**Weierstrass Institute for**
**Applied Analysis and Stochastics**

# Some new results on nonlocal Cahn-Hilliard-Navier-Stokes systems for incompressible binary flow

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ERC Group “Entropy Formulation of Evolutionary Phase Transitions”

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- An isothermal model for the flow of a **mixture of two**
  - viscous
  - incompressible
  - Newtonian fluids
  - of equal density
- Avoid problems related to interface singularities

  $\Longrightarrow$ use a **diffuse interface model**

  $\Longrightarrow$ the classical sharp interface replaced by a **thin interfacial region**
- A partial mixing of the macroscopically immiscible fluids is allowed

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- The original idea of diffuse interface model for fluids: HOHENBERG and HALPERIN, '77
  $\Longrightarrow$ **H-model**
  Later, GURTIN ET AL., '96: continuum mechanical derivation based on microforces
- Models of two-phase or two-component fluids are receiving growing attention (e.g., ABELS, BOYER, GARCKE, GRÜN, GRASSELLI, LOWENGRUB, TRUSKINOVSKI, ...)

In $\Omega \times (0,\infty)$, $\Omega \subset \mathbb{R}^d$, $d = 2,3$

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}$$

$$\mathrm{div}(\boldsymbol{u}) = 0$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}\,(m(\varphi)\nabla\mu)$$

$$\mu = -\epsilon\Delta\varphi + \epsilon^{-1}F'(\varphi)$$

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- $\mu$: **chemical potential** (Cahn-Hilliard), first variation of the (total Helmholtz) free energy

$$E(\varphi) = \int_\Omega \left(\frac{\epsilon}{2}|\nabla\varphi|^2 + \frac{1}{\epsilon}F(\varphi)\right)dx$$

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- $F$ double-well potential: Helmholtz free energy density
  - Singular

$$F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\big((1+s)\log(1+s) + (1-s)\log(1-s)\big)$$

    for all $s \in (-1,1)$, with $0 < \theta < \theta_c$
  - Regular

$$F(s) = (1-s^2)^2 \qquad \forall s \in \mathbb{R}$$

- **Nonlocal free energy** rigorously justified by Giacomin and Lebowitz ('97 & '98) as macroscopic limit of microscopic phase segregation models

$$E(\varphi) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \int_\Omega F(\varphi(x))dx$$

$J : \mathbb{R}^d \to \mathbb{R}$ interaction kernel s.t. $J(x) = J(-x)$ (usually nonnegative and radial)

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- First analytical results on nonlocal CH: Giacomin & Lebowitz '97 and '98; Gajewski '02; Gajewski & Zacharias '03
- Several other contributions on nonlocal Allen-Cahn equations and phase-field systems (notably by Bates et al. and Sprekels et al.)

$$
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&\varphi_t + \boldsymbol{u} \cdot \nabla\varphi = \text{ div } (m(\varphi)\nabla\mu) \\
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&\boldsymbol{u}_t - 2\text{div}(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v} \\
&\text{div}(\boldsymbol{u}) = 0
\end{aligned}
$$

subject to

$$
\begin{aligned}
&\frac{\partial\mu}{\partial n} = 0 \qquad \boldsymbol{u} = 0 \qquad \text{on} \quad \partial\Omega \times (0, \infty) \\
&\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \varphi(0) = \varphi_0 \qquad \text{in} \quad \Omega
\end{aligned}
$$

- Mass is conserved

$$
\overline{\varphi(t)} := |\Omega|^{-1} \int_\Omega \varphi(x, t)dx = \overline{\varphi}_0
$$

- **Constant mobility+ regular potential**
  - $\exists$ **global weak sols in 2D-3D** (Colli, F. & Grasselli, J. Math. Anal. Appl. '12)
  - global attractor in 2D and trajectory attractor in 3D (F. & Grasselli, J. Dynam Differential Equations '12)
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- **Degenerate mobility+ singular potential**
  - $\exists$ and regularity of global weak sols in 2D-3D, global attractor in 2D (F., Grasselli & Rocca, preprint arXiv '13)

## More recent results

- **Constant mobility+ regular or singular potential & degenerate mobility + singular potential**
  - **Uniqueness of global weak sols in 2D**

- **Constant mobility, nonconstant viscosity +regular potential**
  - $\exists$ global unique strong sols in 2D, regularity of global attractor in 2D, convergence to equilibria of weak sols in 2D
  - weak-strong uniqueness in 2D
  - Connectedness and regularity of global attractor, $\exists$ exponential attractor in 2D.

