Optimal Boundary Control for Incompressible Nematic Liquid Crystal Flow in Two Dimensions

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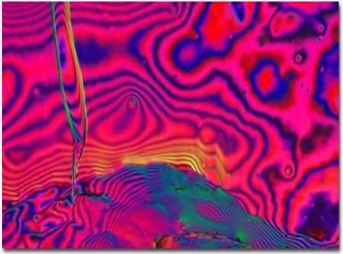
OCERTO 2016, Cortona, June 20–24 2016

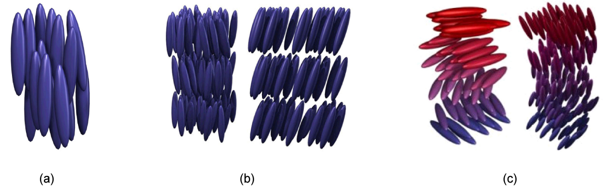
Outline

Liquid crystal



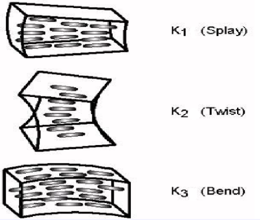
- Liquid crystal – a state of matter between conventional liquid and solid crystal. Liquid crystals may flow like a liquid, but its molecules may be oriented in a crystal-like way
- Special optical properties $\longrightarrow$ real applications (e.g., LCD)

Classification



- Nematic, Smectic, Cholesteric...
- Nematic LC: possesses the orientational order but has NO positional order like the Smectic type.

Static theory: Oseen–Frank model



- Oseen (1925), Frank (1958). Oseen-Frank free energy density:

$$
\begin{aligned}
F(\mathbf{d}) &= \frac{k_1}{2}(\mathrm{div}\mathbf{d})^2 + \frac{k_2}{2}(\mathbf{d}\cdot \mathrm{curl}\mathbf{d})^2 + \frac{k_3}{2}|\mathbf{d}\times \mathrm{curl}\,\mathbf{d}|^2 \\
&\quad + \frac{1}{2}(k_2+k_4)[\mathrm{tr}(\nabla \mathbf{d})^2 - (\mathrm{div}\mathbf{d})^2].
\end{aligned}
$$

- Simplest case $k_1 = k_2 = k_3 = 1,\ k_4 = 0$:

Dirichlet energy $\quad F(\mathbf{d}) = \frac{1}{2}\int_\Omega |\nabla \mathbf{d}|^2 dx:\quad$ for $\mathbf{d}:\Omega \to S^{n-1}$.

Hydrodynamic theory: Ericksen–Leslie model

- Ericksen (1961) and Leslie (1966, 1968).
- Assume the fluid is homogeneous and neglect the inertial term

$$
\begin{aligned}
\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla P &= \nabla\cdot\hat{\sigma},\\
\nabla\cdot\mathbf{v} &= 0,\\
\mathbf{d}\times\left(g + \nabla\cdot\left(\frac{\partial F}{\partial\nabla\mathbf{d}}\right) - \frac{\partial F}{\partial\mathbf{d}}\right) &= \mathbf{0},
\end{aligned}
$$

$\mathbf{v}$ – fluid velocity,
$\mathbf{d}$ – molecular director,
F – Oseen-Frank free energy density,
P – pressure

Constitutive relations

$$
\begin{aligned}
A &= \frac{1}{2}(\nabla \mathbf{v} + \nabla^{\mathrm{tr}} \mathbf{v}) - \text{rate of strain: } \textit{stretching} \\
R &= \frac{1}{2}(\nabla \mathbf{v} - \nabla^{\mathrm{tr}} \mathbf{v}) - \text{skew-symmetric tensor: } \textit{rigid rotation} \\
N &= \mathbf{d}_t + \mathbf{v} \cdot \nabla \mathbf{d} - R\mathbf{d} \\
g_i &= \lambda_1 N_i + \lambda_2 d_j A_{ji}, \quad \text{total kinematic transport of } \mathbf{d} \\
\hat{\sigma}_{ij} &= -P\delta_{ij} - \frac{\partial F}{\partial d_{k,j}} d_{k,i} + \sigma_{ij} \\
\sigma_{ij} &= \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 d_j N_i + \mu_3 d_i N_j + \mu_4 A_{ij} + \mu_5 d_j d_k A_{ki} + \mu_6 d_i d_k A_{kj} \\
& \mu_1, \ldots, \mu_6 - \text{Leslie coefficients}
\end{aligned}
$$

