Mathematical modelling of elastoplastic processes: past, present and future

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Benevento, July 16th, 2013

Hysteresis operators in mechanics

- The equation of motion of a deformable body $\Omega \subset \mathbb{R}^3$ is, in classical continuum mechanics (Landau, Lifschitz, 1953)

$$\rho \mathbf{u}_{tt} = \text{div}\sigma + \mathbf{g} \tag{1}$$

  where $x \in \Omega$, $t > 0$, $\mathbf{u}$ is the displacement vector, ρ is the density, σ is the stress tensor and $\mathbf{g}$ is the applied force
- Well posedness of equation (1) is obtained by coupling suitable initial and boundary conditions and a suitable *constitutive relation* between stress σ and strain ε (defined as the symmetric gradient of $\mathbf{u}$)
- While (1) is a general law, the constitutive relation characterizes specific properties of a concrete material subject to time-dependent loading
- We will deal with the presentation and mathematical properties of some constitutive operators corresponding to models of elasticity, plasticity, elasto-plasticity (single-yield, multi-yield) until some recent models including material fatigue
- Particular care is given to *rate-independent* constitutive operators, while we will not consider viscous, viscoelastic and viscoelastoplastic materials

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Plan of the seminar

- Rheological models
- Models of elastoplasticity
- A new theory of oscillating elastoplastic structures
- The material fatigue
- Main modeling assumption: proportionality between fatigue rate and dissipation rate
- Thermodynamic consistency
- Fatigue and phase transitions
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Rheological models

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A system consisting of
(i) a constitutive relation between σ and ε
(ii) a potential energy $U \geq 0$
is called *rheological model*

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A rheological model is said to be *thermodinamically consistent* if the quantity

$$\dot{q} := \langle \dot{\varepsilon}, \sigma \rangle - \dot{U}$$

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Example: the elastic element $\mathscr{E}$

- In mechanics elastic materials are characterized by linear stress-strain relation and complete reversibility of dynamical processes

- $\sigma = \mathbf{A}\varepsilon$ $\quad$ $\mathbf{A}$ matrix, σ, ε tensors

- Reversibility $\Rightarrow \dot{q} = 0$

- Potential energy

$$U = \frac{1}{2}\langle \mathbf{A}\varepsilon, \varepsilon \rangle$$

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Example: the rigid-plastic element $\mathscr{R}$

- Basic concept in plasticity is the yield surface in the stress space, that can be described as the boundary ∂Z of a closed convex set Z
- the rigid-plastic behaviour consists in two different phases characterized by the instantaneous value σ of the stress
- the material remains rigid as long as $\sigma \in \text{Int}Z$. In this case, no deformation occurs and $\dot{\varepsilon} = 0$. The material becomes plastic if σ reaches ∂Z
- Example in 1D: $Z = [-r, r]$

$$\begin{array}{lll} \dot{\varepsilon} = 0 & -r < \sigma < r & (\dot{\sigma} < 0 \text{ or } \dot{\sigma} > 0) \\ \dot{\varepsilon} \geq 0 & \sigma = r & (\dot{\sigma} = 0) \\ \dot{\varepsilon} \leq 0 & \sigma = -r & (\dot{\sigma} = 0) \end{array}$$

- $\boxed{\dot{\varepsilon}\,\dot{\sigma} = 0}$
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- $\boxed{\dot{\varepsilon}\,\dot{\sigma} = 0}$ **consequence if σ is regular enough**
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- $\sigma \in Z$ (stress does not exceed the threshold ∂Z)
- $U = 0$ (no energy is stored)
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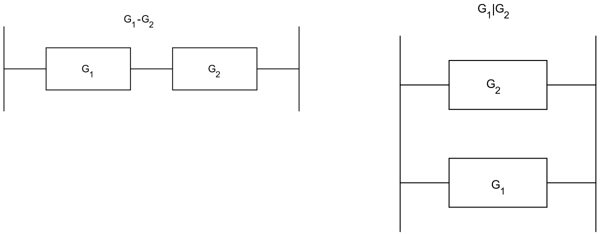
Composition of rheological models