Last results in: F., Gal & Grasselli, WIAS Preprint '14

**Theorem (Colli, F. & Grasselli '12)**

*Assume* $J \in W^{1,1}(\mathbb{R}^d)$ *and that* $\boldsymbol{v} \in L^2(0,T;H^1_{div}(\Omega)')$, $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$. *Then,* $\forall T > 0 \, \exists$ *a weak sol* $[\boldsymbol{u}, \varphi]$ *on* $[0,T]$ *s.t.*

$$\boldsymbol{u} \in L^\infty(0,T;L^2_{div}(\Omega)^d) \cap L^2(0,T;H^1_{div}(\Omega)^d), \qquad \boldsymbol{u}_t \in L^{4/d}(0,T;H^1_{div}(\Omega)')$$
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*which satisfies the energy inequality (identity if* $d=2$*)*

$$\mathcal{E}(\boldsymbol{u}(t),\varphi(t)) + \int_0^t (\nu\|\nabla\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2)d\tau \leq \mathcal{E}(\boldsymbol{u}_0,\varphi_0) + \int_0^t \langle\boldsymbol{v}(\tau),\boldsymbol{u}(\tau)\rangle d\tau$$

*for all* $t > 0$, *where we have set*

$$\mathcal{E}(\boldsymbol{u}(t),\varphi(t)) = \frac{1}{2}\|\boldsymbol{u}(t)\|^2 + \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t)-\varphi(y,t))^2dxdy + \int_\Omega F(\varphi(t))$$

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- **The nonlocal term implies that $\varphi$ is not as regular as for the standard (local) CHNS system**: $\varphi \in L^2(H^1)$ (nonlocal), instead of $\varphi \in L^\infty(H^1)$ (local) $\Longrightarrow$ regularity results and uniqueness of weak sols in 2D difficult issues

- We need stronger assumptions on $J$. In particular $J \in W^{2,1}(\mathbb{R}^2)$ or $J$ **admissible**

**Definition (J. Bedrossian, N. Rodríguez & A. Bertozzi '11)**

*A kernel $J \in W^{1,1}_{loc}(\mathbb{R}^2)$ is admissible if the following conditions are satisfied:*

**(A1)** $J \in C^3(\mathbb{R}^d \setminus \{0\})$;

**(A2)** $J$ *is radially symmetric,* $J(x) = \tilde{J}(|x|)$ *and* $\tilde{J}$ *is non-increasing;*

**(A3)** $\tilde{J}''(r)$ *and* $\tilde{J}'(r)/r$ *are monotone on* $(0, r_0)$ *for some* $r_0 > 0$;

**(A4)** $|D^3 J(x)| \leq C_d |x|^{-d-1}$ *for some* $C_d > 0$

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**Lemma (J. Bedrossian, N. Rodríguez & A. Bertozzi '11)**

*Let $J$ be admissible and $\chi = \nabla J * \psi$. Then, for all $p \in (1, \infty)$, there exists $C_p > 0$ such that*

$$\|\nabla \chi\|_{L^p(\Omega)} \leq C_p \|\psi\|_{L^p(\Omega)}$$

**Theorem (F., Grasselli & Krejčí '13)**

*Assume that* $J \in W^{2,1}(\mathbb{R}^2)$ *or* $J$ *admissible and that*

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Only recently (F., Gal & Grasselli, WIAS Preprint '14) we included

- **Nonconstant viscosity**

$$\nu = \nu(\varphi), \qquad \nu \text{ loc. Lipschitz on } \mathbb{R}, \qquad 0 < \nu_1 \leq \nu(\varphi) \leq \nu_2$$

**How to handle with nonconstant viscosity to get regularity results?**

- We cannot rely on NS regularity in 2D to get $\boldsymbol{u} \in L^2\big(0,T;H^2(\Omega)^2\big)$. Indeed

$$\varphi \text{ weak sol }, \quad \boldsymbol{u} \in H^2(\Omega)^2 \cap H^1_{div}(\Omega)^2 \Longrightarrow \mathrm{div}(\nu(\varphi)D\boldsymbol{u}) \in L^{2-\epsilon}(\Omega)^2$$

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- Approach: (nonloc CH)$\times \mu_t$ and avoid the use of the $H^2-$ norm of $\boldsymbol{u}$. We deduce

$$\frac{d}{dt}\|\nabla\mu\|^2 + c_0\|\varphi_t\|^2 \le Q(R)\big(\|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2\big)\|\nabla\varphi\|^2 + c\|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2\|\nabla\varphi\|^2 + Q(R)$$

$$+ c\big(\|\nabla a\|^2_{L^\infty(\Omega)} + Q(R)\big)\|\nabla\varphi\|^2 + Q(R)\sum_{i,j=1}^{2}\|\partial^2_{ij}a\|^2$$

$$+ c\sum_{i,j=1}^{2}\|\partial_i(\partial_j J * \varphi)\|^2 + c\|J\|^2_{W^{1,1}(\mathbb{R}^2)}\|\varphi_t\|^2_{H^1(\Omega)'} \qquad \|\varphi\|_{L^\infty(Q)} \le R$$