Penalization

- Constraint on director field $|\mathbf{d}| = 1$
  $\Longrightarrow$ highly nonlinear Lagrange multiplier

  Example: assume $k_1 = k_2 = k_3 = 1,\ k_4 = 0$

$$\gamma = -|\nabla \mathbf{d}|^2 + \lambda_2 \mathbf{d}^T A \mathbf{d}.$$

- Penalty function

$$\frac{1}{4\epsilon^2}(|\mathbf{d}|^2 - 1)^2$$

  $\Longrightarrow$ Ginzburg-Landau approximate energy

$$F(\mathbf{d}) = \frac{1}{2}|\nabla \mathbf{d}|^2 + \frac{1}{4\epsilon^2}(|\mathbf{d}|^2 - 1)^2 \quad \text{for } \mathbf{d} : \Omega \to \mathbb{R}^n.$$

Outline

Related Results (I)

(1) Simplified system with penalty (Lin F.-H. 1989)

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \nabla P = -\lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{v} = 0, \\ \mathbf{d}_t + \mathbf{v} \cdot \nabla \mathbf{d} = \eta(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})), \quad \mathbf{f}(\mathbf{d}) = \nabla_{\mathbf{d}} F(\mathbf{d}). \end{cases}$$

$\nabla \mathbf{d} \odot \nabla \mathbf{d}$ denotes the $n \times n$ matrix whose (i,j)-th entry is given by $\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d}$, for $1 \leq i,j \leq n$.

- Well-posedness $\&$ regularity:
  Lin-Liu 1995 & 1996, Dai-Qing 2012
- Long-time behavior:
  Wu 2010, Hu-Wu 2013, Dai-Schonbek 2012 & 2013
- Time-dependent Dirichlet BC for $\mathbf{d}$:
  Climent-Ezquerra et al 2009, Bosia 2012, Grasselli-Wu 2013

Related Results (II)

(1a) Simplified model with constraint $|\mathbf{d}| = 1$

$$\mathbf{d}_t + \mathbf{v} \cdot \nabla \mathbf{d} = \eta(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d})$$

(2) General incompressible E–L system

(3) LC models with general Oseen-Frank energy

(4) Compressible LC models

(5) Nonisothermal LC models

Outline

The state equation

- $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ
- $\mathbf{v} = (v_1, v_2)^{\text{tr}} : \Omega \to \mathbb{R}^2$, $\mathbf{d} = (d_1, d_2)^{\text{tr}} : \Omega \to \mathbb{R}^2$

The PDE system in $\Omega \times \mathbb{R}^+$

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \Delta \mathbf{v} + \nabla P = -\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{v} = 0, \\ \mathbf{d}_t + \mathbf{v} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}). \end{cases} \tag{1}$$

subject to Dirichlet boundary conditions

$$\begin{cases} \mathbf{v}(x,t) = \mathbf{0}, & (x,t) \in \Gamma \times \mathbb{R}^+, & \text{no-slip b.c.} \\ \mathbf{d}(x,t) = \mathbf{h}(x,t), & (x,t) \in \Gamma \times \mathbb{R}^+, & \text{strong anchoring b.c.} \end{cases} \tag{2}$$

and initial conditions

$$\mathbf{v}|_{t=0} = \mathbf{v}_0(x) \ \text{ with } \nabla \cdot \mathbf{v}_0 = 0, \quad \mathbf{d}|_{t=0} = \mathbf{d}_0(x), \qquad x \in \Omega. \tag{3}$$

