Well–posedness of the weak formulation for the phase–field model with memory

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Abstract. A phase–field model based on the Gurtin–Pipkin heat flux law is considered. This model consists in a Volterra integrodifferential equation of hyperbolic type coupled with a nonlinear parabolic equation. The system is then associated with a set of initial and Neumann boundary conditions. The resulting problem was already studied by the authors who proved existence and uniqueness of a smooth solution. A careful and detailed investigation on weak solutions is the goal of this paper, going from the aspects of the approximation to the proof of continuous dependence estimates. In addition, a sufficient condition for the boundedness of the phase variable is given.

1. Introduction. This paper is intended to be a continuation or the weak counterpart of the paper [7], to which we refer at once for a more detailed presentation of the model.

Consider a two–phase material located in a smooth bounded domain $\Omega \subset \mathbf{R}^3$. Denote by ϑ its temperature field and by χ the phase field, which may denote the local proportion of one phase. A quite general version of the standard phase–field model (cf., e.g., [9, 5, 12, 1]) gives rise to the following system of partial differential equations for the pair (ϑ, χ)

$$\partial_t(\vartheta + \lambda(\chi)) + \nabla \cdot \mathbf{q} = \overline{g} \tag{1.1}$$

$$\mu\chi_t - \nu\Delta\chi + \beta(\chi) \ni \overline{\gamma}(\vartheta, \chi) + \lambda'(\chi)\vartheta \tag{1.2}$$

in $\Omega \times \,]-\infty, T[$, where $T > 0$ stands for a fixed final time. Here $\lambda : \mathbf{R} \to \mathbf{R}$ and $\overline{\gamma} : \mathbf{R}^2 \to \mathbf{R}$ are Lipschitz continuous functions, $\nabla\cdot$ denotes the spatial divergence

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operator, $\mathbf{q}$ stands for the heat flux, and $\overline{g}$ for the heat supply. Moreover, μ and ν are positive relaxation parameters and β is a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$.

Of course, to make the description complete a further specification is needed, i.e., the constitutive assumption for the heat flux $\mathbf{q}$. The standard phase–field model assumes the classical Fourier law (see, e.g., [5]). This position leads to a system of coupled parabolic equations. However, other reasonable choices are possible (see [10–11] and the references therein). For instance, in [6–8] the Gurtin–Pipkin law is supposed to hold, namely

$$\mathbf{q}(x,t) = -\int_{-\infty}^{t} k(t-s)\nabla\vartheta(x,s)ds$$

in $\Omega \times]-\infty, T[$, where $k : [0,+\infty[\to \mathbf{R}$ is a given smooth relaxation kernel such that $k(0) > 0$. The same relation is postulated in [1] with different requirements on the kernel k.

Consequently, if the past history of ϑ is known up to $t = 0$, then system (1.1–2) can be considered in $Q := \Omega \times (0,T)$ and rewritten as

$$\partial_t(\vartheta + \lambda(\chi)) - k * \Delta\vartheta = g \tag{1.3}$$

$$\mu\chi_t - \nu\Delta\chi + \beta(\chi) \ni \overline{\gamma}(\vartheta,\chi) + \lambda'(\chi)\vartheta \tag{1.4}$$

where $*$ stands for the usual time convolution product, that is,

$$(a * b)(t) = \int_0^t a(t-s)b(s)\,ds$$

the symbol Δ indicates the standard Laplace spatial operator, and g is a function depending both on the heat supply and on the past history of ϑ. As it is shown below, equation (1.3) is no longer parabolic.

Let us now complement (1.3–4) with initial and boundary conditions. According to [7], we set

$$\vartheta(\cdot,0) = \vartheta_0 \quad \text{and} \quad \chi(\cdot,0) = \chi_0 \qquad \text{in } \Omega \tag{1.5}$$

$$k * \partial_{\mathrm{n}}\vartheta = 0 \quad \text{and} \quad \partial_{\mathrm{n}}\chi = 0 \qquad \text{on } \partial\Omega \times]0,T[. \tag{1.6}$$

where ϑ_0 and χ_0 are prescribed initial data and ∂_{n} is the outer normal derivative to $\partial\Omega$. The boundary condition for ϑ derives from $\mathbf{q}\cdot\mathbf{n} = 0$ ($\mathbf{n}$ is the outward normal vector), with the simplifying (but not restrictive) assumption that the normal derivative of the past history of ϑ vanishes on $\partial\Omega \times]-\infty,0[$. Thus, our framework refers to a system isolated from the very beginning.

In [7] we prove the existence and uniqueness of a strong solution (ϑ,χ) to (1.3–6), i.e., which satisfies equations (1.3–4) and conditions (1.5–6) almost everywhere (at least). It is worth recalling that these results hold for any spatial dimension provided that λ is linear.

Taking advantage of the well–posedness in the strong sense, here we investigate the well–posedness of a weak formulation of Problem (1.3–6) when λ is linear, that is $\lambda(\chi) := \lambda\chi$ with $\lambda \in \mathbf{R}$. In the significant case of $\lambda \neq 0$, system (1.3–4) becomes

$$\partial_t(\vartheta + \lambda\chi) - k * \Delta\vartheta = g \tag{1.7}$$

$$\mu\chi_t - \nu\Delta\chi + \beta(\chi) \ni \overline{\overline{\gamma}}(\vartheta, \chi) \tag{1.8}$$

in Q, where

$$\overline{\overline{\gamma}}(\vartheta, \chi) := \overline{\gamma}(\vartheta, \chi) + \lambda\vartheta \tag{1.9}$$

is still Lipschitz continuous.

On account of the analysis developed in [7], we fix our attention on an equivalent version of (1.7–8), (1.5–6) which appears to be more convenient to deal with. Hence, following [7], we introduce the new variable

$$w := 1 * (\vartheta + \lambda\chi). \tag{1.10}$$

Observe that w_t is the enthalpy density and that

$$\vartheta = w_t - \lambda\chi. \tag{1.11}$$

In order to rewrite (1.7) in terms of w, we use the relationship

$$k * \vartheta = k * (w_t - \lambda\chi) = k(0)w + k' * w - \lambda k * \chi \tag{1.12}$$

which comes immediately from (1.10–11). Then, equations (1.7–8) and conditions (1.5–6) are transformed into

$$w_{tt} - k(0)\Delta w = \Delta(k' * w - \lambda k * \chi) + g \qquad \text{in } Q \tag{1.13}$$

$$\mu\chi_t - \nu\Delta\chi + \beta(\chi) \ni \gamma(w_t, \chi) \qquad \text{in } Q \tag{1.14}$$

$$w(\cdot, 0) = 0, \quad w_t(\cdot, 0) = \eta_0, \quad \text{and} \quad \chi(\cdot, 0) = \chi_0 \qquad \text{in } \Omega \tag{1.15}$$

$$\partial_n w = 0 \quad \text{and} \quad \partial_n\chi = 0 \qquad \text{on } \partial\Omega \times]0, T[. \tag{1.16}$$

where (cf. (1.9) and (1.11))

$$\gamma(w_t, \chi) := \overline{\overline{\gamma}}(w_t - \lambda\chi, \chi) \tag{1.17}$$

preserves the Lipschitz continuity property and

$$\eta_0 := \vartheta_0 + \lambda\chi_0 \tag{1.18}$$

provides the initial enthalpy. Since $k(0) > 0$, the hyperbolic character of (1.13) appears to be evident. Besides, by virtue of (1.12), (1.6) can be easily recovered from (1.16).

The present paper is concerned with the natural weak formulation of Problem (1.13–16). To analyze that, we first introduce rigorously the variational setting and the notion of solution. Actually, we are allowed to consider other boundary conditions for w and χ (independently of their physical interest) as well as very general boundary data for w, since we adopt the framework of the classical Hilbert triplet (V, H, V'). Then we prove a suitable continuous dependence type estimate for weak solutions. This estimate, whose proof uses some technical results which are collected in the Appendix, is the cornerstone of the method. Indeed, it enables us to prove the existence of a weak solution by means of an approximation procedure based on the existence of smooth solutions shown in [7]. It is worth remarking that this argument is quite similar to the one developed in [6]. On the other hand, from the same estimate one can deduce uniqueness and continuous dependence on data. The well–posedness result holds true for any spatial dimension N. Finally, if $N \le 3$, then we can still prove the boundedness of χ in Q as we did for the strong solution, but with a different argument based on a Moser type technique.