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$$\Longrightarrow \varphi \in L^\infty(0,T;H^1(\Omega)), \quad \varphi_t \in L^2(0,T;L^2(\Omega)), \quad \mu \in L^\infty(0,T;H^1(\Omega))$$

- Second step: (NS)$\times\boldsymbol{u}_t$, integrate by parts in time to get

$$\frac{1}{2}\|\boldsymbol{u}_t\|^2 + \frac{d}{dt}\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_t) \leq \frac{1}{2}\|l\|^2 + \int_\Omega |D\boldsymbol{u}|^2\nu'(\varphi)\varphi_t$$

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where $l := -\frac{\varphi^2}{2}\nabla a - (J*\varphi)\nabla\varphi + \boldsymbol{v}$. After some *technical arguments* we are led to

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + \frac{1}{8}\|\boldsymbol{u}_t\|^2 \\ &\leq Q(R,\|\varphi_0\|_V,\|\boldsymbol{u}_0\|)\Big(\|l\|^2 + ((\|\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^{p-2})\|\nabla\boldsymbol{u}\|^2)\|D\boldsymbol{u}\|^2 \\ &+ \|\varphi_t\|^2\|D\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{u}\|^2\Big) \qquad 2 < p < \infty\end{aligned}$$

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Exploiting the regularity obtained at previous step

$$\Longrightarrow \boldsymbol{u}\in L^\infty(0,T;H^1_{div}(\Omega)^2)\cap L^2(0,T;H^2(\Omega)^2)\qquad \boldsymbol{u}_t\in L^2(0,T;L^2_{div}(\Omega)^2)$$

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and then also

$$\varphi_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \qquad \varphi \in L^\infty(0,T;H^2(\Omega))$$

## Constant mobility + regular potentials

**Theorem (F., Gal & Grasselli ’14)**

*Let* $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^2, \varphi_0 \in L^2(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$*. Then,* $\exists$ *a* **unique** *weak sol* $[\boldsymbol{u}, \varphi]$ *corresponding to* $[\boldsymbol{u}_0, \varphi_0]$

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### Degenerate mobility + singular potential

- $\varphi-$dependent mobility in the original derivation of CH eq. (J.W. Cahn & J.E. Hilliard, 1971). Thermodynamically reasonable choice: $m(\varphi) = k(1 - \varphi^2)$
- Key assumption (cf. [Elliot & Garcke '96], [Gajewski & Zacharias '03], [Giacomin & Lebowitz '97,'98]): $mF'' \in C([-1, 1])$

**Theorem (F., Gal & Grasselli '14)**

*Let* $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^2$, $\varphi_0 \in L^\infty(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$ *and* $M(\varphi_0) \in L^1(\Omega)$. *Then,* $\exists$ *a* **unique** *weak sol* $[\boldsymbol{u}, \varphi]$ *corresponding to* $[\boldsymbol{u}_0, \varphi_0]$

$M \in C^2(-1, 1)$ is s.t. $m(s)M''(s) = 1$ for all $s \in (-1, 1)$ and $M(0) = M'(0) = 0$

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## Uniqueness of weak sol in 2D (constant viscosity)

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- A continuous dependence estimate for weak sols in $L^2_{div}(\Omega)^2 \times (H^1(\Omega))'$ also holds

$$
\begin{aligned}
&\|\boldsymbol{u}_2(t) - \boldsymbol{u}_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2_{(H^1(\Omega))'} \\
&+ \int_0^t \Big( c_0 \|\varphi_2(\tau) - \varphi_1(\tau)\|^2 + \frac{\nu}{2} \|\nabla(\boldsymbol{u}_2(\tau) - \boldsymbol{u}_1(\tau))\|^2 \Big) d\tau \\
&\leq \Gamma_1(t) \big( \|\boldsymbol{u}_{02} - \boldsymbol{u}_{01}\|^2 + \|\varphi_{02} - \varphi_{01}\|^2_{(H^1(\Omega))'} \big) + C_\eta \Gamma_2(t) |\overline{\varphi}_{02} - \overline{\varphi}_{01}|
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$|\overline{\varphi}_{01}|, |\overline{\varphi}_{02}| \leq \eta$, with $\Gamma_i \in C(\mathbb{R}^+)$ depending on weak sols norms

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$|\overline{\varphi}_{01}|, |\overline{\varphi}_{02}| \leq \eta$, with $\Gamma_i \in C(\mathbb{R}^+)$ depending on weak sols norms

- **Uniqueness of sol and $\exists$ of the global attractor for the local CH with degenerate mobility are open issues**