- A large variety of models for describing the behaviour of materials can be obtained by composing rheological elements in series or in parallel



series $G_1 - G_2$	**parallel** $G_1 \vert G_2$
$\varepsilon = \varepsilon_1 + \varepsilon_2$	$\varepsilon = \varepsilon_1 = \varepsilon_2$
$\sigma = \sigma_1 = \sigma_2$	$\sigma = \sigma_1 + \sigma_2$
$U = U_1 + U_2$	$U = U_1 + U_2$

- Every combination of thermodynamically consistent elements is still thermodynamically consistent

Examples - Elastoplastic models $\mathscr{E}-\mathscr{R}$, $\mathscr{E}|\mathscr{R}$

- ε^{e}, σ^{e} strain and stress of the elastic element
- ε^{p}, σ^{p} strain and stress of the plastic element

$\mathscr{E}-\mathscr{R}$ in series (stop)	$\mathscr{E}-\mathscr{R}$ in parallel (play)
$\varepsilon=\varepsilon^{\mathrm{e}}+\varepsilon^{\mathrm{p}}$	$\varepsilon=\varepsilon^{\mathrm{e}}=\varepsilon^{\mathrm{p}}$
$\sigma=\sigma^{\mathrm{e}}=\sigma^{\mathrm{p}}$	$\sigma=\sigma^{\mathrm{e}}+\sigma^{\mathrm{p}}$
$\sigma=\mathbf{A}\varepsilon^{\mathrm{e}}$	$\sigma^{\mathrm{e}}=\mathbf{A}\varepsilon$
$\sigma\in Z$	$\sigma^{\mathrm{p}}\in Z$
$\langle\dot{\varepsilon}^{\mathrm{p}},\sigma-\tilde{\sigma}\rangle\geq 0,\quad\forall\tilde{\sigma}\in Z$	$\langle\dot{\varepsilon},\sigma^{\mathrm{p}}-\tilde{\sigma}\rangle\geq 0,\quad\forall\tilde{\sigma}\in Z$
$U=\frac{1}{2}\langle\varepsilon^{\mathrm{e}},\sigma\rangle$	$U=\frac{1}{2}\langle\varepsilon,\sigma^{\mathrm{e}}\rangle$

- In particular, for the **stop**

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$$\langle \dot{\varepsilon} - \mathbf{A}^{-1}\dot{\sigma}, \sigma - \tilde{\sigma} \rangle \geq 0 \qquad \forall \tilde{\sigma} \in Z$$

The multidimensional stop model

- Let Z be a closed convex set, $0 \in \text{Int}Z$, $Z \subset X$ real separable Banach space; given $\varepsilon : [0,T] \to X$ and $\sigma^0 \in Z$, we look for $\sigma : [0,T] \to X$ such that $\sigma(0) = \sigma^0$ and

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- This map is continuous and admits a continuous extension $\mathfrak{s}_Z : Z \times C([0,T];X) \to C([0,T];X)$
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The stop model (1D)

Figure: The stop model.

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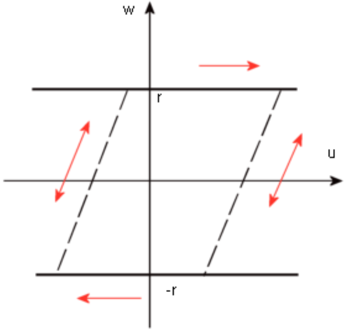

Figure: Hysteretic behaviour of the stop model.