Let us now fix some notation. Set

$$V = H^1(\Omega), \qquad H = L^2(\Omega) \tag{1.19}$$

and consider H as a subspace of V' by means of the usual formula $\langle u, v\rangle = (u, v)_H$ for every $u \in H$ and $v \in V$, where $\langle \cdot, \cdot \rangle$ is the scalar product between V' and V. Besides, define

$$\langle f_\Omega(t), v\rangle = (f_\Omega(t), v) = \int_\Omega g(t) v \qquad \text{for } v \in V$$

and add to $f_\Omega(t)$ a functional $f_\Gamma(t) \in V'$ which may account for possible boundary data. We introduce the spaces

$$\mathcal{W} = C^0([0,T]; V) \cap C^1([0,T]; H) \tag{1.20}$$

$$\mathcal{X} = L^2(0,T; V) \cap C^0([0,T]; H) \tag{1.21}$$

$$\mathcal{F} = L^1(0,T; H) + W^{1,1}(0,T; V') \tag{1.22}$$

endowed with the norms

$$\|v\|_{\mathcal{W}}^2 = \max_{0\le t\le T} \int_\Omega |\nabla v(t)|^2 \, dt + \max_{0\le t\le T} \|v'(t)\|_H^2 + \|v(0)\|_H^2 \tag{1.23}$$

$$\|v\|_{\mathcal{X}}^2 = \iint_Q |\nabla v|^2 + \max_{0\le t\le T} \|v(t)\|_H^2 \tag{1.24}$$

$$\|v\|_{\mathcal{F}} = \inf_{v_1+v_2=v} \Big\{ \|v_1\|_{L^1(0,T;H)} + \|v_2\|_{W^{1,1}(0,T;V')} \Big\} \tag{1.25}$$

respectively, where the infimum in (1.25) is taken over all decompositions of v such that $v_1 \in L^1(0,T;H)$ and $v_2 \in W^{1,1}(0,T;V')$. Observe that

$$\begin{cases} \|v(t)\|_V \le c_T \, \|v\|_{\mathcal{W},t} & \forall\, v \in \mathcal{W} \\ \|v(t)\|_{V'} \le c_T \, \|v\|_{W^{1,1}(0,T;V')} & \forall\, v \in W^{1,1}(0,T;V') \end{cases} \tag{1.26}$$

for any $t \in [0,T]$, the quantity $\|v\|_{\mathcal{W},t}$ being defined by

$$\|v\|^2_{\mathcal{W},t} = \max_{0 \le s \le t} \int_\Omega |\nabla v(s)|^2 \, ds + \max_{0 \le s \le t} \|v'(s)\|^2_H + \|v(0)\|^2_H \tag{1.27}$$

and c_T denoting a positive constant which depends only on T .

Here are the assumptions on the data. We require that

$$\lambda, \mu, \nu \in \mathbf{R}, \qquad \mu, \nu > 0 \tag{1.28}$$

$$k \in W^{2,1}(0,T), \qquad k(0) > 0 \tag{1.29}$$

$$f = f_\Omega + f_\Gamma \in \mathcal{F}. \tag{1.30}$$

In addition, fix a function $\phi : \mathbf{R} \to [0, +\infty]$ such that

$$\phi \quad \text{is convex, proper, lower–semicontinuous, and} \quad \phi(0) = 0. \tag{1.31}$$

Let β coincide with the subdifferential

$$\beta = \partial\phi \tag{1.32}$$

and note that $0 \in \beta(0)$ since $\phi(0) = \min \phi$. The Cauchy data are suppose to fulfill

$$\vartheta_0, \chi_0 \in H \qquad \text{and} \qquad \phi(\chi_0) \in L^1(\Omega). \tag{1.33}$$

Finally, in order to avoid unessential technicalities, for $\gamma : \mathbf{R}^2 \to \mathbf{R}$ we assume that

$$\gamma \in C^1(\mathbf{R}^2) \quad \text{with bounded partial derivatives and} \quad \gamma(0,0) = 0 \tag{1.34}$$

although our procedure works for a Lipschitz continuous γ .

We now state our results.

Theorem 1.1 (Existence). *Let* (1.28–34) *hold. Then there exists a triplet* (w, χ, ξ) *such that*

$$w \in \mathcal{W}, \qquad w'' \in L^1(0,T;H) + L^\infty(0,T;V') \tag{1.35}$$

$$\chi \in \mathcal{X}, \qquad \chi' \in L^2(0,T;V'), \qquad \phi(\chi) \in L^\infty(0,T;L^1(\Omega)) \tag{1.36}$$

$$\xi \in L^2(Q), \qquad \xi \in \beta(\chi) \quad \text{a.e. in } Q \tag{1.37}$$

$$\langle w''(t), v \rangle + k(0) \int_\Omega \nabla w(t) \cdot \nabla v \tag{1.38}$$

$$= - \int_\Omega \nabla(k' * w)(t) \cdot \nabla v + \lambda \int_\Omega \nabla(k * \chi)(t) \cdot \nabla v + \langle f(t), v \rangle$$

$$\forall v \in V, \quad \text{for a.a. } t \in]0,T[$$

$$\mu \langle \chi'(t), v \rangle + \nu \int_\Omega \nabla\chi(t) \cdot \nabla v + \int_\Omega \xi(t) v = \int_\Omega \gamma(w'(t), \chi(t)) v \tag{1.39}$$

$$\forall v \in V, \quad \text{for a.a. } t \in]0,T[$$

$$w(0) = 0, \qquad w'(0) = \eta_0, \qquad \text{and} \qquad \chi(0) = \chi_0. \tag{1.40}$$

Moreover, the following estimate holds

$$\begin{cases} \|w\|_{\mathcal{W}}^2 + \|\chi\|_{\mathcal{X}}^2 + \|\chi\|_{H^1(0,T;V')}^2 + \|\xi\|_{L^2(Q)}^2 + \|\phi(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \\ \le c\Big(\|f\|_{\mathcal{F}}^2 + \|\eta_0\|_H^2 + \|\chi_0\|_H^2 + \|\phi(\chi_0)\|_{L^1(\Omega)}\Big) \end{cases} \tag{1.41}$$

for some constant c *depending only on* $\|k\|_{W^{2,1}(0,T)}$, $k(0)$, λ, μ, ν, Ω, T, *and on the Lipschitz constant of* γ.

Remark 1.2. It is easy to see that (1.29), (1.35), and (1.36) imply

$$k' * w,\ \ k * \chi \in C^0([0,T];V)$$

so that all the integrals in (1.38) make sense.

Theorem 1.3 (Continuous dependence). *Let* $\{f_i, \eta_{0i}, \chi_{0i}, \phi_i, \beta_i\}$, $i = 1, 2$, *be two sets of data satisfying* (1.30–33) *and let* $\{w_i, \chi_i, \xi_i\}$ *denote corresponding solutions of* (1.35–40). *Then the following estimate holds*

$$\begin{cases} \|w_1 - w_2\|_{\mathcal{W}}^2 + \|\chi_1 - \chi_2\|_{\mathcal{X}}^2 + \sup\limits_{t\in[0,T]} \left(\iint_{Q_t} (\xi_1 - \xi_2)(\chi_1 - \chi_2)\right)^+ \\ \le c\left(\|f_1 - f_2\|_{\mathcal{F}}^2 + \|\eta_{01} - \eta_{02}\|_H^2 + \|\chi_{01} - \chi_{02}\|_H^2 + \inf\limits_{\zeta} \|\xi_1 - \zeta\|_{L^2(Q)}^2\right) \end{cases} \tag{1.42}$$

where the infimum is taken over all $\zeta \in L^2(Q)$ *such that* $\zeta \in \beta_2(\chi_1)$ *a.e. in* Q *and* c *has the same dependences as the constant in* (1.41). *In particular, Problem* (1.35–40) *has a unique solution.*

In (1.42) one finds the positive part of an integral over

$$Q_t = \Omega \times]0, t[, \qquad t \in [0,T].$$

The notation Q_t is used also in the sequel where, however, we write Q in place of Q_T, as before. Moreover, in the proofs we employ the same symbol c for different constants depending, in general, on the parameters specified in the first statement. Let us end the Introduction by recalling some well–known and useful results, namely, the Young theorem

$$\|a * b\|_{L^r(0,T)} \le \|a\|_{L^p(0,T)} \|b\|_{L^q(0,T)} \tag{1.43}$$

with $1 \le p, q, r \le \infty$ and $1/r = (1/p) + (1/q) - 1$, and the formulas

$$a * b = a(0) * b + a_t * 1 * b \qquad \text{and} \qquad (a * b)_t = a(0)b + a_t * b \tag{1.44}$$

which hold whenever they make sense.

2. The key estimate. In this section we prove an estimate which turns to be basic in the proofs of the main theorems stated above.

