Master Program in Electronic Engineering

Advanced Mathematical Methods for Engineers

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1. Let $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$ and consider the following Cauchy Problem

$$\begin{cases} y'(x) = |y(x)|(1 - y(x)) \\ y(x_0) = y_0. \end{cases}$$

1.1) Discuss local and global existence and uniqueness of solutions.

1.2) Draw a qualitative graph of solutions for $y_0 \in [0, 1]$.

1.3) Find the explicit solutions (with the respective domains) in the three cases:

a) $(x_0, y_0) = (0, 1/2)$,
b) $(x_0, y_0) = (0, -1)$,
c) $(x_0, y_0) = (0, 2)$.

2. Given $\alpha \in \mathbf{R} \setminus \{3/2\}$ and the following ODE system

$$\begin{cases} x' = -x + 2y \\ y' = (1 + \alpha)x - 5y, \end{cases}$$

find the values of α such that the null solution $(0, 0)$ is asymptotically stable for the system. Moreover in case $\alpha = -1$ compute explicitely the solutions $(x(t), y(t))$.

3. Consider in $[0, 1]$ the sequence of functions (for $\alpha, \beta > 0$)

$$f_n(x) = n^\alpha x^\beta e^{-n^2 x^2}.$$

a) Compute the pointwise limit of f_n as $n \to \infty$.

b) Compute the $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx$ in case $\alpha < 1 + \beta$.

c) Establish if it is true or not that $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx$ in case $\alpha \geq 1 + \beta$.

4. Using the method of separation of variables, determine the solution of the following Initial-Boundary Value Problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) - \frac{\pi^2}{2\ell^2} u(x,t) = 0 & \text{for } (x,t) \in (0,\ell) \times \mathbb{R}^+ \\ u(x,0) = 3\sin\left(\frac{5\pi x}{\ell}\right) & \text{for } x \in (0,\ell) \\ u(0,t) = u(\ell,t) = 0 & \text{for } t \in \mathbb{R}^+ \end{cases}$$

where $\ell \in \mathbb{R}^+$.