OPTIMAL CONTROL OF SOLUTIONS TO THE CAUCHY PROBLEM FOR THE SOBOLEV TYPE EQUATION OF HIGHER ORDER

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STRUCTURING

- Introduction
- Abstract Theory
- The Boussinesq – L$\grave{o}$ve Mathematical Model

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0x + y + Cu, \tag{1}$$

where $A, B_{n-1}, \ldots, B_0 \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y})$, $C \in \mathcal{L}(\mathfrak{U}; \mathfrak{Y})$, the functions $u : [0, \tau) \subset \mathbf{R}_+ \to \mathfrak{U}$, $y : [0, \tau) \subset \mathbf{R}_+ \to \mathfrak{Y}\,(\tau < \infty)$, $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ are Hilbert spaces.

$$x^{(m)}(0) = x_m, \quad m = \overline{0, n-1}. \tag{2}$$

$$P(x^{(m)}(0) - x_m) = 0, \quad m = \overline{0, n-1}. \tag{3}$$

$$J(\hat{x}, \hat{u}) = \min_{(x,u) \in \mathfrak{X} \times \mathfrak{U}_{ad}} J(x, u). \tag{4}$$

RELEVANCE OF RESEARCH

S.L. Sobolev

M.O. Korpusov, A.I. Kozhanov, G.V. Demidenko

N.A. Sidorov, B.V. Loginov, A.V. Sinitsin, M.V. Falaleev, I.V. Melnikova, A.I. Filinkov, M.A. Alshansky

Yu.E. Boyarintsev, V.F. Chistyakov, A.A. Sheglova
A.V. Keller

R. Showalter, A. Favini, A. Yagi

G.A. Sviridyuk, T.G. Sukacheva, V.E. Fedorov

Models

The de Gennes mathematical model of acoustic waves in a smectic

$$\frac{\partial^2}{\partial t^2}\Delta_3 u = \alpha_1 \frac{\partial^2}{\partial z^2}\Delta_2 u,\ \ \alpha_1 > 0,$$

where $\Delta_3 = \Delta_2 + \frac{\partial^2}{\partial z^2},\ \ \Delta_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

$$u(x_1, x_2, z, t) = v(x_1, x_2, z)\exp(-i\omega t)$$

$$\frac{\partial^2}{\partial z^2}(\Delta_2 v + \alpha_2 v) + \alpha_2 \Delta_2 v = 0,\ \ \alpha_2 = \omega^2 \alpha_1^{-1}$$

The mathematical model of fluctuations in the DNA molecule

$$\begin{array}{ll} u(x,0) = u_0(x), & \dot{u}(x,0) = u_1(x), \\ v(x,0) = v_0(x), & \dot{v}(x,0) = v_1(x), \end{array} \quad x \in \Omega,$$

$$u(x,t) = v(x,t) = 0, (x,t) \in \partial\Omega \times \mathbb{R},$$

$$\begin{cases} (b+\Delta)\ddot{u} = a\Delta u + f(u,v) + w_1, \\ (b+\Delta)\ddot{v} = d\Delta v + g(u,v) + w_2. \end{cases}$$

Models

The mathematical model of shallow water waves propagation

$$(\lambda - \Delta)\ddot{u} = \alpha^2 \Delta u + f,$$
$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = u_1(x), \quad x \in \Omega,$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}.$$

The mathematical model of ion-acoustic waves in plasma

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} + \omega_{B_i}^2\right)(\Delta_3 \Phi - \frac{1}{r_D^2}\Phi) + \omega_{p_i}^2 \frac{\partial^2}{\partial t^2}\Delta_3 \Phi + \omega_{B_i}^2 \omega_{p_i}^2 \frac{\partial^2 \Phi}{\partial x_3^2} = 0$$

$$(\Delta - \lambda)v_{tttt} + (\Delta - \lambda')v_{tt} + \alpha \frac{\partial^2 u}{\partial x_3^2} = 0$$

$$\frac{\partial^2}{\partial t^2}(\Delta \Phi - \Phi) + \Delta \Phi = 0$$

The Boussinesq – Love model

$$(\lambda - \Delta)u_{tt} = \alpha(\Delta - \lambda^{'})u_t + \beta(\Delta - \lambda^{''})u + g$$