Lemma 2.1 (Main lemma). *Let* $(\widehat{f}, \widehat{\eta}_0, \widehat{\chi}_0, \widehat{\phi}, \widehat{\beta})$ *and* $(\widetilde{f}, \widetilde{\eta}_0, \widetilde{\chi}_0, \widetilde{\phi}, \widetilde{\beta})$ *be two sets of data satisfying* (1.30–33) *and let* $(\widehat{w}, \widehat{\chi}, \widehat{\xi})$ *and* $(\widetilde{w}, \widetilde{\chi}, \widetilde{\xi})$ *denote any corresponding solutions to Problem (1.35–40). We set*

$$\eta_0 = \widehat{\eta}_0 - \widetilde{\eta}_0, \qquad \chi_0 = \widehat{\chi}_0 - \widetilde{\chi}_0 \tag{2.1}$$

and, for almost all $t \in \,]0, T[$,

$$\begin{cases} f(t) = \widehat{f}(t) - \widetilde{f}(t), \quad w(t) = \widehat{w}(t) - \widetilde{w}(t), \\ \chi(t) = \widehat{\chi}(t) - \widetilde{\chi}(t), \quad \xi(t) = \widehat{\xi}(t) - \widetilde{\xi}(t), \quad \alpha(t) = \displaystyle\iint_{Q_t} \xi\chi. \end{cases} \tag{2.2}$$

Then the following estimate holds

$$\begin{aligned} &\|w\|^2_{\mathcal{W}} + \|\chi\|^2_{\mathcal{X}} + \sup_{t\in[0,T]} \alpha^+(t) \\ &\leq c\left(\|f\|^2_{\mathcal{F}} + \|\eta_0\|^2_H + \|\chi_0\|^2_H + \sup_{0\leq t\leq T} \alpha^-(t) \right). \end{aligned} \tag{2.3}$$

Proof. For every $t \in [0,T]$ and $\varepsilon > 0$, we consider the solution $w_\varepsilon(t) \in V$ to the elliptic variational problem (see the Appendix)

$$\int_\Omega w_\varepsilon(t)\zeta + \varepsilon^2 \int_\Omega \big(\nabla w_\varepsilon(t)\cdot\nabla\zeta + w_\varepsilon(t)\zeta\big) = \int_\Omega w(t)\zeta \qquad \forall\, \zeta \in V. \tag{2.4}$$

Since w belongs at least to the space $W^{2,1}(0,T;V')$, this construction gives a function $w_\varepsilon : [0,T] \to V$ which lies in $W^{2,1}(0,T;V)$. We have moreover

$$\int_\Omega w'_\varepsilon(t)\zeta + \varepsilon^2 \int_\Omega \big(\nabla w'_\varepsilon(t)\cdot\nabla\zeta + w'_\varepsilon(t)\zeta\big) = \int_\Omega w'(t)\zeta \qquad \forall\, \zeta \in V \quad \forall\, t \in [0,T]$$

since $w \in C^1([0,T];H)$, while the corresponding equation involving $w''_\varepsilon(t)$ and $w''(t)$ has to be written using the scalar product between V' and V on its right hand side and holds for almost all $t \in \,]0,T[$.

Now, we consider equation (1.38). Observing that it can be seen as the difference of the corresponding equations for the two sets of data and solutions, we choose $v = w'_\varepsilon(t)$, integrate with respect to t, and take the limit as $\varepsilon \to 0$ using Propositions 6.1–3 of the Appendix. For simplicity we treat each term separately. Thanks to Proposition 6.2, the first integral is given by

$$\lim_{\varepsilon\to 0} \int_0^t \langle w''(s), w'_\varepsilon(s)\rangle \, ds = \frac{1}{2}\, \|w'(t)\|^2_H - \frac{1}{2}\, \|\eta_0\|^2_H$$

and, in view of Proposition 6.3, the second one becomes

$$\lim_{\varepsilon\to 0} k(0) \iint_{Q_t} \nabla w \cdot \nabla w_\varepsilon' = \frac{k(0)}{2} \int_\Omega |\nabla w(t)|^2.$$

Next, we deal with the right hand side. The first term can be integrated by parts and transformed using (1.44). Then its limit can be handled using Proposition 6.1 and estimated with the help of (1.43). More precisely, we have

$$\begin{aligned}
&- \lim_{\varepsilon\to 0} \iint_{Q_t} (k' * \nabla w) \cdot \nabla w_\varepsilon' \\
&= \lim_{\varepsilon\to 0} \iint_{Q_t} (k' * \nabla w)' \cdot \nabla w_\varepsilon - \lim_{\varepsilon\to 0} \int_\Omega (k' * \nabla w)(t) \cdot \nabla w_\varepsilon(t) \\
&= \lim_{\varepsilon\to 0} \iint_{Q_t} (k'(0)\nabla w + k'' * \nabla w) \cdot \nabla w_\varepsilon - \lim_{\varepsilon\to 0} \int_\Omega (k' * \nabla w)(t) \cdot \nabla w_\varepsilon(t) \\
&= \iint_{Q_t} (k'(0)\nabla w + k'' * \nabla w) \cdot \nabla w - \int_\Omega (k' * \nabla w)(t) \cdot \nabla w(t) \\
&\leq c \iint_{Q_t} |\nabla w|^2 + c \iint_{Q_t} |k'' * \nabla w|^2 + \frac{1}{2\sigma} \int_\Omega |(k' * \nabla w)(t)|^2 + \frac{\sigma}{2} \int_\Omega |\nabla w(t)|^2 \\
&\leq \frac{c}{\sigma} \iint_{Q_t} |\nabla w|^2 + \frac{\sigma}{2} \int_\Omega |\nabla w(t)|^2
\end{aligned}$$

where $\sigma \in]0,1[$ is arbitrary. Let us say one word on a detail, namely

$$\int_\Omega |(k' * \nabla w)(x,t)|^2 \, dx \leq \int_\Omega \|k'\|_{L^2(0,t)}^2 \, \|\nabla w(x,\cdot)\|_{(L^2(0,t))^N}^2 \leq \|k'\|_{L^2(0,T)}^2 \iint_{Q_t} |\nabla w|^2.$$

The next integral can be treated similarly. Indeed,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \lambda \iint_{Q_t} (k * \nabla\chi) \cdot \nabla w_\varepsilon' \\
&= -\lambda \lim_{\varepsilon\to 0} \iint_{Q_t} (k * \nabla\chi)' \cdot \nabla w_\varepsilon + \lim_{\varepsilon\to 0} \int_\Omega (k * \nabla\chi)(t) \cdot \nabla w_\varepsilon(t) \\
&= -\lambda \lim_{\varepsilon\to 0} \iint_{Q_t} \big(k(0)\nabla\chi + k' * \nabla\chi\big) \cdot \nabla w_\varepsilon + \lambda \lim_{\varepsilon\to 0} \int_\Omega (k * \nabla\chi)(t) \cdot \nabla w_\varepsilon(t) \\
&= -\lambda \iint_{Q_t} k(0)\nabla\chi \cdot \nabla w - \lambda \iint_{Q_t} (k' * \nabla\chi) \cdot \nabla w + \lambda \int_\Omega (k * \nabla\chi)(t) \cdot \nabla w(t) \\
&\leq \frac{1}{2} \iint_{Q_t} |\nabla\chi|^2 + c \iint_{Q_t} |\nabla w|^2 + \frac{\sigma}{2} \int_\Omega |\nabla w(t)|^2 + \frac{\lambda^2}{2\sigma} \int_\Omega |(k * \nabla\chi(t)|^2
\end{aligned}$$

because of

$$\begin{aligned}
\iint_{Q_t} |(k' * \nabla\chi) \cdot \nabla w| &\leq \|k' * \nabla\chi\|_{(L^2(Q_t))^N} \, \|\nabla w\|_{(L^2(Q_t))^N} \\
&\leq \|k'\|_{L^1(0,T)} \, \|\nabla\chi\|_{(L^2(Q_t))^N} \, \|\nabla w\|_{(L^2(Q_t))^N} \, .
\end{aligned}$$

On the other hand, for fixed x and t, one has

$$|(k * \nabla\chi)(x,t)| \le \|k\|_{L^2(0,t)} \left(\int_0^t |\nabla\chi(x,s)|^2\, ds\right)^{1/2}$$

so that

$$\int_\Omega |(k * \nabla\chi)(t)|^2 \le \|k\|^2_{L^2(0,T)} \iint_{Q_t} |\nabla\chi|^2.$$

Therefore

$$\lim_{\varepsilon\to 0} \lambda \iint_{Q_t} (k * \nabla\chi) \cdot \nabla w'_\varepsilon$$
$$\le \frac{1}{2} \iint_{Q_t} |\nabla\chi|^2 + \frac{\sigma}{2} \int_\Omega |\nabla w(t)|^2 + c \iint_{Q_t} |\nabla w|^2 + \frac{\lambda^2 \|k\|^2_{L^2(0,T)}}{2\sigma} \iint_{Q_t} |\nabla\chi|^2.$$