### Consequences

- the nonlocal CHNS system generates a **semigroup** $S(t)$ of *closed* operators: $[\boldsymbol{u}(t), \varphi(t)] = S(t)[\boldsymbol{u}_0, \varphi_0]$ on the (metric) phase-space

$$\mathcal{X}_\eta = L^2_{div}(\Omega)^2 \times \mathcal{Y}_\eta \quad \mathcal{Y}_\eta = \{\varphi \in L^2(\Omega) : F(\varphi) \in L^1(\Omega), |\bar{\varphi}| \leq \eta\}$$

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- The global attractor in $\mathcal{X}_\eta$ for $S_\eta(t)$ is **connected**
- **Smoothing property** for the difference of two sols in $L^2_{div}(\Omega)^2 \times L^2(\Omega)$

**Theorem (F., Gal & Grasselli '14)**

*For every $\eta \geq 0$ the dynamical system $\big(\mathcal{X}_\eta, S(t)\big)$ possesses an* **exponential attractor** $\mathcal{M}_\eta$, *i.e., a compact set in $\mathcal{X}_\eta$ s.t.*

**(i)** *Positively invariance:* $S(t)\mathcal{M} \subset \mathcal{M}\ \forall t \geq 0$

**(ii)** *Finite dimensionality: dim$_F$* $\mathcal{M} < \infty$

**(iii)** *Exponential attraction:* $\exists J : \mathbb{R}^+ \to \mathbb{R}^+$ *increasing and* $\kappa > 0$ *s.t.,* $\forall R > 0$ *and* $\forall \mathcal{B} \subset \mathcal{X}_\eta$ *with* $\sup_{z\in\mathcal{B}} \mathbf{d}_{\mathcal{X}_\eta}(z, 0) \leq R$ *there holds*

$$\textit{dist}(S(t)\mathcal{B}, \mathcal{M}) \leq J(R)e^{-\kappa t}$$

## Constant mobility+regular potential

Problem **(CP)**: minimize the **cost functional**

$$J(y,\boldsymbol{v}) := \frac{\beta_1}{2}\|\boldsymbol{u}-\boldsymbol{u}_Q\|^2_{L^2(Q)^2} + \frac{\beta_2}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)} + \frac{\beta_3}{2}\|\boldsymbol{u}(T)-\boldsymbol{u}_\Omega\|^2 + \frac{\beta_4}{2}\|\varphi(T)-\varphi_\Omega\|^2 + \frac{\gamma}{2}\|\boldsymbol{v}\|^2_{L^2(Q)^2}$$

where $y := [\boldsymbol{u},\varphi]$ solves

**(nlocCHNS)**

$$\begin{aligned}
&\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}\\
&\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta\mu\\
&\mu = a\varphi - J*\varphi + F'(\varphi)\\
&\mathrm{div}(\boldsymbol{u}) = 0\\
&\partial_{\boldsymbol{n}}\mu = 0 \qquad \boldsymbol{u} = 0 \qquad \text{on } \partial\Omega\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \varphi(0) = \varphi_0
\end{aligned}$$

and the external body force density $\boldsymbol{v}$, which plays the role of the **control**, belongs to a suitable closed, bounded and convex subset of the **space of controls**

$$\mathcal{V} := L^2(0,T;L^2_{div}(\Omega)^2)$$

- Introducing the space

$$\mathcal{H} := \left[L^\infty(0,T;H^1_{div}(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2)\right] \times L^\infty(0,T;H^2(\Omega))$$

then, the **control-to-state map**

$$S : \mathcal{V} \to \mathcal{H}, \qquad \boldsymbol{v} \in \mathcal{V} \mapsto S(\boldsymbol{v}) := y := [\boldsymbol{u}, \varphi] \in \mathcal{H}$$

where $y := [\boldsymbol{u}, \varphi]$ is the unique strong sol to Problem **(nloc CHNS)** corresponding to $\boldsymbol{v} \in \mathcal{V}$ and to fixed initial data $\boldsymbol{u}_0 \in H^1_{div}(\Omega)^2$, $\varphi_0 \in H^2(\Omega)$, is well defined

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- Set of **admissible controls**

$$\mathcal{V}_{ad} := \left\{\boldsymbol{v} \in \mathcal{V} : \ v_{a,i}(x,t) \le v_i(x,t) \le v_{b,i}(x,t), \ \text{a.e. } (x,t) \in Q, \ \ i = 1,2\right\}$$

with $\boldsymbol{v}_a, \boldsymbol{v}_b \in \mathcal{V} \cap L^\infty(Q)^2$ prescribed

