Adv. Math. Meth. 25 July 2023

1) $f(x,y) = -xy - x\sqrt{y} \in C^0$ and
$f_y(x,y) = -x + \dfrac{x}{2y^{1/2}} \in C^0(\mathbb{R}\times(0,+\infty))$
$\Rightarrow \exists!$ local solution $\forall K>0$.
For $K=0$ we have only local existence
but the solution could be not unique
If $K<0$ the equation has no
meaning.
It is a Bernoulli type equation:
we make the substitution:
$z = y^{1/2} > 0 \Rightarrow$

$$2z' = -xz + x$$ which

has solution $z = 1 + Ce^{-x^2/4} \Rightarrow$
$y(x) = (1 + C e^{-x^2/4})^2$.
Substituting $y(0) = K = (1+C)^2$
$\Rightarrow 1+C = \sqrt{K} \Rightarrow C = \sqrt{K} - 1$.

2) The system can be rewritten as ②

$$X' = AX \text{ with}$$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$\det(A - \lambda I) = (1-\lambda)^3 \Rightarrow$
$\lambda = 1$ is eigenvalue with multiplicity $m = 3$.

We can find the eigenvectors v:

$(A - I)v = 0 \Rightarrow$

$$\begin{cases} 3y + 2z = 0 \\ y = 0 \\ x \in \mathbb{R} \end{cases} \Rightarrow v = (x, 0, 0) \quad \forall x \in \mathbb{R}$$

$\Rightarrow v_1 = (1, 0, 0)$

$(A - I)v = v_1$ gives

$$\begin{cases} 3y + 2z = 1 \\ y = 0 \\ x \in \mathbb{R} \end{cases} \Rightarrow v_2 = \left(0, 0, \tfrac{1}{2}\right)$$

$(A - I)v = v_2$ gives

$$\begin{cases} 3y + 2z = 0 \\ 2y = \tfrac{1}{2} \\ x \in \mathbb{R} \end{cases} \Rightarrow v = \left(x, \tfrac{1}{4}, -\tfrac{3}{8}\right) \quad \forall x \in \mathbb{R}$$

$\Rightarrow v_3 = \left(0, \tfrac{1}{4}, -\tfrac{3}{8}\right)$

Hence we get $A = PBP^{-1}$ where (3)

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{3}{8} \end{pmatrix}$$

$$\Rightarrow X(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/3 \\ 0 & 1/2 & -3/8 \end{pmatrix} \begin{pmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} c_1e^t + c_2te^t + \frac{1}{2}c_3t^2e^t \\ \frac{1}{4}c_3e^t \\ \frac{1}{2}c_2e^t - \frac{3}{8}c_3e^t + \frac{1}{2}c_3te^t \end{pmatrix}$$

$\forall c_1, c_2, c_3 \in \mathbb{R}$

Which can be also written as:

$$X(t) = c_1e^tv_1 + c_2e^t(tv_1 + v_2) + c_3\left(\frac{1}{2}t^2v_1 + tv_2 + v_3\right)e^t$$

3) $f_n \to 0$ pointwise

because as $n \to \infty$ $f_n(x) \sim \frac{x^3}{n} \frac{\pi}{2}$
$x > 0$

and for $x = 0$ $f_n = 0$

$\Rightarrow f \equiv 0$ on $[0, +\infty)$

$$\sup g_n = \frac{3^{3/4}}{4\sqrt[4]{n}} \xrightarrow[n\to\infty]{} 0$$

$$g_n'(x) = \frac{3x^2 \cdot (n + x^4) - 4x^3 \cdot x^3}{(n + x^4)^2}$$

$$= \frac{3x^2 \cdot n + 3x^6 - 4x^6}{(\quad)^2}$$

$$= \frac{-x^6 + 3x^2 \cdot n}{(\quad)} = 0$$

$\Leftrightarrow$ $x^4 = 3n$ $\Rightarrow$ $x_n = 3^{1/4} \cdot n^{1/4}$

and $g_n(x_n) = \frac{3^{3/4} \cdot n^{3/4}}{n + 3n} = \frac{3^{3/4}}{4n^{1/4}}$

$\Rightarrow$ $0 \leq f_n(x) \leq \frac{\pi}{2} g_n(x) \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$ $\sup |f_n - 0| \xrightarrow[n\to\infty]{} 0$

4) a) $\forall v \in \mathcal{D}(\mathbb{R})$ we have

$$\langle (\psi f)', v\rangle = -\langle \psi f, v'\rangle = -\langle f, \psi v'\rangle$$

and $\langle \psi' f, v\rangle = \langle f, \psi' v\rangle$

$$\langle \psi f', v\rangle = \langle f', \psi v\rangle = -\langle f, (\psi v)'\rangle$$

$$= -\langle f, \psi' v\rangle - \langle f, \psi v'\rangle$$

and summing up we get the result.

b) Applying a), we get:

$$\underset{\substack{\uparrow \\ x\delta(x)=0}}{0} = (x\delta)' = \delta + x\delta' \Rightarrow$$

$$x\delta' = -\delta$$