To treat the last term, take a decomposition of f, i.e. $f = f_1 + f_2$ with $f_1 \in L^1(0,T;H)$ and $f_2 \in W^{1,1}(0,T;V')$. The integral involving f_1 gives

$$\lim_{\varepsilon\to 0} \iint_{Q_t} f_1 w'_\varepsilon = \iint_{Q_t} f_1 w' \le \int_0^t \|f_1(s)\|_H \, \|w'(s)\|_H \, ds$$

while the second one is estimated as follows

$$\lim_{\varepsilon\to 0} \int_0^t \langle f_2(s), w'_\varepsilon(s)\rangle \, ds$$
$$= - \lim_{\varepsilon\to 0} \int_0^t \langle f'_2(s), w_\varepsilon(s)\rangle \, ds + \lim_{\varepsilon\to 0} \langle f_2(t), w_\varepsilon(t)\rangle$$
$$= - \int_0^t \langle f'_2(s), w(s)\rangle \, ds + \langle f_2(t), w(t)\rangle$$
$$\le \int_0^t \|f'_2(s)\|_{V'} \, \|w(s)\|_V \, ds + \frac{\sigma}{2} \|w(t)\|^2_V + \frac{1}{2\sigma} \|f_2(t)\|^2_{V'}.$$

Collecting all the inequalities we have derived, observing that $w(0) = 0$, and using (1.26) and (1.27), we get

$$\begin{cases} \min\{1, k(0)\} \|w\|^2_{\mathcal{W},t} \\ \le \|\eta_0\|^2_H + 2\sigma \int_\Omega |\nabla w(t)|^2 + \iint_{Q_t} |\nabla\chi|^2 + c \iint_{Q_t} |\nabla w|^2 \\ \qquad + \frac{\lambda^2 \|k\|^2_{L^2(0,T)}}{\sigma} \iint_{Q_t} |\nabla\chi|^2 + \sigma c_T^2 \|w\|^2_{\mathcal{W},t} \\ \qquad + \frac{c_T^2}{\sigma} \|f_2\|^2_{W^{1,1}(0,T;V')} + c \int_0^t \left(\|f_1(s)\|_H + \|f'_2(s)\|_{V'}\right) \|w\|_{\mathcal{W},s} \, ds \end{cases} \tag{2.5}$$

for any $t \in [0,T]$ and any $\sigma \in]0,1[$.

Let us come to the second equation (1.39). Write it down for both sets of data and solutions and take the difference. Then choose $v = \chi(t)$ and integrate in time. Splitting $\alpha(t)$ into its positive and negative parts, we get

$$\mu \int_0^t \langle \chi'(s), \chi(s) \rangle \, ds + \nu \iint_{Q_t} |\nabla \chi|^2 + \alpha^+(t) = \alpha^-(t) + \iint_{Q_t} \bigl(\gamma(\widehat{w}\,', \widehat{\chi}) - \gamma(\widetilde{w}\,', \widetilde{\chi})\bigr)\chi.$$

Therefore, by (1.34) it is straightforward to infer that

$$\begin{cases} \mu \, \|\chi(t)\|_H^2 + 2\nu \displaystyle\iint_{Q_t} |\nabla \chi|^2 + 2\alpha^+(t) \\ \le 2 \displaystyle\sup_{0 \le s \le T} \alpha^-(s) + \mu \, \|\chi_0\|_H^2 + c \iint_{Q_t} |\chi|^2 + c \iint_{Q_t} |w'|^2 \end{cases} \tag{2.6}$$

for any $t \in [0,T]$.

To conclude, fix $\sigma \in]0,1[$ and then $M > 0$ such that

$$\sigma(2 + c_T^2) < \min\{1, k(0)\} \qquad \text{and} \qquad 2\nu M > \frac{\lambda^2 \, \|k\|_{L^2(0,T)}^2}{\sigma}.$$

Multiply (2.6) by M and add the resulting inequality to (2.5). Applying the Gronwall lemma in the form of [2] and taking the infimum with respect to all admissible decompositions of f , one recovers (2.3).

3. Existence. In this section we prove the existence of a solution to Problem (1.35–40) and we do that starting from the existence result given in [7], where stronger assumptions on data are made. Therefore, we approximate the data of Problem (1.35–40) with smoother ones, which depend on a positive parameter ε subject to tend to 0 .

Let us choose three families $\{f_\varepsilon\}$, $\{\eta_{0\varepsilon}\}$, and $\{\chi_{0\varepsilon}\}$ such that

$$f_\varepsilon \in W^{1,1}(0,T;H) \qquad f_\varepsilon \to f \quad \text{in } \mathcal{F} \tag{3.1}$$

$$\eta_{0\varepsilon} \in V \qquad \eta_{0\varepsilon} \to \eta_0 \quad \text{in } H \tag{3.2}$$

$$\chi_{0\varepsilon} \in V \qquad \chi_{0\varepsilon} \to \chi_0 \quad \text{in } H. \tag{3.3}$$

and such that for every nonnegative convex function $\psi : \mathbf{R} \to \mathbf{R}$ there holds

$$\int_\Omega \psi(\chi_{0\varepsilon}) \le \int_\Omega \psi(\chi_0). \tag{3.4}$$

For instance, $\chi_{0\varepsilon}$ could be the solution to the elliptic problem (see the Appendix)

$$\chi_{0\varepsilon} \in V \qquad \text{and} \qquad \int_\Omega \bigl(\chi_{0\varepsilon} v + \varepsilon^2 (\nabla \chi_{0\varepsilon} \cdot \nabla v + \chi_{0\varepsilon} v)\bigr) = \int_\Omega \chi_0 v \quad \forall\, v \in V. \tag{3.5}$$

Then (3.4) is fulfilled, as one can see with a simple modification of [3, pp. 281–282]. It is also worth noting that (3.5) implies $\chi_{0\varepsilon} \in H^2(\Omega)$ and $\partial_{\mathrm{n}}\chi_{0\varepsilon} = 0$ on $\partial\Omega$. These last properties are used in Section 5.

Then we approximate the function ϕ and the graph β defining, for $s \in \mathbf{R}$,

$$\phi_\varepsilon(s) = \min_{r\in\mathbf{R}} \left\{ \frac{1}{2\varepsilon}|s-r|^2 + \phi(r) \right\} \qquad \text{and} \qquad \beta_\varepsilon(s) = \phi_\varepsilon'(s). \tag{3.6}$$

Hence β_ε is the Yosida regularization of β and ϕ_ε satisfies

$$0 \le \phi_\varepsilon(s) \le \phi(s) \qquad \text{and} \qquad \phi_\varepsilon(s) \nearrow \phi(s) \qquad \forall\, s \in \mathbf{R} \tag{3.7}$$

as shown in [4, p. 39]. In particular $\phi_\varepsilon(0) = \beta_\varepsilon(0) = 0$ and β_ε is Lipschitz continuous. Combining the inequalities (3.4) and (3.7) we deduce the upper bound

$$\int_\Omega \phi_\varepsilon(\chi_{0\varepsilon}) \le \int_\Omega \phi(\chi_0) \qquad \forall\, \varepsilon > 0. \tag{3.8}$$

Recalling [7, Theorem 3.2], we get the existence of a unique smooth solution $(w_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ to Problem (1.35–40), where the data are replaced with their approximations. Now we derive *a priori* estimates, then we find a sequence $\{\varepsilon_n\}$ such that the corresponding sequence of solutions converges to a triplet (w, χ, ξ) and we show that (w, χ, ξ) solves Problem (1.35–40). Note that uniqueness implies that the whole family of approximate solutions converges.