By $\vec{B}$ denote the pencil formed by operators $B_{n-1},\ldots,B_0$.

$$\rho^A(\vec{B})=\{\mu\in\mathbb{C}:(\mu^nA-\mu^{n-1}B_{n-1}-\ldots-\mu B_1-B_0)^{-1}\in\mathcal{L}(\mathfrak{Y};\mathfrak{X})\},\quad \sigma^{\mathrm{A}}(\vec{\mathrm{B}})=\mathbb{C}\backslash\rho^{\mathrm{A}}(\vec{\mathrm{B}}).$$

$$R^A_\mu(\vec{B})=(\mu^nA-\mu^{n-1}B_{n-1}-\ldots-\mu B_1-B_0)^{-1}.$$

Definition 1.

The pencil $\vec{B}$ is called *polynomially bounded with respect to operator* A (or *polynomially A-bounded*), if

$$\exists a\in\mathbb{R}_+\quad\forall\mu\in\mathbb{C}\quad(|\mu|>a)\Rightarrow(R^A_\mu(\vec{B})\in\mathcal{L}(\mathfrak{Y};\mathfrak{X})).$$

$$\int\limits_\gamma \mu^m R^A_\mu(\vec{B})d\mu\equiv\mathbb{O},m=\overline{0,n-2},\qquad (A)$$

where $\gamma=\{\mu\in\mathbb{C}:|\mu|=r>a\}$.

Lemma 1.

If the pencil $\vec{B}$ is polynomially A-bounded, and condition (A) is satisfied, then the following operators

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^A_\mu(\vec{B})\mu^{n-1}A d\mu, \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} \mu^{n-1} A R^A_\mu(\vec{B}) d\mu$$

are projectors in $\mathfrak{X}$ and $\mathfrak{Y}$, respectively.

$\mathfrak{X}^0 = \ker P$, $\mathfrak{Y}^0 = \ker Q$, $\mathfrak{X}^1 = \text{im } P$, $\mathfrak{Y}^1 = \text{im } Q$, $\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1$, $\mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1$. By A^k (B^k_l) we denote the restriction of the operator A (B_l) to $\mathfrak{X}^k$, $k = 0, 1$; $l = \overline{0, n-1}$.

Theorem 1.

Let the assumptions of Lemma 1 be satisfied. Then
(i) $A^k \in \mathcal{L}(\mathfrak{X}^k; \mathfrak{Y}^k)$, $k = 0, 1$;
(ii) $B^k_l \in \mathcal{L}(\mathfrak{X}^k; \mathfrak{Y}^k)$, $k = 0, 1$, $l = 0, 1, \dots, n-1$;
(iii) *there exists an operator* $(A^1)^{-1} \in \mathcal{L}(\mathfrak{Y}^1; \mathfrak{X}^1)$;
(iv) *there exists an operator* $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{Y}^0; \mathfrak{X}^0)$.

Let us construct the operators $H_0 = (B^0_0)^{-1} A^0$, $H_m = (B^0_0)^{-1} B^0_{n-m}$, $m = \overline{1, n-1}$, $S_m = (A^1)^{-1} B^1_m$, $m = \overline{0, n-1}$.

Definition 2.

Introduce the family of operators $\{K_q^1, K_q^2, \dots, K_q^n\}$ as follows:

$$K_0^s = \mathbb{O}, s \neq n, K_0^n = \mathbb{I},\ K_1^1 = H_0,\ K_1^2 = -H_{n-1}, \dots, K_1^s = -H_{n+1-s}, \dots, K_1^n = -H_1,$$
$$K_q^1 = K_{q-1}^n H_0,\ K_q^2 = K_{q-1}^1 - K_{q-1}^n H_{n-1}, \dots,\ K_q^s = K_{q-1}^{s-1} - K_{q-1}^n H_{n+1-s}, \dots,$$
$$K_q^n = K_{q-1}^{n-1} - K_{q-1}^n H_1, q = 2, 3, \dots$$

Definition 3.