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- Introducing the **reduced cost functional** $f(\boldsymbol{v}) := J\big(S(\boldsymbol{v}), \boldsymbol{v}\big)$, for all $\boldsymbol{v} \in \mathcal{V}$, then

$$\textbf{(CP)} \Longleftrightarrow \min_{\boldsymbol{v} \in \mathcal{V}_{ad}} f(\boldsymbol{v})$$

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$$\textbf{(CP)} \iff \min_{\boldsymbol{v} \in \mathcal{V}_{ad}} f(\boldsymbol{v})$$

**Theorem**

*Problem **(CP)** admits a sol $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$, with associated state $\overline{y} := [\overline{\boldsymbol{u}}, \overline{\varphi}] := S(\overline{\boldsymbol{v}})$*

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$$\boldsymbol{\xi}_t - \nu\Delta\boldsymbol{\xi} + (\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{\xi} + (\boldsymbol{\xi} \cdot \nabla)\overline{\boldsymbol{u}} + \nabla\widetilde{\pi} = \big(a\eta - J * \eta + F''(\overline{\varphi})\eta\big)\nabla\overline{\varphi} + \overline{\mu}\nabla\eta + \boldsymbol{h}$$

$$\eta_t + \overline{\boldsymbol{u}} \cdot \nabla\eta = -\boldsymbol{\xi} \cdot \nabla\overline{\varphi} + \Delta\big(a\eta - J * \eta + F''(\overline{\varphi})\eta\big)$$

$$\mathrm{div}(\boldsymbol{\xi}) = 0$$

$$\boldsymbol{\xi} = 0, \qquad \frac{\partial}{\partial \boldsymbol{n}}\big(a\eta - J * \eta + F''(\overline{\varphi})\eta\big) = 0 \qquad \text{on } \Sigma := \partial\Omega \times (0,T)$$

$$\boldsymbol{\xi}(0) = \eta(0) = 0$$

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## Proposition

*For every $\boldsymbol{h} \in \mathcal{V}$ the linearized problem above has a unique sol satisfying*

$$\boldsymbol{\xi} \in C\big([0,T]; L^2_{div}(\Omega)^2\big) \cap L^2\big(0,T; H^1_{div}(\Omega)^2\big), \quad \eta \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$$

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**Remark.** States $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}]$ need to be **strong sols** to **(nloc CHNS)**

**Differentiability of the control-to-state operator.** Set

$$\mathcal{Z} := \left[C\big([0,T]; L^2_{div}(\Omega)^2\big) \cap L^2\big(0,T; H^1_{div}(\Omega)^2\big)\right] \times \left[C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))\right]$$

**Theorem**

*The control-to-state operator $S : \mathcal{V} \to \mathcal{Z}$ is Frechét differentiable on $\mathcal{V}$ and the Frechét derivative $S'(\overline{\boldsymbol{v}}) \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ is given by*

$$S'(\overline{\boldsymbol{v}})\boldsymbol{k} = [\boldsymbol{\xi}^{\boldsymbol{k}}, \eta^{\boldsymbol{k}}], \qquad \forall \boldsymbol{k} \in \mathcal{V},$$

*where $[\boldsymbol{\xi}^{\boldsymbol{k}}, \eta^{\boldsymbol{k}}]$ is the unique sol to the linearized system at $[\overline{\boldsymbol{u}}, \overline{\varphi}] = S(\overline{\boldsymbol{v}})$ and corresponding to $\boldsymbol{k} \in \mathcal{V}$*

**Key tool for the proof: stability estimates**

**Lemma (Stability estimate I — F., Gal & Grasselli '14)**

*Let* $\boldsymbol{u}_{0i} := \boldsymbol{u}_i(0) \in H^1_{div}(\Omega)^2$, $\varphi_{0i} := \varphi_i(0) \in H^2(\Omega)$, $\boldsymbol{v}_i \in L^2(0,T;L^2_{div}(\Omega)^2)$ *and let* $[\boldsymbol{u}_i, \varphi_i]$ *be the corresponding (unique) strong sols,* $i = 1, 2$*. Then, we have*

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2_{div}(\Omega)^2)} + \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^2(0,T;H^1_{div}(\Omega)^2)} + \|\varphi_2 - \varphi_1\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \|\varphi_2 - \varphi_1\|^2_{L^2(0,T;H^1(\Omega))} \leq \Lambda_1 \big(\|\boldsymbol{u}_{20} - \boldsymbol{u}_{10}\|^2 + \|\varphi_{20} - \varphi_{10}\|^2 + \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{\mathcal{V}}\big)$$

*where*

$$\Lambda_1 = \Lambda_1 \big(\|\nabla \boldsymbol{u}_{01}\|, \|\varphi_{01}\|_{H^2(\Omega)}, \|\boldsymbol{v}_1\|_{\mathcal{V}}, \|\nabla \boldsymbol{u}_{02}\|, \|\varphi_{02}\|_{H^2(\Omega)}, \|\boldsymbol{v}_2\|_{\mathcal{V}}\big)$$