A priori estimates. We apply Lemma 2.1 with

$$\widehat{f} = f_\varepsilon, \quad \widehat{\eta}_0 = \eta_{0\varepsilon}, \quad \widehat{\chi}_0 = \chi_{0\varepsilon}, \quad \widehat{\phi} = \phi_\varepsilon, \quad \widehat{\beta} = \beta_\varepsilon, \quad \widehat{w} = w_\varepsilon, \quad \widehat{\chi} = \chi_\varepsilon, \quad \widehat{\xi} = \xi_\varepsilon$$
$$\widetilde{f} = 0, \quad \widetilde{\eta}_0 = 0, \quad \widetilde{\chi}_0 = 0, \quad \widetilde{\phi} = \phi_\varepsilon, \quad \widetilde{\beta} = \beta_\varepsilon, \quad \widetilde{w} = 0, \quad \widetilde{\chi} = 0, \quad \widetilde{\xi} = 0.$$

Since the corresponding α is non negative, we deduce immediately

$$\|w_\varepsilon\|_{\mathcal{W}}^2 + \|\chi_\varepsilon\|_{\mathcal{X}}^2 + \iint_Q \xi_\varepsilon \chi_\varepsilon \le c \left(\|f_\varepsilon\|_{\mathcal{F}}^2 + \|\eta_{0\varepsilon}\|_H^2 + \|\chi_{0\varepsilon}\|_H^2 \right). \tag{3.9}$$

Consider equation (1.39) written down for the approximate solution. Then take $v = \xi_\varepsilon$ and integrate over $[0,t]$ where t is arbitrary in $[0,T]$. Observe that this choice of the test function is allowed as $\xi_\varepsilon = \beta_\varepsilon(\chi_\varepsilon)$ and β_ε is Lipschitz continuous. The left hand side of the resulting equality is estimated from below as follows

$$\begin{aligned}
&\mu \iint_{Q_t} \chi_\varepsilon' \xi_\varepsilon + \nu \iint_{Q_t} \nabla\chi_\varepsilon \cdot \nabla\xi_\varepsilon + \iint_{Q_t} |\xi_\varepsilon|^2 \\
&= \mu \iint_{Q_t} \partial_t \phi_\varepsilon(\chi_\varepsilon) + \nu \iint_{Q_t} \beta_\varepsilon'(\chi_\varepsilon)|\nabla\chi_\varepsilon|^2 + \iint_{Q_t} |\xi_\varepsilon|^2 \\
&\ge \mu \int_\Omega \phi_\varepsilon(\chi_\varepsilon(t)) - \mu \int_\Omega \phi_\varepsilon(\chi_{0\varepsilon}) + \iint_{Q_t} |\xi_\varepsilon|^2 \\
&\ge \mu \int_\Omega \phi_\varepsilon(\chi_\varepsilon(t)) - \mu \int_\Omega \phi(\chi_0) + \iint_{Q_t} |\xi_\varepsilon|^2
\end{aligned}$$

in view of (3.8), while the right hand side is easily estimated from above using (1.34),

$$\begin{aligned}\iint_{Q_t} \gamma(w_\varepsilon', \chi_\varepsilon)\xi_\varepsilon &\le c \iint_{Q_t} (|w_\varepsilon'| + |\chi_\varepsilon|)\,|\xi_\varepsilon| \\ &\le \frac{1}{2}\iint_{Q_t} |\xi_\varepsilon|^2 + c\iint_{Q_t} \left(|w_\varepsilon'|^2 + |\chi_\varepsilon|^2\right).\end{aligned}$$

Thus we get

$$\mu \int_\Omega \phi_\varepsilon(\chi_\varepsilon(t)) + \frac{1}{2}\iint_{Q_t} |\xi_\varepsilon|^2 \le c \iint_{Q_t} \left(|w_\varepsilon'|^2 + |\chi_\varepsilon|^2\right) + \mu \int_\Omega \phi(\chi_0). \tag{3.10}$$

In view of (3.9) and (3.10), by comparison in the approximated version of (1.39), we achieve a uniform bound for $\{\partial_t \chi_\varepsilon\}$ in the space $L^2(0,T;V')$.

Therefore, accounting for the Lipschitz continuity of γ, we can combine the above estimates with (3.1–3) to infer that

$$\{w_\varepsilon\} \quad \text{is bounded in} \quad L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H) \tag{3.11}$$

$$\{\chi_\varepsilon\} \quad \text{is bounded in} \quad L^2(0,T;V) \cap C^0([0,T];H) \cap H^1(0,T;V') \tag{3.12}$$

$$\{\xi_\varepsilon\} \quad \text{is bounded in} \quad L^2(Q) \tag{3.13}$$

$$\{\phi_\varepsilon(\chi_\varepsilon)\} \quad \text{is bounded in} \quad L^\infty(0,T;L^1(\Omega)) \tag{3.14}$$

$$\{\gamma(w_\varepsilon', \chi_\varepsilon)\} \quad \text{is bounded in} \quad L^\infty(0,T;H). \tag{3.15}$$

Moreover, we have that

$$\begin{cases} \limsup_{\varepsilon\to 0}\Big(\|w_\varepsilon\|_{\mathcal{W}}^2 + \|\chi_\varepsilon\|_{\mathcal{X}}^2 + \|\chi_\varepsilon\|_{H^1(0,T;V')}^2 \\ \qquad\qquad + \|\xi_\varepsilon\|_{L^2(Q)}^2 + \|\phi_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))}\Big) \\ \le c\left(\|f\|_{\mathcal{F}}^2 + \|\eta_0\|_H^2 + \|\chi_0\|_H^2 + \|\phi(\chi_0)\|_{L^1(\Omega)}\right) \end{cases} \tag{3.16}$$

where c depends on $\|k\|_{W^{2,1}(0,T)}$, $k(0)$, λ, μ, ν, T, and on the Lipschitz constant of γ, only. We point out that (3.16) entails (1.41) once we verify that the approximate solutions tend to a solution to Problem (1.35–40).

Weak convergent subsequence. Thanks to (3.11–15) and to well–known compactness results, there exist a sequence $\varepsilon_n \searrow 0$ and a triplet (w, χ, ξ) such that, setting $w_n = w_{\varepsilon_n}$, $\chi_n = \chi_{\varepsilon_n}$, and $\xi_n = \xi_{\varepsilon_n}$, we have

$$w_n \to w, \qquad \chi_n \to \chi, \qquad \xi_n \to \xi, \qquad \text{and} \qquad \gamma(w_n', \chi_n) \to \omega$$

weakly or weakly* in the appropriate spaces. Observe that

$$w \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H) \tag{3.17}$$

$$\chi \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap H^1(0,T;V') \tag{3.18}$$

$$\xi \in L^2(Q) \tag{3.19}$$

$$\omega \in L^\infty(0,T;H). \tag{3.20}$$

Due to the Aubin lemma (see, e.g., [14, p. 58]), χ_n converges to χ strongly in $L^2(0,T;H) = L^2(Q)$. Recalling then [3, Prop. 1.1, p. 42], one can easily check that

$$\xi \in \beta(\chi) \qquad \text{a.e. in } Q.$$

Besides, it is straightforward to derive the equations

$$\langle\!\langle w'', v\rangle\!\rangle + k(0) \iint_Q \nabla w \cdot \nabla v$$

$$= - \iint_Q \nabla(k' * w) \cdot \nabla v + \lambda \iint_Q \nabla(k * \chi) \cdot \nabla v + \langle\!\langle f, v\rangle\!\rangle$$

$$\langle\!\langle \mu\chi', v\rangle\!\rangle + \nu \iint_Q \nabla\chi \cdot \nabla v + \iint_Q \xi v = \iint_Q \omega v$$

for any $v \in \mathcal{D}(0,T;V)$, where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ stands for the duality pairing between the spaces $\mathcal{D}'(0,T;V')$ and $\mathcal{D}(0,T;V)$. By comparison, we deduce that $w'' \in L^1(0,T;V')$. Therefore, the previous equations can be set in the form (1.38) and

$$\mu \langle \chi'(t), v\rangle + \nu \int_\Omega \nabla\chi(t) \cdot \nabla v + \int_\Omega \xi(t) v = \int_\Omega \omega(t) v \tag{3.21}$$
$$\forall\, v \in V, \quad \text{for a.a. } t \in \,]0,T[.$$

Hence, to prove that (w, χ, ξ) is a solution to Problem (1.35–40), it suffices to show that

$$w_n \to w \text{ and } \chi_n \to \chi \text{ strongly in } \mathcal{W} \text{ and in } \mathcal{X}, \text{ respectively.} \tag{3.22}$$

Indeed, (3.22) and (1.34) allow us to identify ω, namely

$$\omega = \gamma(w', \chi) \quad \text{a.e. in } Q\,.$$

Moreover, (3.22) yield at once the initial conditions (1.40) and, thanks to (3.14), (3.7), and to the Fatou lemma, the regularity requirement $\phi(\chi) \in L^\infty(0,T;L^1(\Omega))$ of (1.36).