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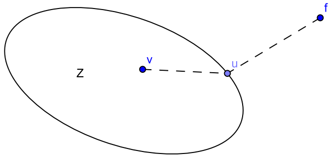

Tangential and normal cone on a convex set

Tangential and normal cone

$$T_Z(\sigma) = \{x \in X : \langle x, y\rangle \leq 0 \quad \forall y \in N_Z(\sigma)\}$$
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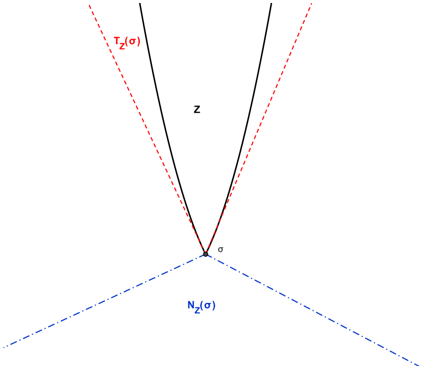

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- $\boxed{\dot{\varepsilon} - \dot{\sigma} \in N_Z(\sigma)}$
- $\boxed{\dot{\sigma} \in T_Z(\sigma)}$ (otherwise a contradiction)

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- $\boxed{\dot{\varepsilon} - \dot{\sigma} \in N_Z(\sigma)}$
- $\boxed{\dot{\sigma} \in T_Z(\sigma)}$
- $\dot{\varepsilon} = (\dot{\varepsilon} - \dot{\sigma}) + \dot{\sigma}$ unique orthogonal decomposition into the **normal** and **tangential** components

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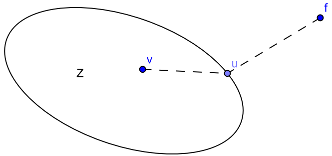

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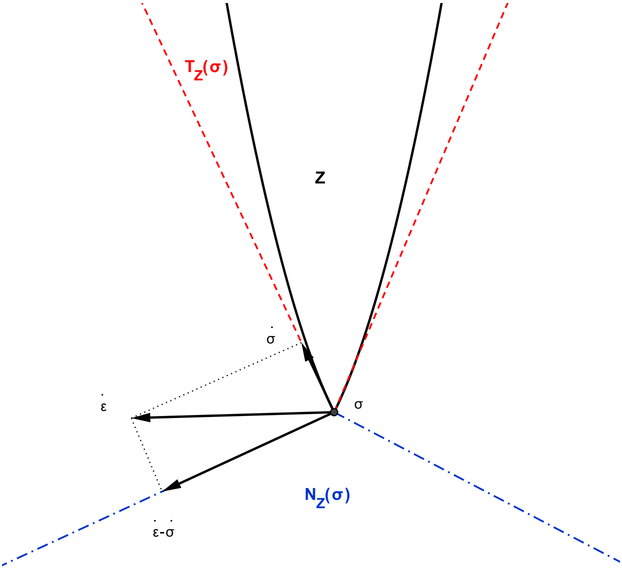

Other models of (perfect) plasticity

- Basic concept in plasticity is the yield surface in the stress space, that can be described as the boundary ∂Z of a closed convex set Z
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- In concrete experiments, the transition between the elastic and the plastic regime is smooth
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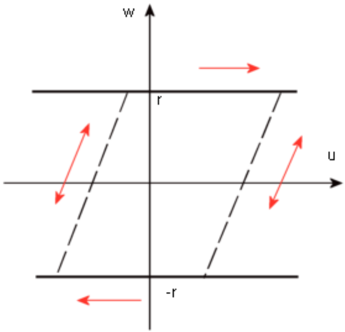

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The stop operators and their combinations

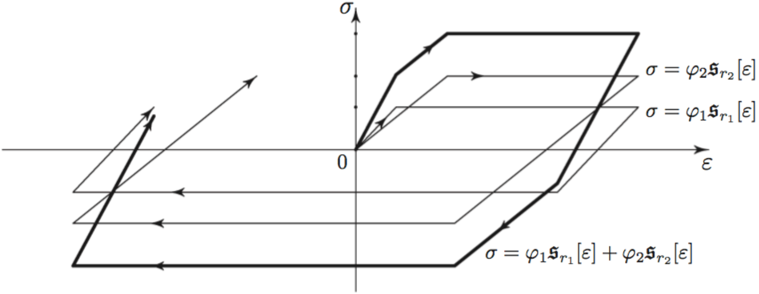

Figure: the stop operators and their combinations.