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**Remak.** To prove Frechét differentiability of $S : \mathcal{V} \to \mathcal{Z}$ we need an improved stability estimate

**Key tool for the proof: stability estimates**

**Lemma (Stability estimate II)**

*Let* $\boldsymbol{u}_{0i} := \boldsymbol{u}_i(0) \in H^1_{div}(\Omega)^2$, $\varphi_{0i} := \varphi_i(0) \in H^2(\Omega)$, $\boldsymbol{v}_i \in L^2(0,T;L^2_{div}(\Omega)^2)$ *and let* $[\boldsymbol{u}_i, \varphi_i]$ *be the corresponding (unique) strong sols,* $i = 1, 2$. *Then, we have*

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**Sketch of the proof of differentiability of** $S : \mathcal{V} \to \mathcal{Z}$. Let $\overline{\boldsymbol{v}} \in \mathcal{V}$ be fixed, $\overline{y} := [\overline{\boldsymbol{u}}, \overline{\varphi}] = S(\overline{\boldsymbol{v}})$, and consider a perturbation $\boldsymbol{h} \in \mathcal{V}$. Set

$$y^{\boldsymbol{h}} := [\boldsymbol{u}^{\boldsymbol{h}}, \varphi^{\boldsymbol{h}}] := S(\overline{\boldsymbol{v}} + \boldsymbol{h})$$
$$\boldsymbol{p}^{\boldsymbol{h}} := \boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}} - \boldsymbol{\xi}^{\boldsymbol{h}}, \qquad q^{\boldsymbol{h}} := \varphi^{\boldsymbol{h}} - \overline{\varphi} - \eta^{\boldsymbol{h}}$$

- Then, $\boldsymbol{p}^{\boldsymbol{h}}, q^{\boldsymbol{h}}$ solve

$$\boldsymbol{p}_t - \nu\Delta\boldsymbol{p} + (\boldsymbol{p}\cdot\nabla)\overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{p} + \big((\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla\big)(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}) + \nabla\pi^{\boldsymbol{h}}$$
$$= a(\varphi^{\boldsymbol{h}} - \overline{\varphi})\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) - \big(J*(\varphi^{\boldsymbol{h}} - \overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + (aq - J*q)\nabla\overline{\varphi}$$
$$+ (a\overline{\varphi} - J*\overline{\varphi})\nabla q + \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + F'(\overline{\varphi})\nabla q$$
$$+ \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big)\nabla\overline{\varphi} \tag{0.1}$$

$$q_t + (\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + \boldsymbol{p}\cdot\nabla\overline{\varphi} + \overline{\boldsymbol{u}}\cdot\nabla q$$
$$= \Delta\big(aq - J*q + F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big) \tag{0.2}$$

- Then, $\boldsymbol{p}^{\boldsymbol{h}}, q^{\boldsymbol{h}}$ solve

$$\begin{aligned}
&\boldsymbol{p}_t - \nu\Delta\boldsymbol{p} + (\boldsymbol{p}\cdot\nabla)\overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{p} + \big((\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla\big)(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}) + \nabla\pi^{\boldsymbol{h}} \\
&= a(\varphi^{\boldsymbol{h}} - \overline{\varphi})\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) - \big(J*(\varphi^{\boldsymbol{h}} - \overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + (aq - J*q)\nabla\overline{\varphi} \\
&+ (a\overline{\varphi} - J*\overline{\varphi})\nabla q + \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + F'(\overline{\varphi})\nabla q \\
&+ \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big)\nabla\overline{\varphi} \qquad (0.1)
\end{aligned}$$

$$\begin{aligned}
&q_t + (\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + \boldsymbol{p}\cdot\nabla\overline{\varphi} + \overline{\boldsymbol{u}}\cdot\nabla q \\
&= \Delta\big(aq - J*q + F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big) \qquad (0.2)
\end{aligned}$$