The point ∞ is called:
(i) *a removable singular point of the* A-resolvent of pencil $\vec{B}$, if $K_1^1 = K_1^2 = \dots = K_1^n \equiv \mathbb{O}$;
(ii) *a pole* of order $p \in \mathbb{N}$ of the A-resolvent of pencil $\vec{B}$, if $K_p^s \not\equiv \mathbb{O}$, for some s, but $K_{p+1}^s \equiv \mathbb{O}$ for any $s = \overline{1, n}$;
(iii) *an essentially singular point* of the A-resolvent of pencil $\vec{B}$, if $K_p^n \not\equiv \mathbb{O}$ for any $p \in \mathbb{N}$.

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0 x. \qquad (5)$$

$$X_m^t = \frac{1}{2\pi i} \int\limits_{\gamma} R_\mu^A(\vec{B})(\mu^{n-m-1}A - \mu^{n-m-2}B_{n-1} - \ldots - B_{m+1})e^{\mu t} d\mu, \quad m = \overline{0, n-1}.$$

Lemma 2.

(i) *For every* $m = \overline{0, n-1}$ *the operator function* X_m^t *is a propagator of (5).*
(ii) *For every* $m = \overline{0, n-1}$ *the operator function* X_m^t *is an entire function.*
(iii)

$$\left. \frac{d^l}{dt^l} X_m^t \right|_{t=0} = \begin{cases} P, \ l = m; \\ \mathbb{O}, \ l \neq m; \end{cases} \quad \text{for all } m = \overline{0, n-1}, \ l = 0, 1, \ldots .$$

Definition 4.

The set $\mathcal{P} \subset \mathfrak{X}$ is called a *phase space* of equation (5), if
(i) every solution $x = x(t)$ of (5) lies in $\mathcal{P}$, i.e. $x(t) \in \mathcal{P} \quad \forall t \in \mathbb{R}$.
(ii) for arbitrary $x_m \in \mathcal{P}$, $m = \overline{0, n-1}$, there exists a unique solution to (2), (5).

Theorem 2.

If the pencil $\vec{B}$ is polynomially A-bounded, condition (A) is satisfied and ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of A-resolvent, then the phase space of equation (5) coincides with the image of projector P.

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0 x + y. \tag{6}$$

$$\mathcal{M}_y^m = \{x \in \mathfrak{X} : \ (\mathbb{I} - P)x = -\sum_{l=0}^{p} K_l^n (B_0^0)^{-1} \frac{d^{l+m}}{dt^{l+m}} (\mathbb{I} - Q)y(0)\}, \quad m = \overline{0, n-1}.$$

Theorem 3.

If the pencil $\vec{B}$ is polynomially A-bounded, condition (A) is satisfied and ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of the A-resolvent of pencil $\vec{B}$, the function $y : [0, \tau) \to \mathfrak{Y}$ is such that $y^0 = (\mathbb{I} - Q)y \in C^{p+n}([0, \tau); \mathfrak{Y}^0)$ and $y^1 = Qy \in C([0, \tau); \mathfrak{Y}^1)$, then for arbitrary $x_m \in \mathcal{M}_y^m$, $m = \overline{0, n-1}$ there exists a unique classical solution to problem (2), (6) for $t \in [0, \tau)$ given by

$$x(t) = -\sum_{q=0}^{p} K_q^n (B_0^0)^{-1} \frac{d^q}{dt^q} y^0(t) + \sum_{m=0}^{n-1} X_m^t P x_m + \int_0^t X_{n-1}^{t-s} (A^1)^{-1} y^1(s) ds. \tag{7}$$

STRONG SOLUTION

$$x^{(m)}(0) = x_m, \quad m = \overline{0, n-1}. \tag{2}$$

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0 x + y. \tag{6}$$

Definition 5.

The vector-function $x \in H^n(\mathfrak{X}) = \{x \in L_2(0,\tau;\mathfrak{X}) : \ x^{(n)} \in L_2(0,\tau;\mathfrak{X})\}$ is called a *strong solution* to (6), if it turns the equation to an identity almost everywhere on interval $(0,\tau)$. A strong solution $x = x(t)$ of (6) is called a strong solution to (2),(6), if condition (2) holds.

$$H^{p+n}(\mathfrak{Y}) = \{v \in L_2(0,\tau;\mathfrak{Y}) : v^{(p+n)} \in L_2(0,\tau;\mathfrak{Y}), p \in \{0\} \cup \mathbb{N}\}.$$

Theorem 4.

