Advanced Math. Meth. for Engineers

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1) $f(x,y) = \frac{y}{x+1}(1-3xy) \in C^1(\mathrm{dom} f)$

$\mathrm{dom} f : x \neq -1 \Rightarrow (-1,+\infty) \ni 2$

$\Rightarrow \exists!$ local solution $y: \mathcal{U}(2) \to \mathbb{R}$

The equation is of Bernoulli type, we do not know if the solution will be global because f is not globally Lip. or sublinear.

$y(x) = \frac{1}{z(x)}$ ← substitution ⇒

$$\begin{cases} z' + \frac{z}{x+1} = \frac{3x}{x+1} \\ z(2) = \alpha/2 \end{cases}$$ ⇐ linear

$$\Rightarrow z(x) = \frac{3}{x+1}\left\{\frac{\alpha}{2} + \int_2^x \tau \, d\tau\right\} =$$

$$= \frac{3(x^2+\alpha-4)}{2(x+1)}$$

$$\Rightarrow y(x) = \frac{2(x+1)}{3(x^2+\alpha-4)}$$

To find dom y_α we need to impose $x^2+\alpha-4 \neq 0$. In order to have $\mathrm{dom}(y) \supset [0,3]$, we need to find α : $x^2+\alpha-4 \neq 0$ on $[0,3]$

This is obviously true if $\alpha > 4$ and false for $\alpha = 4$. If $\alpha < 0$ we have

$x^2 + \alpha - 4 = 0 \iff x = \pm\sqrt{4-\alpha} \Rightarrow$

$x^2 + \alpha - 4 \neq 0$ on $[0,3] \iff \sqrt{4-\alpha} > 3$, i.e.

$\alpha < -5$.

And so we get that $[0,3] \subset \mathrm{dom}\, y_\alpha$

$\iff \alpha < -5$ or $\alpha > 4$

2) $\begin{cases} x' = f(x,y) = 0 \\ y' = g(x,y) = 0 \end{cases} \iff$

$P_0 = (0,0)$, $P_1 = (2,1)$, $P_2 = (-2,1)$,

$\underline{F} = \begin{pmatrix} f \\ g \end{pmatrix}$ $\quad D\underline{F} = \begin{pmatrix} 1-y^3 & -3xy^2 \\ -2xy & 4-x^2 \end{pmatrix}$

$DF(P_0) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow P_0$ is unstable

$DF(P_1) = \begin{pmatrix} 0 & -6 \\ -4 & 0 \end{pmatrix}$

$Tr = 0$, $Det = -24 \Rightarrow \lambda_\pm = \pm 2\sqrt{6}$

$\Rightarrow P_1$ is unstable

$DF(P_2) = \begin{pmatrix} 0 & 6 \\ 4 & 0 \end{pmatrix}$ $\quad Tr = 0$ $Det = -24$

$\Rightarrow \lambda_\pm = \pm 2\sqrt{6} \Rightarrow P_2$ is unstable

3) a), b) $\lim_{n\to\infty} f_n(x) \overset{(*)}{=} \lim_{n\to\infty} \frac{4}{3}\cdot\frac{1}{16+x^2} = \frac{4}{3}\cdot\frac{1}{16+x^2}$

$0 \le f_n(x) \le \frac{n}{3x}\cdot\frac{4x}{n}\cdot\frac{1}{16+x^2} =$

$= \frac{4}{3}\cdot\frac{1}{16+x^2} \in L^1(0,+\infty)$

$\Rightarrow f_n \in L^1(0,+\infty)$

(*) because $\arctan\left(\frac{4x}{n}\right) \underset{n\to\infty}{\sim} \frac{4x}{n}$ $\forall x>0$

c) $|f_n(x)| \le \frac{4}{3}\frac{1}{16+x^2} = g(x) \in L^1(0,+\infty)$

$\underset{\text{by Lebesgue}}{\Rightarrow} \lim_{n\to\infty}\int_0^{+\infty} f_n(x)\,dx =$

$= \int_0^{+\infty} f(x)\,dx =$

$= \int_0^{+\infty} \frac{4}{3}\frac{1}{16+x^2}\,dx$

$= \frac{1}{3}\arctan\left(\frac{x}{4}\right)\Big|_0^{+\infty}$

$= \frac{\pi}{6}$

4) $u'(x) = 1 - \frac{1}{1+x^2} \Rightarrow$ (4)

$$\hat{u'} = \hat{1} - \mathcal{F}\left(\frac{1}{1+x^2}\right) = 2\pi\delta_0 - \pi e^{-|\xi|}$$

because

$$\mathcal{F}(e^{-|x|}) = \int_{-\infty}^{+\infty} e^{-|x|} e^{-i\xi x}\, dx$$

$$= \int_{-\infty}^{0} e^{x} e^{-i\xi x} dx +$$

$$+ \int_{0}^{+\infty} e^{-x} e^{-i\xi x} dx$$

$$= \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2}$$

$$\Rightarrow e^{-|x|} = \frac{1}{2\pi} \int \frac{2}{1+\xi^2} e^{i\xi x} d\xi$$

$$\Rightarrow$$

$$\pi e^{-|x|} = \int_{\mathbb{R}} \frac{e^{+i\xi x}}{1+\xi^2} d\xi$$

$$\Rightarrow i\xi \hat{u} = 2\pi\delta_0 - \pi e^{-|\xi|}$$

Division : we solve first $i\xi \hat{u}_1 = 0$

$\Rightarrow \hat{u}_1 = c\delta_0$

then $i\xi \hat{u}_2 = 2\pi\delta \Rightarrow$

$\xi \hat{u}_2 = -2\pi i \delta \Rightarrow$

$\hat{u}_2 = 2\pi i \delta'$

Then $i\xi\,\hat{u}_3 = -\pi e^{-|\xi|} \Rightarrow$

$\xi\,\hat{u}_3 = \pi i\, e^{-|\xi|} \Rightarrow$

$\hat{u}_3 = \pi i\, pv\left(\frac{e^{-|\xi|}}{\xi}\right)$ and

we get

$$\hat{u} = c\delta_0 + 2\pi i\,\delta' + i\pi\, pv\left(\frac{e^{-|\xi|}}{\xi}\right)$$

but $\hat{u}$ is odd iff $c = 0$

$$\Rightarrow \hat{u} = 2\pi i\,\delta' + i\pi\, pv\left(\frac{e^{-|\xi|}}{\xi}\right).$$