- Let us test (0.1) by $\boldsymbol{p}$ in $L^2_{div}(\Omega)^2$ and (0.2) by $q$ in $L^2(\Omega)$. After some technical arguments we are led to

$$\frac{d}{dt}\big(\|\boldsymbol{p}^{\boldsymbol{h}}\|^2 + \|q^{\boldsymbol{h}}\|^2\big) + \nu\|\nabla\boldsymbol{p}^{\boldsymbol{h}}\|^2 + c_0\|\nabla q^{\boldsymbol{h}}\|^2 \le \alpha(t)\|\boldsymbol{p}^{\boldsymbol{h}}\|^2 + \overline{\Gamma}\|q^{\boldsymbol{h}}\|^2 + \beta_{\boldsymbol{h}}(t)$$

$$\overline{\Gamma} = \overline{\Gamma}\big(\|\nabla\boldsymbol{u}_0\|, \|\varphi_0\|_{H^2(\Omega)}, \|\overline{\boldsymbol{v}}\|_{\mathcal{V}}\big)$$

and $\alpha, \beta_{\boldsymbol{h}} \in L^1(0,T)$ given by

$$\alpha := \overline{\Gamma}\big(1 + \|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\big)$$

$$\begin{aligned}\beta_{\boldsymbol{h}} := \overline{\Gamma}\big(&\|\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}\|^2 \|\nabla(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\|^2 + \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|^2 \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|^2_{H^1(\Omega)} \\ &+ \|\nabla(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\|^2 \|\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\|^2 + \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|^4_{H^1(\Omega)} + \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|^2_{H^1(\Omega)} \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|^2_{H^2(\Omega)}\big)\end{aligned}$$

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\end{aligned}
$$

- Thanks to Stability estimate II we have

$$
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and so by Gronwall lemma ($\boldsymbol{p}^{\boldsymbol{h}}(0) = q^{\boldsymbol{h}}(0) = 0$)

$$\|\boldsymbol{p}^{\boldsymbol{h}}\|^2_{L^\infty(0,T;L^2_{div}(\Omega)^2)} + \nu\|\boldsymbol{p}^{\boldsymbol{h}}\|^2_{L^2(0,T;H^1_{div}(\Omega)^2)} + \|q^{\boldsymbol{h}}\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$+ c_0\|q^{\boldsymbol{h}}\|^2_{L^2(0,T;H^1(\Omega))} \leq \overline{\Gamma}\|\boldsymbol{h}\|^4_{\mathcal{V}}$$

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$$+ c_0\|q^{\boldsymbol{h}}\|^2_{L^2(0,T;H^1(\Omega))} \leq \overline{\Gamma}\|\boldsymbol{h}\|^4_{\mathcal{V}}$$

$$\Longrightarrow \frac{\|S(\overline{\boldsymbol{v}} + \boldsymbol{h}) - S(\overline{\boldsymbol{v}}) - [\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]\|_{\mathcal{Z}}}{\|\boldsymbol{h}\|_{\mathcal{V}}} \leq \overline{\Gamma}\|\boldsymbol{h}\|_{\mathcal{V}} \to 0 \quad \text{as } \boldsymbol{h} \to 0 \text{ in } \mathcal{V}$$

**Remark.** The weaker differentiability property of the control-to-state map from $\mathcal{V}$ with values in

$$\left[C\big([0,T];L^2_{div}(\Omega)^2\big)\cap L^2\big(0,T;H^1_{div}(\Omega)^2\big)\right]\times\left[C([0,T];H^1(\Omega)')\cap L^2(0,T;L^2(\Omega))\right]$$

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easier to establish: test (0.2) by $(-\Delta)^{-1}q$ and use only Stability estimate I

Nevertheless, with this weaker differentiability we get necessary conditions for existence of the optimal control for the control problem associated to the “incomplete"cost functional

$$J(y,\boldsymbol{v}):=\frac{\beta_1}{2}\|\boldsymbol{u}-\boldsymbol{u}_Q\|^2_{L^2(Q)^2}+\frac{\beta_2}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)}+\frac{\beta_3}{2}\|\boldsymbol{u}(T)-\boldsymbol{u}_\Omega\|^2 + \frac{\gamma}{2}\|\boldsymbol{v}\|^2_{L^2(Q)^2}$$

If $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ is an optimal control for Problem **(CP)**, then

$$f'(\overline{\boldsymbol{v}})(\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

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$$f'(\boldsymbol{v}) = J'_y\big(S(\boldsymbol{v}), \boldsymbol{v}\big)S'(\boldsymbol{v}) + J'_{\boldsymbol{v}}\big(S(\boldsymbol{v}), \boldsymbol{v}\big)$$

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## Corollary

*Let $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ be an optimal control for Problem **(CP)** with associated state $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}] := S(\overline{\boldsymbol{v}})$. Then*

$$\beta_1 \int_0^T \int_\Omega (\overline{\boldsymbol{u}} - \boldsymbol{u}_Q) \cdot \boldsymbol{\xi}^{\boldsymbol{h}} + \beta_2 \int_0^T \int_\Omega (\overline{\varphi} - \varphi_Q) \eta^{\boldsymbol{h}} + \beta_3 \int_\Omega (\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega) \cdot \boldsymbol{\xi}^{\boldsymbol{h}}(T)$$
$$+ \beta_4 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega) \eta^{\boldsymbol{h}}(T) + \gamma \int_0^T \int_\Omega \overline{\boldsymbol{v}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