If the pencil $\vec{B}$ is polynomially A-bounded, condition (A) is satisfied, then for arbitrary $y \in H^{p+n}(\mathfrak{Y})$ and $x_m \in \mathcal{M}_y^m$, $m = \overline{0, n-1}$ there exists a unique strong solution to (2), (6).

Corollary 1.

If the pencil $\vec{B}$ is polynomially A-bounded, condition (A) is satisfied, then for arbitrary $x_m \in \mathfrak{X}$, $m = \overline{0, n-1}$ and $y \in H^{p+n}(\mathfrak{Y})$ there exists a unique strong solution to problem (3), (6).

$$\overset{\circ}{H}{}^{p+n}(\mathfrak{U}) = \{u \in L_2(0,\tau;\mathfrak{U}) : u^{(p+n)} \in L_2(0,\tau;\mathfrak{U}), u^{(q)}(0) = 0, q = \overline{0,p}\}, \quad p \in \{0\} \cup \mathbb{N}.$$

A closed and convex subset $\mathfrak{U}_{ad}$ of $\overset{\circ}{H}{}^{p+n}(\mathfrak{U})$ is called *a set of admissible controls.*

Definition 6.

The vector-function $\hat{u} \in \overset{\circ}{H}{}^{p+n}_{\partial}(\mathfrak{U})$ is called an *optimal control of solutions* to (1), (3), if relation (4) holds.

$$J(\hat{x}, \hat{u}) = \min_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x, u), \tag{4}$$

where

$$J(x, u) = \mu \sum_{q=0}^{n} \int_0^{\tau} ||x^{(q)} - \tilde{x}^{(q)}||^2 dt + \nu \sum_{q=0}^{p+n} \int_0^{\tau} \left\langle N_q u^{(q)}, u^{(q)} \right\rangle_{\mathfrak{U}} dt. \tag{8}$$

Here $\mu, \nu > 0$, $\mu + \nu = 1$, $N_q \in \mathcal{L}(\mathfrak{U})$, $q = 0, 1, \ldots, p + n$, are self-adjoint positively defined operators, and $\tilde{x}(t)$ is the target state of the system.

Theorem 5.

If the pencil $\vec{B}$ is polynomially A-bounded, condition (A) is satisfied, then for arbitrary $y \in H^{p+n}(\mathfrak{Y})$ and $x_m \in \mathcal{M}_y^m$, $m = \overline{0, n-1}$ there exists a unique optimal control of solutions to (1), (3).

$$(\lambda - \Delta)x_{tt} = \alpha(\Delta - \lambda')x_t + \beta(\Delta - \lambda'')x + u, \tag{9}$$

$$x(s,t) = 0, \quad (s,t) \in \partial\Omega \times \mathbb{R}. \tag{10}$$

$$x(s,0) = x_0(s),\ x_t(s,0) = x_1(s),\ s \in \Omega. \tag{11}$$

Put $\mathfrak{X} = \{x \in W_2^{l+2}(\Omega) : x(s) = 0, s \in \partial\Omega\}, \mathfrak{Y} = W_2^l(\Omega)$, $A = \lambda - \Delta$, $B_1 = \alpha(\Delta - \lambda')$, $B_0 = \beta(\Delta - \lambda'')$, $C = \mathbb{I}$.

Lemma 2. *Let one of the following conditions be fulfilled:*
(i) $\lambda \notin \sigma(\Delta)$;
(ii) $(\lambda \in \sigma(\Delta) \wedge (\lambda \neq \lambda')$;
(iii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$.
Then for all $\alpha, \beta \in \mathbb{R}\setminus\{0\}$ *the pencil* $\vec{B}$ *is polynomially* A*-bounded.*

$$(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda'' - \lambda_k) = 0,\ k \in \mathbb{N}. \tag{12}$$

Remark 1. If $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$ then condition (A) doesn't hold.
Remark 2. If $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$ then ∞ is a pole of order $p = 1$ of the A-resolvent of $\vec{B}$.