Further strong convergences and end of the proof. Here we show that $\{w_n\}$ and $\{\chi_n\}$ are Cauchy sequences in $\mathcal{W}$ and in $\mathcal{X}$, respectively, that is to say (3.22) holds. To this aim, we exploit once more the Main lemma with

$$\widehat{f} = f_n, \quad \widehat{\eta}_0 = \eta_{0n}, \quad \widehat{\chi}_0 = \chi_{0n}, \quad \widehat{\phi} = \phi_n, \quad \widehat{\beta} = \beta_n$$
$$\widetilde{f} = f_m, \quad \widetilde{\eta}_0 = \eta_{0m}, \quad \widetilde{\chi}_0 = \chi_{0m}, \quad \widetilde{\phi} = \phi_m, \quad \widetilde{\beta} = \beta_m$$
$$\widehat{w} = w_n, \quad \widehat{\chi} = \chi_n, \quad \widehat{\xi} = \xi_n$$
$$\widetilde{w} = w_m, \quad \widetilde{\chi} = \chi_m, \quad \widetilde{\xi} = \xi_m.$$

This application yields

$$\begin{cases} \|w_{nm}\|_{\mathcal{W}}^2 + \|\chi_{nm}\|_{\mathcal{X}}^2 + \sup_{0\le t\le T} \alpha_{nm}^+(t) \\ \le c\left(\|f_{nm}\|_{\mathcal{F}}^2 + \|\eta_{0nm}\|_H^2 + \|\chi_{0nm}\|_H^2 + \sup_{0\le t\le T} \alpha_{nm}^-(t)\right) \end{cases} \tag{3.23}$$

where

$$\alpha_{nm}(t) = \iint_{Q_t} \xi_{nm}\chi_{nm}$$

and $w_{nm} = w_n - w_m$, $\chi_{nm} = \chi_n - \chi_m$, and so on. The constant c does not depend on n and m . Due to (3.1–3), the first three terms in the right hand side of (3.23) tend to 0 as $n, m \to \infty$. Hence, if we check that

$$\lim_{n,m\to\infty} \alpha_{nm} = 0 \qquad \text{uniformly in } [0,T] \tag{3.24}$$

the strong convergences (3.22) follows.

Note that the family $\{\alpha_{nm}\}$ is equibounded because of (3.12) and (3.13). Since $\xi_{nm} \to 0$ weakly and $\chi_{nm} \to 0$ strongly in $L^2(Q)$, we realize that

$$\lim_{n,m\to\infty} \alpha_{nm}(t) = 0 \qquad \forall\, t \in [0,T]\,.$$

Therefore, to prove (3.24), one just needs to show that the family $\{\alpha_{nm}\}$ is equicontinuous. In fact, from

$$\alpha_{nm}'(t) = \int_\Omega \bigl(\xi_n(t) - \xi_m(t)\bigr)\bigl(\chi_n(t) - \chi_m(t)\bigr) \qquad \text{for a.a. } t \in \,]0,T[$$

one deduces that

$$\int_0^T |\alpha_{nm}'(t)|^2\, dt \le 2 \sup_j \|\chi_j\|_{\mathcal{X}}^2 \int_0^T \left(\|\xi_n(t)\|_H^2 + \|\xi_m(t)\|_H^2\right) dt.$$

As the right hand side is bounded by a constant independent of n and m by virtue of (3.12) and (3.13), one infers that $\{\alpha_{nm}\}$ is bounded in $H^1(0,T)$ and the equicontinuity is proved.

4. Continuous dependence and uniqueness. Let us observe first of all that the uniqueness result of Theorem 1.3 actually follows from the estimate (1.42). Indeed, if $\beta_1 = \beta_2$, the last term in the right hand side vanishes, the choice $\zeta = \xi_1$ being allowed. Hence we conclude immediately that $w_1 = w_2$, $\chi_1 = \chi_2$. Consequently, writing down equation (1.39) for both the solutions and taking the difference, we infer

$$\int_\Omega (\xi_1 - \xi_2)(t) v = 0 \qquad \forall\, v \in V$$

for almost all $t \in \,]0,T[$. As $\xi_1(t),\ \xi_2(t) \in H$ for a.a. $t \in \,]0,T[$ and V is dense in H, we get $\xi_1 = \xi_2$.

To prove the continuous dependence estimate (1.42), we apply Lemma 2.1 with

$$\begin{array}{lllll} \widehat{f} = f_1, & \widehat{\eta}_0 = \eta_{01}, & \widehat{\chi}_0 = \chi_{01}, & \widehat{\phi} = \phi_1, & \widehat{\beta} = \beta_1 \\ \widetilde{f} = f_2, & \widetilde{\eta}_0 = \eta_{02}, & \widetilde{\chi}_0 = \chi_{02}, & \widetilde{\phi} = \phi_2, & \widetilde{\beta} = \beta_2. \end{array}$$

This gives

$$\|w\|_{\mathcal{W}}^2 + \|\chi\|_{\mathcal{X}}^2 + \sup_{0 \le t \le T} \alpha^+(t) \le c \left(\|f\|_{\mathcal{F}}^2 + \|\eta_0\|_H^2 + \|\chi_0\|_H^2 + \sup_{0 \le t \le T} \alpha^-(t) \right)$$

where, e.g., $w = w_1 - w_2$ and

$$\alpha(t) = \iint_{Q_t} \xi\chi.$$

Now we estimate the last term on the right hand side. For any $t \in [0,T]$ and any admissible ζ, we have

$$\alpha(t) = \iint_{Q_t} (\xi_1 - \zeta)\chi + \iint_{Q_t} (\zeta - \xi_2)\chi \ge \iint_{Q_t} (\xi_1 - \zeta)\chi$$

because $\zeta \in \beta_2(\chi_1)$ a.e. in Q. Then, for any $\sigma > 0$, it turns out that

$$\begin{aligned} \alpha^-(t) &\le \left(\iint_{Q_t} (\xi_1 - \zeta)\chi \right)^- \le \left| \iint_{Q_t} (\xi_1 - \zeta)\chi \right| \le \iint_{Q} |\xi_1 - \zeta|\, |\chi| \\ &\le \sigma \|\chi\|_{L^2(Q)}^2 + \frac{1}{4\sigma} \|\xi_1 - \zeta\|_{L^2(Q)}^2 \le \sigma T \|\chi\|_{\mathcal{X}}^2 + \frac{1}{4\sigma} \|\xi_1 - \zeta\|_{L^2(Q)}^2 . \end{aligned}$$

The result follows immediately choosing σ small enough and taking the infimum over all admissible ζ.

Remark 4.1. A straightforward modification in the above procedure leads to

$$\alpha^-(t) \le \sigma \|\chi\|_{\mathcal{X}}^2 + \frac{1}{4\sigma} \|\xi_1 - \zeta\|_{L^1(0,T;H)}^2 .$$

Therefore the last term in (1.42) can be replaced with $\inf_\zeta \|\xi_1 - \zeta\|_{L^1(0,T;H)}^2$.

Remark 4.2. The infimum appearing in (1.42) is related to how close β_1 and β_2 are to each other and it reduces to $\|\xi_1 - \beta_2(\chi_1)\|_{L^2(Q)}^2$ when β_2 is single–valued.

Remark 4.3. In the case when β_1 is a smooth function β defined on the whole $\mathbf{R}$ and β_2 is the $\varepsilon-$ Yosida approximation β_ε of β, a sharp estimate of the norm of $\beta(\chi_1) - \beta_\varepsilon(\chi_1)$ can be found. Indeed, we assume that either β is Lipschitz continuous or χ is bounded and β is locally Lipschitz continuous. Then, we write χ and χ_ε in place of χ_1 and χ_2, respectively, and term L either the Lipschitz constant of β in the first case or the Lipschitz constant of β in $[-M,M]$ in the second one, where $M = \|\chi\|_{L^\infty(Q)}$. Hence, the definition of β_ε implies (see [4, p. 28])

$$\beta_\varepsilon(\chi) = \frac{\chi - \chi^\varepsilon}{\varepsilon} \qquad \text{a.e. in } Q$$

where χ^ε solves

$$\chi^\varepsilon + \varepsilon\beta(\chi^\varepsilon) = \chi \qquad \text{a.e. in } Q.$$

Consequently, $|\chi_\varepsilon| \le |\chi|$ and $\beta_\varepsilon(\chi) = \beta(\chi^\varepsilon)$, so that

$$|\beta(\chi) - \beta_\varepsilon(\chi)| = |\beta(\chi) - \beta(\chi^\varepsilon)| \le L|\chi - \chi^\varepsilon| = \varepsilon L|\beta(\chi^\varepsilon)| \le \varepsilon L^2|\chi^\varepsilon| \le \varepsilon L^2|\chi|$$

whence we deduce

$$\|\beta(\chi) - \beta_\varepsilon(\chi)\|_{L^2(Q)} \le \varepsilon L^2 \|\chi\|_{L^2(Q)} .$$

If β is just Hölder continuous of exponent σ (globally or locally, respectively), then a simple modification of the above argument shows that the last term has order ε^σ.

Remark 4.4. In the context of Remark 4.3, it is natural to wonder whether the other terms in the right hand side of (1.42) have order ε or ε^σ, provided that the approximations are properly chosen. The procedure described in the appendix gives a regularization u_ε of a given u and it can be applied both to the Cauchy data and, by means of a pointwise definition, to the function f as well. In fact, we have (see Proposition 6.1)

$$\|u - u_\varepsilon\|_H \le 2\|u\|_H \qquad \text{and} \qquad \|u - u_\varepsilon\|_H \le \varepsilon\|u\|_V$$

whether u belongs to the space H or V, respectively. The standard interpolation theory (cf., e.g., [16]) allows us to deduce

$$\|u - u_\varepsilon\|_H \le c\varepsilon^\sigma \|u\|_{H^\sigma(\Omega)} \qquad \text{if} \quad u \in H^\sigma(\Omega) \quad \text{and} \quad 0 < \sigma < 1.$$

5. A boundedness result. Establishing the boundedness of χ can be useful both from the physical and the mathematical viewpoint. Concerning the latter, it is worth recalling Remark 4.3. Here we give a sufficient condition for χ to be bounded, that is

Theorem 5.1. *Assume* $N = 3$, (1.28–34), *and* $\chi_0 \in L^\infty(\Omega)$. *If the triplet* (w, χ, ξ) *is the solution to Problem* (1.35–40), *then* $\chi \in L^\infty(Q)$.