*where $[\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$ is the unique sol to the linearized system corresponding to $\boldsymbol{h} = \boldsymbol{v} - \overline{\boldsymbol{v}}$*

**Aim**: eliminate $\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}$ from the previous inequality. Hence, introduce the **adjoint system**

$$
\begin{aligned}
\widetilde{\boldsymbol{p}}_t = & -\nu\Delta\widetilde{\boldsymbol{p}} - (\overline{\boldsymbol{u}}\cdot\nabla)\widetilde{\boldsymbol{p}} + (\widetilde{\boldsymbol{p}}\cdot\nabla^T)\overline{\boldsymbol{u}} + \widetilde{q}\nabla\overline{\varphi} - \beta_1(\overline{\boldsymbol{u}} - \boldsymbol{u}_Q) \\
\widetilde{q}_t = & -(a\Delta\widetilde{q} + \nabla J \dot{*} \nabla\widetilde{q} + F''(\overline{\varphi})\Delta\widetilde{q}) - \overline{\boldsymbol{u}}\cdot\nabla\widetilde{q} \\
& - \big(a\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi} - J*(\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}) + F''(\overline{\varphi})\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}\big) + \widetilde{\boldsymbol{p}}\cdot\nabla\overline{\mu} - \beta_2(\overline{\varphi} - \varphi_Q) \\
& \mathrm{div}(\widetilde{\boldsymbol{p}}) = 0 \\
& \widetilde{\boldsymbol{p}} = 0, \qquad \frac{\partial\widetilde{q}}{\partial\boldsymbol{n}} = 0 \quad \text{on } \Sigma \\
& \widetilde{\boldsymbol{p}}(T) = \beta_3(\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega), \qquad \widetilde{q}(T) = \beta_4(\overline{\varphi}(T) - \varphi_\Omega)
\end{aligned}
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\widetilde{q}_t =& -(a\Delta\widetilde{q} + \nabla J \dot{*} \nabla\widetilde{q} + F''(\overline{\varphi})\Delta\widetilde{q}) - \overline{\boldsymbol{u}}\cdot\nabla\widetilde{q}\\
&- \left(a\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi} - J*(\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}) + F''(\overline{\varphi})\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}\right) + \widetilde{\boldsymbol{p}}\cdot\nabla\overline{\mu} - \beta_2(\overline{\varphi} - \varphi_Q)\\
\text{div}(\widetilde{\boldsymbol{p}}) =& \, 0\\
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\widetilde{\boldsymbol{p}}(T) =& \, \beta_3(\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega), \qquad \widetilde{q}(T) = \beta_4(\overline{\varphi}(T) - \varphi_\Omega)
\end{aligned}$$

**Proposition**

*The adjoint system has a unique weak sol* $\widetilde{\boldsymbol{p}}, \widetilde{q}$ *satisfying*

$$\widetilde{\boldsymbol{p}} \in C\big([0,T]; L^2_{div}(\Omega)^2\big) \cap L^2\big(0,T; H^1_{div}(\Omega)^2\big), \quad \widetilde{q} \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$$

**Theorem**

*Let $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ be an optimal control for Problem* ***(CP)*** *with associated state $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}] = S(\overline{\boldsymbol{v}})$ and adjoint state $[\widetilde{\boldsymbol{p}}, \widetilde{q}]$. Then*

$$\gamma \int_0^T \int_\Omega \overline{\boldsymbol{v}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) + \int_0^T \int_\Omega \widetilde{\boldsymbol{p}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

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- The system **(nloc CHNS)**, written for $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, the adjoint system and the above variational inequality form together the first order necessary optimality conditions

**Theorem**

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- The system **(nloc CHNS)**, written for $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, the adjoint system and the above variational inequality form together the first order necessary optimality conditions
- Since $\mathcal{V}_{ad}$ is a nonempty, closed and convex subset of $L^2(Q)^2$, then the above variational inequality with $\gamma > 0$ is equivalent to

$$\overline{\boldsymbol{v}} = P_{\mathcal{V}_{ad}}\Big(\Big\{ -\frac{\widetilde{\boldsymbol{p}}}{\gamma} \Big\}\Big)$$

where $P_{\mathcal{V}_{ad}}$ is the orthogonal projector in $L^2(Q)^2$ onto $\mathcal{V}_{ad}$

- Optimal control for nonlocal CHNS in 2D with degenerate mobility+singular potential
- unmatched densities (Abels, Garcke & Grün '12 for the local CHNS)
- compressible models
- **non-isothermal model(s)**

  (Eleuteri, Rocca & Schimperna preprint '14 for the local CHNS)
- multicomponent models