Corollary 2. *Let the vector-function* $u \in C^1((0,\tau);\mathfrak{F}) \cap C([0,\tau];\mathfrak{F})$ *and*

(i) $\lambda \notin \sigma(\Delta)$. *Then for arbitrary* $x_0, x_1 \in \mathfrak{U}$ *there exists a unique solution to (9) – (11), given by*

$$x(t) = \sum_{k=1}^{\infty} \left[\frac{\mu_k^1(\lambda-\lambda_k)+\alpha(\lambda'-\lambda_k)}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)} e^{\mu_k^1 t} + \frac{\mu_k^2(\lambda-\lambda_k)+\alpha(\lambda'-\lambda_k)}{(\lambda-\lambda_k)(\mu_k^2-\mu_k^1)} e^{\mu_k^2 t} \right] < \varphi_k, x_0 > \varphi_k +$$

$$\sum_{k=1}^{\infty} \frac{e^{\mu_k^1 t}-e^{\mu_k^2 t}}{(\mu_k^1-\mu_k^2)} < \varphi_k, x_1 > \varphi_k + \sum_{k=1}^{\infty} \int_0^t \frac{e^{\mu_k^1 (t-s)}-e^{\mu_k^2 (t-s)}}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)} < \varphi_k, u(s) > \varphi_k ds, \ t \in (0,\tau).$$

(ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$. *Then for arbitrary* $x_0, x_1 \in \mathfrak{X}$ *such that*

$$< \varphi_k, x_0 > + \frac{< \varphi_k, u(0) >}{\beta(\lambda''-\lambda_k)} = < \varphi_k, x_1 > + \frac{< \varphi_k, u'(0) >}{\beta(\lambda''-\lambda_k)} = 0, \ \ \lambda = \lambda_k$$

there exists a unique solution to (9) – (11), given by

$$x(t) = -\sum_{\lambda=\lambda_k} \frac{< \varphi_k, u(t) >}{\beta(\lambda''-\lambda_k)} \varphi_k - \sum_{\lambda=\lambda_k} \frac{< \varphi_k, u'(t) >}{\beta(\lambda''-\lambda_k)} \varphi_k + \sum{}' \left[\frac{\mu_k^1(\lambda-\lambda_k)+\alpha(\lambda'-\lambda_k)}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)} + \right.$$

$$\left. + \frac{\mu_k^2(\lambda-\lambda_k)+\alpha(\lambda'-\lambda_k)}{(\lambda-\lambda_k)(\mu_k^2-\mu_k^1)} e^{\mu_k^2 t} \right] < \varphi_k, x_0 > \varphi_k +$$

$$+ \sum{}' \frac{e^{\mu_k^1 t}-e^{\mu_k^2 t}}{(\mu_k^1-\mu_k^2)} < \varphi_k, x_1 > \varphi_k + \sum{}' \int_0^t \frac{e^{\mu_k^1 (t-s)}-e^{\mu_k^2 (t-s)}}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)} < \varphi_k, u(s) > \varphi_k ds, \ t \in (0,\tau),$$

where prime at the sum means the absence of summand with index k *such that* $\lambda = \lambda_k$.

Consider the penalty functional

$$J(x,u) = \mu \sum_{q=0}^{2} \int_0^{\tau} \|x^{(q)} - \tilde{x}^{(q)}\|^2 dt + \nu \sum_{q=0}^{p+2} \int_0^{\tau} \left\langle N_q u^{(q)}, u^{(q)} \right\rangle_{\mathfrak{U}} dt. \tag{13}$$

Corollary 2. *For all* $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ *and* $\lambda \in \mathbb{R}$ *such that* $\lambda \notin \sigma(\Delta)$ *or* $\lambda \in \sigma(\Delta) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$, *for arbitrary* $\tau \in \mathbb{R}_+, x_k, \in \mathfrak{X}^1, k = 0, 1,$ *there exists a unique solution* $(\hat{x}, \hat{u})$ *to optimal control problem for the Boussinesq – Lòve equation (9) with conditions (10), (11) minimizing functional (12).*

Thank you for your attention!