Proof. We work on the approximating problem, maintaining the notation of Section 3. Letting f_ε, $\eta_{0\varepsilon}$, $\chi_{0\varepsilon}$ fulfill (3.1–3) and choosing $\chi_{0\varepsilon}$ as in (3.5), Theorem 3.3 and

Remark 2.4 of [7] entail that the component χ_ε of the solution belongs to $H^1(Q)\cap L^\infty(Q)$ and it satisfies the initial condition $\chi_\varepsilon(0)=\chi_{0\varepsilon}$ as well as, for any $t\in[0,T]$ and any $\eta\in L^2(0,T;V)\cap L^\infty(Q)$, the variational equation

$$\iint_{Q_t}\big(\mu(\partial_t\chi_\varepsilon)\eta+\nu\nabla\chi_\varepsilon\cdot\nabla\eta+\beta_\varepsilon(\chi_\varepsilon)\eta-F_\varepsilon\eta\big)=0,\quad \text{where}\quad F_\varepsilon=\gamma(w_\varepsilon',\chi_\varepsilon). \tag{5.1}$$

Moreover, in our existence proof we have shown that $\{\chi_\varepsilon\}$ converges to χ. On the other hand, we have

$$\|F_\varepsilon\|_{C^0([0,T];H)}\le c\left(\|w_\varepsilon'\|_{C^0([0,T];H)}+\|\chi_\varepsilon\|_{C^0([0,T];H)}\right)$$

and

$$\|\chi_{0\varepsilon}\|_{L^\infty(\Omega)}\le\|\chi_0\|_{L^\infty(\Omega)}$$

where c does not depend on ε. The first one follows from (1.34), the second one is a consequence of the maximum principle applied to the solution $\chi_{0\varepsilon}$ of (3.5). Therefore, to get the result it suffices to determine an upper bound for χ_ε in $L^\infty(Q)$ in terms of the norms of F_ε and $\chi_{0\varepsilon}$ in $L^\infty(0,T;H)$ and in $L^\infty(\Omega)$, respectively.

To this goal, we apply an argument developed in [13]. Taking $q=2$, $r=\infty$, and $\kappa_1=1/4$, the assumption (7.2) of [13, p. 181] is fulfilled. Thus we just need to check that the Moser type argument used in [13, pp. 189-191] is fit for the present case, in spite of the nonlinear term $\beta_\varepsilon(\chi_\varepsilon)$ and of the different boundary conditions.

First of all note that, for any $c>0$, the function $c\chi_\varepsilon$ satisfies an equation which has the same structure as (5.1) does. Then we assume $\|\chi_{0\varepsilon}\|_{L^\infty(\Omega)}\le 1$ without loss of generality and take $\eta=|\chi_\varepsilon|^{2s-1}\operatorname{sign}(\chi_\varepsilon)$ as test function, where $s\ge 1$ is arbitrary. We obtain

$$\begin{aligned}\frac{\mu}{2s}\int_\Omega|v(t)|^2+\frac{\nu(2s-1)}{s^2}\iint_{Q_t}|\nabla v|^2+\iint_{Q_t}\beta_\varepsilon(\chi_\varepsilon)|\chi_\varepsilon|^{2s-1}\operatorname{sign}(\chi_\varepsilon)\\ \le\frac{\mu}{2s}\int_\Omega|\chi_{0\varepsilon}|^{2s}+\iint_{Q_t}|F_\varepsilon|\,|v|^{(2s-1)/s}\end{aligned}$$

where $v=|\chi_\varepsilon|^s$. Since β_ε is nondecreasing and $\beta_\varepsilon(0)=0$, the last integral on the left hand side is nonnegative. Thus it is not difficult to infer

$$\|v\|_{\mathcal{X}}^2\le cs^2\left(1+\iint_Q|F_\varepsilon|\,|v|^{(2s-1)/s}\right) \tag{5.2}$$

where c does not depend on ε and s. Hence, estimating the right hand side, we can deduce the inequality [13, (7.25), p. 190] in the same way, where $\bar q=4$ and $\bar r=2$. Now, following [13], we set $\widehat q=14/3$, $\widehat r=7/3$ and observe that the inequality

$$\|v\|_{L^{7/3}(0,T;L^{14/3}(\Omega))}\le C\,\|v\|_{\mathcal{X}} \tag{5.3}$$

i.e., (3.4) of [13, p. 75], holds even though v does not vanish on the boundary, provided that C, depending on Ω and T, is properly chosen. Therefore we can proceed exactly as in [13, pp. 190-191] and conclude.

6. Appendix. We give here an abstract version of the regularization procedure we have used in deriving the Main lemma of Section 2 in the framework of a real Hilbert triplet (V, H, V').

Let us fix two Hilbert spaces H and V and denote by $|\cdot|$ and $\|\cdot\|$ their norms and by $(\cdot,\cdot)$ and $(\!(\cdot,\cdot)\!)$ their scalar products, respectively. We assume that V is a dense linear subspace of H, the inclusion of V into H being continuous, and consider H as embedded into the dual space V' of V by means of the usual formula

$$\langle u, v\rangle = (u, v) \qquad \forall\, u \in H \quad \forall\, v \in V$$

where $\langle\cdot,\cdot\rangle$ is the scalar product between V' and V. In the sequel we denote by $\|\cdot\|_*$ the associated norm in V'.

We introduce the identity or injection operator $I : V \to V'$ and the canonical isomorphism J from V onto V' given by the Riesz theorem, i.e. defined by

$$\langle Ju, v\rangle = (\!(u, v)\!) \qquad \forall\, u, v \in V.$$

Thus we have $\|Jv\|_* = \|v\|$ for any $v \in V$ and

$$(\!(u, v)\!)_* = \left\langle u, J^{-1}v\right\rangle \qquad \forall\, u, v \in V'.$$

For $\varepsilon > 0$ and $u \in V'$ the problem

$$u_\varepsilon \in V \qquad \text{and} \qquad (I + \varepsilon^2 J)u_\varepsilon = u \tag{6.1}$$

has a unique solution u_ε. Indeed (6.1) is equivalent to

$$u_\varepsilon \in V \qquad \text{and} \qquad (u_\varepsilon, v) + \varepsilon^2 (\!(u_\varepsilon, v)\!) = \langle u, v\rangle \quad \forall\, v \in V \tag{6.2}$$

and the Lax–Milgram theorem applies. The behavior of u_ε as $\varepsilon \to 0$ is well known (cf. [15]). We summarize a list of results in

Proposition 6.1. For any $u \in V'$, we have

$$\|u_\varepsilon\|_* \le \|u\|_* \qquad \text{and} \qquad u_\varepsilon \to u \quad \text{in} \;\; V' \tag{6.3}$$

$$\varepsilon\, |u_\varepsilon| \le \|u\|_* \qquad \text{and} \qquad \varepsilon u_\varepsilon \to 0 \quad \text{in} \;\; H \tag{6.4}$$

$$\varepsilon^2\, \|u_\varepsilon\| \le \|u\|_* \qquad \text{and} \qquad \varepsilon^2 u_\varepsilon \to 0 \quad \text{in} \;\; V. \tag{6.5}$$

Moreover, if $u \in H$, then we have

$$\|u - u_\varepsilon\|_* \le \varepsilon\, |u| \tag{6.6}$$

$$|u_\varepsilon| \le |u| \qquad \text{and} \qquad u_\varepsilon \to u \quad \text{in} \;\; H \tag{6.7}$$

$$\varepsilon\, \|u_\varepsilon\| \le |u| \qquad \text{and} \qquad \varepsilon u_\varepsilon \to 0 \quad \text{in} \;\; V. \tag{6.8}$$

Finally, if $u \in V$, then we have

$$\|u-u_\varepsilon\|_* \le \varepsilon^2\,\|u\| \tag{6.9}$$

$$|u-u_\varepsilon| \le \varepsilon\,\|u\| \tag{6.10}$$

$$\|u_\varepsilon\| \le \|u\| \qquad \text{and} \qquad u_\varepsilon \to u \quad \text{in} \;\; V. \tag{6.11}$$

