Master Program in Electronic Engineering

Advanced Mathematical Methods for Engineers

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1. Let $k \in \mathbf{R}$, consider the following Cauchy Problem

$$\begin{cases} y'(x) = y(x)(y(x)-1)^{1/3} \\ y(0) = k. \end{cases}$$

a) Discuss local and global existence and uniqueness of solutions, depending on k.

b) Draw the graph of the solutions, defining the domain, studying the monotonicity, the convexity, and limits at the extrema of the domain for

b1) $k < 0$,

b2) $k \in (0,1)$,

b3) $k > 1$,

b4) $k = 1$.

2. Given the ODE

$$x'' + k(x-2)x' + \tan(x) = 0,$$

prove that the origin $x = 0$ is asymptotically stable if $k < 0$.

3. Compute, justifying the passages, the following

$$\lim_{n\to+\infty} \int_0^{n+\sqrt{n}} \frac{\log(1+x)}{x^2+n^2+1}\, dx\,.$$

4. Let $g \in C^1([0,\pi])$, find "formally" the solution u, using the method of separation of variables, of the following problem:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 \quad 0 < x < \pi,\ t > 0 \\ u(x,0) = g(x) \quad 0 \le x \le \pi \\ u(0,t) = 0, \quad u_x(\pi,t) = 0 \quad t > 0. \end{cases}$$

Then compute the solution in case $g(x) = 2\sin\left(\frac{3}{2}x\right)$.