Master Program in Electronic Engineering

**Advanced Mathematical Methods for Engineers**

**January 24, 2023**

**1.** Let $\alpha \in \mathbf{R} \setminus \{0\}$, consider the following Cauchy Problem

$$\begin{cases} y'(x) = \dfrac{y}{x+1}(1 - 3xy) \\ y(2) = 2/\alpha. \end{cases}$$

a) Discuss local and global existence and uniqueness of solutions, depending on $\alpha$.

b) Find explicitly the solutions $y_\alpha$ (depending on $\alpha$).

c) Find the values of the parameter $\alpha$ such that $[0, 3] \subset \mathrm{dom}(y_\alpha)$.

**2.** Given the ODE system

$$\begin{cases} x' = x(1 - y^3) \\ y' = y(4 - x^2), \end{cases}$$

find the stationary points and discuss their stability.

**3.** Let, for $x > 0$ and $n \in \mathbf{N} \setminus \{0\}$,

$$f_n(x) := \frac{n}{3x} \arctan\left(\frac{4x}{n}\right) \frac{1}{16 + x^2}.$$

Then

a) Find the pointwise limit $f$ of $f_n$ as $n$ tends to $+\infty$.

b) Prove that $f_n \in L^1(0, +\infty)$ for every $n$.

c) Compute, justifying the passages, the $\lim_{n\to+\infty} \int_0^{+\infty} f_n(x)\, dx$ .

**4.** Let $u(x) = x - \arctan(x) \in \mathcal{S}'(\mathbf{R})$. Then compute $\mathcal{F}(u)$ in $\mathcal{S}'$.
*Hint:* Compute first $\mathcal{F}(u')$ and $\mathcal{F}(\mathrm{e}^{-|x|})$.