac{\partial}{\partial t}\mathscr{U}[\varepsilon,\theta,\chi] + \mathrm{div}\mathbf{q} = \langle\sigma,\varepsilon_t\rangle \quad \text{(energy conservation)}$$
$$\frac{\partial}{\partial t}\mathscr{S}[\varepsilon,\theta,\chi] + \mathrm{div}\frac{\mathbf{q}}{\theta} \geq 0, \quad \text{(Clausius-Duhem inequality)}$$

- **Evolution equation for m:**

$$\left(C - \langle\mathbf{B}'(m)\varepsilon,\varepsilon\rangle\right)m_t = -h(\chi_t) + \int_0^\infty \langle\partial_t(\varepsilon - \mathfrak{s}_{rZ}[\varepsilon])\mathfrak{s}_{rZ}[\varepsilon]\rangle\varphi(\theta,r)\,\mathrm{d}r$$

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- Rainflow method for fatigue evaluation in elastoplastic materials (uniaxial cyclic loading) allow to consider dissipated energy as a measure for fatigue
- The solution cannot be expected to exist globally: singularities (thermal shocks) occur in finite time
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- The time of failure can be shifted and considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found
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