Master Program in Electronic Engineering

Advanced Mathematical Methods for Engineers

January 26, 2021

1. Let $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, consider the following Cauchy Problem

$$\begin{cases} y'(x) = x^5(e^{4-y^2} - 1) \\ y(x_0) = y_0. \end{cases}$$

a) Discuss local and global existence of solutions.

b) Draw the graph of the solutions, studying the monotonicity, the convexity and the existence of maxima and minima, asymptots and limits at the extrema of the domain.

2. Given the following ODE system

$$\begin{cases} x' = 3x + y \\ y' = -x + y \end{cases}$$

2.1) Solve the system.

2.2) Find the bounded solutions on $(-\infty, 0]$.

3. Consider, for $x \in (0, +\infty)$ the sequence of functions f_n:

$$f_n(x) = x^n e^{-nx}.$$

a) Prove that $f_n \in L^1(0, +\infty)$ for every $n \in \mathbf{N}$.

b) Compute the pointwise limit f of f_n as $n \to \infty$.

c) Prove that $f_n(x) \leq f_1(x)$ for $x > 0$ and for every $n \geq 1$.

d) Compute the $\lim_{n\to\infty} \int_0^{+\infty} f_n(x)\, dx$, justifying the computations.

4. Find the solution u, using the method of separation of variables, of the following problem:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = tx \quad 0 < x < \pi,\ t > 0 \\ u(x,0) = 1 \quad 0 \leq x \leq \pi \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0 \quad t > 0. \end{cases}$$

Hint: Find first the functions $v_k(x)$ in the definition of solution u_0 of the homogeneous equation ($u_0(x,t) = \sum t_k(t) v_k((x))$ and use then the method of variations of aritrary constants writing the solution of the non-homogeneous equation as $u(x,t) = \sum c_k(t) v_k(x)$ and find c_k imposing the equation and the initial condition and writing down $f(x) = x$ in cos-series.