Prandtl-Ishlinskii model v.s. Von Mises model

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- ☺ simple
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Prandtl-Ishlinskii model v.s. Von Mises model

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New theory of oscillating elastoplastic beams and plates

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- They demonstrated that the three-dimensional single-yield von Mises criterion leads after a dimensional reduction to a multi-yield Prandtl-Ishlinskii operator where the weight function φ **can be explicitly determined!**
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A plate section with grey plasticized zone

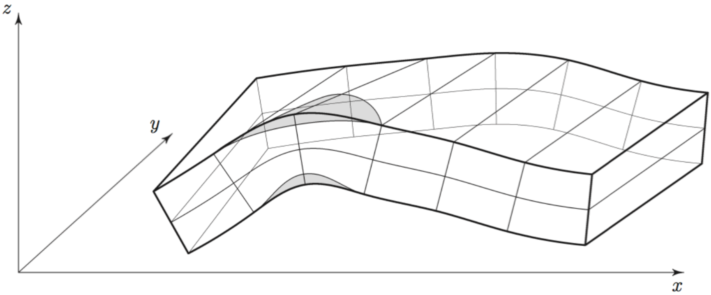

Figure: A plate section with grey plasticized zone.

Motivation for the material fatigue

- ✎ It is well known that plastic deformations lead to energy dissipation and material fatigue
- ✎ Material fatigue is manifested by material softening, heat release, material failure in finite time
- ✎ Very important: take into account the effects of energy exchange between heat and mechanical energy, thermal stresses, and material fatigue
- ✎ In particular great importance for the applications: methods for estimating the lifetime of oscillating thermoelastoplastic structures under material fatigue
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The system of PDEs

- The resulting system from the theory developed by Krejčí and coworkers (starting point of our new project) is the following:

$$\partial_{tt} w - \partial_{tt}\Delta w + \mathbf{D}_2^* \sigma \;=\; g\,,$$

$$\sigma \;=\; \mathbf{B}\varepsilon + \int_0^\infty \mathfrak{s}_{rZ}[\varepsilon](t)\,\varphi(r)\,\mathrm{d}r$$

$$\varepsilon \;=\; \mathbf{D}_2 w$$

  where $\mathbf{D}_2$ is the second derivative operator $(\partial_{xx}, \partial_{yy}, \partial_{xy})$ and $\mathbf{D}_2^*$ is its adjoint

- We introduce a positive parameter $\theta > 0$ indicating the absolute temperature and $m(x,t) \geq 0$ a parameter which represents the material fatigue accumulated in the point x in the time interval $[0,t]$

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Coupling with the energy balance laws

- To the previous system we associate the **specific free energy**

$$\begin{aligned}\mathscr{F}[\theta,\varepsilon] \quad &= \quad c_V\theta(1-\log(\theta/\theta_c)) + \frac{1}{2}\langle \mathbf{B}(m)\varepsilon,\varepsilon\rangle \\ &+\frac{1}{2}\int_0^\infty \langle \mathfrak{s}_{rZ}[\varepsilon],\mathfrak{s}_{rZ}[\varepsilon]\rangle\,\varphi(\theta,r)\,\mathrm{d}r - \beta(\theta-\theta_c)\langle\varepsilon,\mathbf{1}\rangle\,,\end{aligned}$$