Now we deal with time dependent elements. Given $u \in L^1(0,T;V')$ we define u_ε using (6.1) pointwise, i.e. by means of

$$u_\varepsilon(t) \in V \qquad \text{and} \qquad (I+\varepsilon^2 J)u_\varepsilon(t) = u(t) \qquad \text{for a.a. } t \in \,]0,T[. \tag{6.12}$$

Thanks to (6.5), it is clear that $u_\varepsilon \in L^1(0,T;V)$. Moreover, if $u' \in L^1(0,T;V')$, differentiating (6.12), we obtain $u'_\varepsilon \in L^1(0,T;V)$ and

$$(I+\varepsilon^2 J)u'_\varepsilon(t) = u'(t) \qquad \text{for a.a. } t \in \,]0,T[.$$

Proposition 6.2. Assume $u \in C^0([0,T];H) \cap W^{1,1}(0,T;V')$. Then

$$\lim_{\varepsilon\to 0} \int_0^t \langle u'(s), u_\varepsilon(s)\rangle \, ds = \frac{1}{2}|u(t)|^2 - \frac{1}{2}|u(0)|^2 \qquad \forall\, t \in [0,T]. \tag{6.13}$$

Proof. For any fixed $t \in [0,T]$ we have

$$\begin{aligned}\int_0^t \langle u'(s), u_\varepsilon(s)\rangle \, ds &= \int_0^t \langle u'(s)-u'_\varepsilon(s), u_\varepsilon(s)\rangle \, ds + \int_0^t (u'_\varepsilon(s), u_\varepsilon(s))\, ds \\ &= \int_0^t \langle u'(s)-u'_\varepsilon(s), u_\varepsilon(s)\rangle \, ds + \frac{1}{2}|u_\varepsilon(t)|^2 - \frac{1}{2}|u_\varepsilon(0)|^2\end{aligned}$$

since $u_\varepsilon \in W^{1,1}(0,T;H)$. For $u \in C^0([0,T];H)$, the limit as $\varepsilon \to 0$ of the last two terms gives the right hand side of (6.13) and the lemma is proved if we show that the last integral tends to 0. Using the fact that $u_\varepsilon \in W^{1,1}(0,T;V)$, we have

$$\begin{aligned}\int_0^t \langle u'(s)-u'_\varepsilon(s), u_\varepsilon(s)\rangle \, ds &= \int_0^t \big\langle \varepsilon^2 J u'_\varepsilon(s), u_\varepsilon(s)\big\rangle \, ds = \varepsilon^2 \int_0^t ((u'_\varepsilon(s), u_\varepsilon(s)))\, ds \\ &= \frac{\varepsilon^2}{2}\,\|u_\varepsilon(t)\|^2 - \frac{\varepsilon^2}{2}\,\|u_\varepsilon(0)\|^2 = \frac{1}{2}\,\|\varepsilon u_\varepsilon(t)\|^2 - \frac{1}{2}\,\|\varepsilon u_\varepsilon(0)\|^2\end{aligned}$$

and this quantity tends to 0 owing to (6.8) and for $u \in C^0([0,T];H)$.

Proposition 6.3. Let $a(\cdot,\cdot)$ be a bilinear continuous symmetric form on $V \times V$ satisfying the compatibility condition

$$a(Jv_1, v_2) = a(v_1, Jv_2) \qquad \forall\, v_1, v_2 \in J^{-1}(V) = J^{-1}(I(V)). \tag{6.14}$$

Then, if $u \in C^0([0,T];V) \cap W^{1,1}(0,T;V')$, we have

$$\lim_{\varepsilon\to 0} \int_0^t a(u(s), u'_\varepsilon(s))\, ds = \frac{1}{2}a(u(t),u(t)) - \frac{1}{2}a(u(0),u(0)) \qquad \forall\, t \in [0,T]. \tag{6.15}$$

Proof. Note firstly that the linear subspace $W = J^{-1}(V)$ of V is dense in V. Indeed, if $v \in V$ and $\delta > 0$, defining v_δ by $v_\delta = (I + \delta^2 J)^{-1} v$, it turns out that $v_\delta \in W$, since $\delta^2 J v_\delta = v - v_\delta \in V$. Moreover, $v_\delta \to v$ in V as $\delta \to 0$, thanks to (6.11). It follows that W is dense in H and we can consider the new Hilbert triplet (W, H, W'), which is compatible with the previous one, that is

$$W \subseteq V \subseteq H \subseteq V' \subseteq W'.$$

We now define the form b on $W \times W$ by setting

$$b(w_1, w_2) = a(Jw_1, w_2), \qquad w_1, w_2 \in W$$

and observe that b is bilinear and continuous. Moreover, b is symmetric since a is symmetric and (6.14) holds.

For any fixed $t \in [0, T]$, we have

$$\int_0^t a(u(s), u'_\varepsilon(s))\, ds = \int_0^t a(u_\varepsilon(s), u'_\varepsilon(s))\, ds + \int_0^t a(u(s) - u_\varepsilon(s), u'_\varepsilon(s))\, ds.$$

Let us consider the two last terms separately. As $u_\varepsilon \in W^{1,1}(0, T; V)$, it follows

$$\int_0^t a(u_\varepsilon(s), u'_\varepsilon(s))\, ds = \frac{1}{2} a(u_\varepsilon(t), u_\varepsilon(t)) - \frac{1}{2} a(u_\varepsilon(0), u_\varepsilon(0))$$

and this right hand side tends to the right hand side of (6.15) since we know that $u \in C^0([0, T]; V)$.

Therefore, to prove the lemma, we have to show that the other integral tends to 0. In order to do that, we state an auxiliary formula, starting firstly with an arbitrary $w \in W^{1,1}(0, T; W)$. Using the properties of the form b we obtain

$$\int_0^t a(Jw(s), w'(s))\, ds = \int_0^t b(w(s), w'(s))\, ds$$
$$= \frac{1}{2} b(w(t), w(t)) - \frac{1}{2} b(w(0), w(0)) = \frac{1}{2} a(Jw(t), w(t)) - \frac{1}{2} a(Jw(0), w(0)).$$

Assume now only $w \in C^0([0, T]; W) \cap W^{1,1}(0, T; V)$ and consider the approximation w_δ of w defined by (6.12), that is $w_\delta = (I + \delta^2 J)^{-1} w$. Then $w_\delta \in W^{1,1}(0, T; W)$, so that the last conclusion applies to w_δ. We deduce that

$$\int_0^t a(Jw(s), w'(s))\, ds = \lim_{\delta \to 0} \int_0^t a(Jw_\delta(s), w'_\delta(s))\, ds$$
$$= \frac{1}{2} \lim_{\delta \to 0} a(Jw_\delta(t), w_\delta(t)) - \frac{1}{2} \lim_{\delta \to 0} a(Jw_\delta(0), w_\delta(0))$$
$$= \frac{1}{2} a(Jw(t), w(t)) - \frac{1}{2} a(Jw(0), w(0)).$$

Hence the same formula holds even in the weaker assumption made on w. Taking $w = u_\varepsilon$, we get

$$\int_0^t a(u(s) - u_\varepsilon(s), u_\varepsilon'(s))\, ds = \int_0^t a(\varepsilon^2 J u_\varepsilon(s), u_\varepsilon'(s))\, ds$$
$$= \frac{\varepsilon^2}{2} a(Ju_\varepsilon(t), u_\varepsilon(t)) - \frac{\varepsilon^2}{2} a(Ju_\varepsilon(0), u_\varepsilon(0))$$
$$= \frac{1}{2} a(u(t) - u_\varepsilon(t), u_\varepsilon(t)) - \frac{1}{2} a(u(0) - u_\varepsilon(0), u_\varepsilon(0))$$

and the last two terms tend to 0 by (6.11), as $u \in C^0([0,T];V)$.

Remark 6.4. The compatibility condition (6.14) holds, for instance, if the form a satisfies the weak coercivity inequality

$$a(v,v) + \lambda_0 |v|^2 \geq \alpha_0 \|v\|^2 \qquad \forall\, v \in V$$

for some $\lambda_0 \in \mathbf{R}$ and $\alpha_0 > 0$, provided we introduce in V the new norm

$$\|v\|_\bullet^2 = a(v,v) + \lambda_0 |v|^2$$

which is equivalent to the previous one. In this case, the isomorphism J is given by $J = I + \lambda_0 A$, where A is the operator from V to V' associated with a by

$$\langle Au, v\rangle = a(u,v), \qquad u, v \in V.$$

Therefore, for any $v_1, v_2 \in J^{-1}(V)$, we have

$$a(Jv_1, v_2) = \langle AJv_1, v_2\rangle = \langle A^2 v_1, v_2\rangle + \lambda_0 \langle Av_1, v_2\rangle$$

and (6.14) follows, since A is a symmetric operator.

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