Advanced Math. Meth. for Eng. (1)

January 31, 2018

Solutions

1.1) The function $f(x,y) = |y|(1-y)$ is locally Lip. because

$$|f(x,y_1) - f(x,y_2)| \le \big||y_1| - |y_2|\big| + \big||y_1|y_1 - |y_2|y_2\big| \le |y_1 - y_2|\,(1 + |y_1| + |y_2|) \le L\,|y_1 - y_2|$$

$\forall (x,y_1),(x,y_2) \in \mathcal{U}(x_0,y_0)$ with $L = 1 + 2\sup_{y\in\mathcal{U}} |y|$ "$\mathcal{U}(x_0)\times\mathcal{U}$"

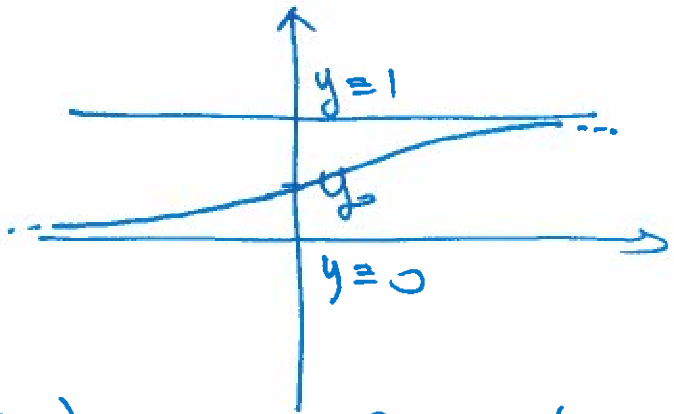
$\Rightarrow \exists!$ local solution $y : \mathrm{dom}(y) \subseteq \mathbb{R} \to \mathbb{R}$.

we do not know if the solution is global.

1.2) $y \equiv 0$ and $y \equiv 1$ are solutions if $y_0 = 0$ and $y_0 = 1$ respectively

and by monotonicity and the asymptots theorem we get : $\mathrm{dom}(y) = \mathbb{R}$ $\forall y_0 \in [0,1)$

and



$\lim_{x\to-\infty} y(x) = 0$, $\lim_{x\to+\infty} y(x) = 1$

1.3) For $y_0 \neq 0$, $y_0 \neq 1$ we can separate the variables and we have: (2)

$$\frac{dy}{|y|(1-y)} = dx$$

If $y > 0 \Rightarrow y(x) = \frac{K}{K+e^{-t}}$, $K \in \mathbb{R}$ solves the ODE

If $y < 0$ $y(x) = \frac{1}{1+Ke^{t}}$ solves the ODE

$\Rightarrow$

a) $y = \frac{2}{2-e^{-t}}$, $t > -\log 2$

a) $y = \frac{1}{1+e^{-t}}$, $t \in \mathbb{R}$

b) $y = \frac{1}{1-2e^{t}}$, $t > -\log 2$

2) $A = \begin{pmatrix} -1 & 2 \\ (1+\alpha) & -5 \end{pmatrix}$

$0 = \det(A - \lambda I) = (+1+\lambda)\cdot(+5+\lambda) - 2\cdot(1+\alpha)$

$= 5 + \lambda + 5\lambda + \lambda^2 - 2 - 2\alpha = 0$

$= \lambda^2 + 6\lambda + 3 - 2\alpha = 0$

$\lambda_{1,2} = -3 \pm \sqrt{9-3+2\alpha}$

$= -3 \pm \sqrt{6+2\alpha}$

If $\alpha = -3$ $\lambda_1 = \lambda_2 = -3 < 0 \Rightarrow (0,0)$ is asymptotically stable

If $\alpha < -3$ $\quad Re(\lambda_{1,2}) < 0 \Rightarrow$ (3)
$(0,0)$ is asymptotically stable

If $\alpha > -3$ $\quad \lambda_1 \neq \lambda_2 \in \mathbb{R}$

If $-3 < \alpha < 3/2 \Rightarrow (0,0)$ is asympt. stable

If $\alpha > 3/2$ it is not

In case $\alpha = -1$, we are in the case $-3 < \alpha < 3/2$
$\Rightarrow \lambda_{1,2} = -1, -5$ and the eigenvectors
are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$c_1, c_2 \in \mathbb{R}$

3) a) $\lim\limits_{n \to +\infty} f_n(x) = 0 =: f(x) \quad \forall \alpha, \beta > 0$
$\forall x \in [0,1]$

b) we try to use the Lebesgue theorem to pass to the limit under integral:
The function $f_n(x) = (nx)^\alpha x^{\beta-\alpha} e^{-n^2x^2}$
and $(nx)^\alpha e^{-n^2x^2}$ is bdd

$$\Rightarrow 0 \leq f_n(x) \leq c\, x^{\beta-\alpha} = \underbrace{\frac{c}{x^{\alpha-\beta}}}_{g(x)}$$

$g \in L^1(0,1)$
iff $\alpha - \beta < 1 \Rightarrow \alpha < 1 + \beta$

Hence if $\alpha < 1+\beta \Rightarrow$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx = 0$$

c) If $\alpha \geq 1+\beta \Rightarrow$

$$\int_0^1 f_n(x) = \int_0^n t^\alpha e^{-t^2}\left(\frac{t}{n}\right)^{\beta-\alpha}\frac{dt}{n}$$

$$= \frac{1}{n^{\beta-\alpha+1}} \underbrace{\int_0^n t^\beta e^{-t^2}\,dt}$$

$\downarrow$ 1 if $\beta-\alpha+1=0$

$\downarrow$ $+\infty$ if $\beta-\alpha+1<0$

$n\to+\infty$ $\downarrow$ because $\ell>0$ $t^\beta e^{-t^2} \in L^1(0,+\infty)$

In any case $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f = 0$

4) Set $u(x,t) = e^{at}\,w(x,t)$

where $a = \dfrac{\pi^2}{2\ell^2} \Rightarrow$

$$u_t = e^{at}(aw + w_t),\quad u_x = e^{at} w_x,$$

$$u_{xx} = e^{at} w_{xx} \Rightarrow w \text{ solves}$$

$$w_t - w_{xx} = 0, \quad w(0,t) = w(L,t) = 0$$

$$w(x,0) = 3\,\mathrm{sen}\left(\frac{5\pi x}{\ell}\right)$$

$\Rightarrow$

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{\ell^2}t} \operatorname{sen}\left(\frac{n\pi x}{\ell}\right) \quad (5)$$

Imposing $w(x,0) = 3 \operatorname{sen}\left(\frac{5\pi x}{\ell}\right)$

we get $b_5 = 3$, $b_n = 0 \ \forall n \neq 5$

and $u(x,t) = e^{\frac{\pi^2}{2\ell^2}t} \cdot 3 e^{-\frac{25\pi^2}{\ell^2}t} \operatorname{sen}\left(\frac{5\pi x}{\ell}\right)$

$$= 3 e^{\frac{\pi^2}{\ell^2}\cdot\left(\frac{1}{2}-25\right)t} \operatorname{sen}\left(\frac{5\pi x}{\ell}\right)$$