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$$\partial_t \mathscr{U}[\theta, \varepsilon] + \operatorname{div} \mathbf{q} = \langle \sigma, \partial_t \varepsilon \rangle ,$$

where $\mathbf{q}$ is the heat flux vector

- We derive the evolution law for the fatigue parameter which has to be compatible with the **Second Principle of Thermodynamics**, which we state in the form of the **Clausius-Duhem inequality**

$$\psi := \partial_t \mathscr{S}[\theta, \varepsilon] + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) \geq 0,$$

where ψ is the entropy production

- This implies that the **dissipation rate**

$$\begin{aligned} \mathscr{D} &= \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathscr{S}[\theta, \varepsilon] - \partial_t \mathscr{F}[\theta, \varepsilon] \\ &= -\frac{1}{2} \langle \mathbf{B}'(m)\varepsilon, \varepsilon \rangle \, \partial_t m + \int_0^\infty \langle \partial_t(\varepsilon - \mathfrak{s}_{rZ}[\varepsilon]), \mathfrak{s}_{rZ}[\varepsilon] \rangle \, \varphi(\theta, r) \, \mathrm{d}r \end{aligned}$$

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- The integral is non negative by virtue of the variational inequality which defines the stop operator
- The fatigue accumulation rate $\partial_t m$ should be nonnegative. Hence, it suffices to assume that $\mathbf{B}'(m)$ is a negative semidefinite matrix (softening!)
- The system will be complete by assuming the **linear Fourier law** between the heat flux and the temperature gradient

$$\mathbf{q} = -\kappa \nabla \theta \, ,$$

- **Fundamental assumption:** proportionality between the rate of fatigue $\partial_t m$ and the dissipation rate $\mathscr{D}$

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The rainflow algorithm

- **Rainflow method for cyclic fatigue accumulation** - M. Endo (1968)
- The analysis of the rainflow algorithm in uniaxial processes has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy
- With an undamaged material, we associate fatigue value 0 , and 1 corresponds to total failure. Values between 0 and 1 quantify the degree of fatigue
- Experimental (decreasing!) curve $n(b)$ (the so-called Wöhler line) determines how many closed cycles of amplitude b lead to total failure
- With each closed cycle of amplitude b we associate the contribution $d(b) = \frac{1}{n(b)}$ of the individual cycle to total fatigue
- Therefore, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude (the contribution of the loop to total damage) taken from the Wöler line

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- **Rainflow method for cyclic fatigue accumulation** - M. Endo (1968)
- The analysis of the rainflow algorithm in uniaxial processes has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy
- With an undamaged material, we associate fatigue value 0 , and 1 corresponds to total failure. Values between 0 and 1 quantify the degree of fatigue
- Experimental (decreasing!) curve $n(b)$ (the so-called Wöhler line) determines how many closed cycles of amplitude b lead to total failure
- With each closed cycle of amplitude b we associate the contribution $d(b) = \frac{1}{n(b)}$ of the individual cycle to total fatigue
- Therefore, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude (the contribution of the loop to total damage) taken from the Wöler line

The rainflow algorithm

- This corresponds to the mechanism of energy dissipation: the number associated with a closed loop is its area in this case
- Let a real loading process consist of a sequence of cycles with amplitudes b_j, $j = 1, \ldots, n$. The Palmgren-Miner additivity rule states that the total fatigue can be computed as the sum of all individual contributions $D_n = \sum_{j=1}^{n} d(b_j)$
- When the contribution of a closed cycle to total fatigue is evaluated, the cycle is removed from the history. At the end of the process, the remaining *residual* of the input signal contains no more closed cycles
- For a numerical treatment of large data sets it is important to assume that oscillations of very small amplitude cause no damage
- The rainflow method is then *stable with respect to small measurement errors independently of the number of cycles*
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In multiassial processes?

- Drawback: the rainflow method is exclusively uniaxial - no closed cycles in the vector case
- In multiaxial loading processes, the concept of closed loop is meaningless, and no counterpart of the rainflow algorithm is known
- On the other hand, the notion of energy dissipation is independent of the experimental setting: experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak
- In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry).

Fundamental assumption

Assuming that the fatigue rate is proportional to the dissipation rate, and that the material parameters depend on the fatigue parameter

- Assumption realistic: plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid

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Evolution equation for the fatigue

$$\begin{aligned}\mathscr{D} &= \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathscr{S}[\theta, \varepsilon] - \partial_t \mathscr{F}[\theta, \varepsilon] \\ &= -\frac{1}{2} \left\langle \mathbf{B}'(m)\varepsilon, \varepsilon \right\rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - \mathfrak{s}_{rZ}[\varepsilon]), \mathfrak{s}_{rZ}[\varepsilon] \rangle \, \varphi(\theta, r) \, \mathrm{d}r\end{aligned}$$

- The integral is non negative by virtue of the variational inequality which defines the stop operator
- The fatigue accumulation rate $\partial_t m$ should be nonnegative. Hence, it suffices to assume that $\mathbf{B}'(m)$ is a negative semidefinite matrix (softening!)
- **Fundamental assumption:** proportionality between the rate of fatigue $\partial_t m$ and the dissipation rate $\mathscr{D}$

$$\left(\frac{1}{C(\theta)} + \frac{1}{2} \langle \mathbf{B}'(m)\varepsilon, \varepsilon \rangle \right) \partial_t m = \int_0^\infty \langle \partial_t (\varepsilon - \mathfrak{s}_{rZ}[\varepsilon]), \mathfrak{s}_{rZ}[\varepsilon] \rangle \, \varphi(\theta, r) \, \mathrm{d}r$$

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- **Motivation:**
  - possibility to account also for decreasing fatigue rate (in view of engineering applications)
  - the material can be partially repaired by local melting
- **How to achieve this goal:**
- Phase transition equation in the form of melting-solidification law

$$\alpha \chi_t \in -\partial_\chi \mathscr{F}[\varepsilon, \theta, \chi] \qquad \chi \in [0,1]$$

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$$\alpha\chi_t \in -\partial_\chi \mathscr{F}[\varepsilon, \theta, \chi] \qquad \chi \in [0,1]$$

$\chi_0 \in [0,1]$ some initial condition, $A(x,t) := \int_0^t \frac{1}{\alpha}\left(\frac{L}{\theta_c}(\theta - \theta_c)\right)(x,\tau)\,d\tau$

$$\boxed{\chi \in \mathfrak{s}_{[0,1]}[\chi_0, A]}$$

$\mathfrak{s}_{[0,1]}$ is a shifted stop

Thermodynamical consistency

- If we introduce $\mathscr{F}[\varepsilon,\theta,\chi]$ **specific free energy**, $\mathscr{S}[\varepsilon,\theta,\chi]$ **specific entropy** and $\mathscr{U}[\varepsilon,\theta,\chi]$ **internal energy** we are able to show that the <u>first and second principles of thermodynamics</u> are satisfied

$$\frac{\partial}{\partial t}\mathscr{U}[\varepsilon,\theta,\chi] + \text{div}\,\mathbf{q} = \langle \sigma, \varepsilon_t \rangle \qquad \text{(energy conservation)}$$

$$\frac{\partial}{\partial t}\mathscr{S}[\varepsilon,\theta,\chi] + \text{div}\,\frac{\mathbf{q}}{\theta} \geq 0, \qquad \text{(Clausius-Duhem inequality)}$$

- **Evolution equation for m:**

$$(C + \mathscr{K}[m,\theta,\varepsilon])\, m_t = -h(\chi_t) + \mathscr{D}[m,\theta,\varepsilon]$$

- where

$$\mathscr{D}[m,\theta,\varepsilon] := \int_0^\infty \varphi(m,\theta,r)\langle \mathbf{K}\mathfrak{s}_{rZ}[\varepsilon], (\varepsilon - \mathfrak{s}_{rZ}[\varepsilon])_t\rangle\, \mathrm{d}r$$

$$\mathscr{K}[m,\theta,\varepsilon] := -\frac{1}{2}\langle \mathbf{B}'(m)\varepsilon, \varepsilon\rangle - \frac{1}{2}\int_0^\infty \varphi_m(m,\theta,r)\langle \mathbf{K}\mathfrak{s}_{rZ}[\varepsilon], \mathfrak{s}_{rZ}[\varepsilon]\rangle\, \mathrm{d}r.$$

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