Basic settings

- Function spaces

$$\mathbf{H} = \overline{\mathcal{V}}^{\mathbf{L}^2(\Omega)}, \quad \mathbf{V} = \overline{\mathcal{V}}^{\mathbf{H}_0^1(\Omega)}, \quad \mathcal{V} = \{\mathbf{v} \in C_0^\infty(\Omega, \mathbb{R}^n) : \nabla \cdot \mathbf{v} = 0\}$$

- $T \in (0, +\infty)$ is a given finite final time.

$$Q := \Omega \times (0, T), \quad \Sigma := \Gamma \times (0, T).$$

- $\beta_i \geq 0$ $(i = 1, 2, 3, 4)$ and $\gamma \geq 0$ are given constants that do not vanish simultaneously.
- Target functions

$$\mathbf{v}_Q \in L^2(0, T; \mathbf{H}), \quad \mathbf{d}_Q \in \mathbf{L}^2(Q), \quad \mathbf{v}_\Omega \in \mathbf{H}, \quad \mathbf{d}_\Omega \in \mathbf{L}^2(\Omega)$$

The optimal boundary control problem

(CP) Minimize the tracking type cost functional:

$$\begin{aligned}\mathcal{J}((\mathbf{v},\mathbf{d}),\mathbf{h}) \quad := \quad & \frac{\beta_1}{2}\|\mathbf{v}-\mathbf{v}_Q\|^2_{\mathbf{L}^2(Q)} + \frac{\beta_2}{2}\|\mathbf{d}-\mathbf{d}_Q\|^2_{\mathbf{L}^2(Q)} \\ & + \frac{\beta_3}{2}\|\mathbf{v}(T)-\mathbf{v}_\Omega\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta_4}{2}\|\mathbf{d}(T)-\mathbf{d}_\Omega\|^2_{\mathbf{L}^2(\Omega)} \\ & + \frac{\gamma}{2}\|\mathbf{h}\|^2_{\mathbf{L}^2(\Sigma)},\end{aligned}$$

subject to the boundary control constraint $\mathbf{h}(x,t)$
and the state constraints $(\mathbf{v},\mathbf{d})$ due to the state problem (1)–(3).

- Joint work with C. Cavaterra (Milan) and E. Rocca (Pavia)

Outline

Global strong solutions

Theorem (Cavaterra-Rocca-Wu, 2016)

Let $n = 2$*. For any* $T > 0$*, assume*

$$\mathbf{h} \in L^2(0, T; \mathbf{H}^{\frac{5}{2}}(\Gamma)), \quad \partial_t \mathbf{h} \in L^4(0, T; \mathbf{H}^{\frac{1}{2}}(\Gamma)).$$

For any $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$ *satisfying* $\mathbf{d}_0|_\Gamma = \mathbf{h}|_{t=0}$*,*
problem (1)–(3) admits a ***unique global strong solution*** $(\mathbf{v}, \mathbf{d})$ *such that*

$$\mathbf{v} \in C([0, T]; \mathbf{V}) \cap L^2(0, T; \mathbf{H}^2(\Omega)),$$
$$\mathbf{d} \in C([0, T]; \mathbf{H}^2(\Omega)) \cap L^2(0, T; \mathbf{H}^3(\Omega)).$$

Moreover, for $t \in [0, T]$*,*

$$\|\mathbf{v}(t)\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{d}(t)\|_{\mathbf{H}^2(\Omega)} + \int_0^t \Big(\|\mathbf{v}(\tau)\|^2_{\mathbf{H}^2(\Omega)} + \|\mathbf{d}(\tau)\|^2_{\mathbf{H}^3(\Omega)}\Big) d\tau \le C_T,$$

where C_T *is a positive constant depending on* $\|\mathbf{v}_0\|_{\mathbf{H}^1}$*,* $\|\mathbf{d}_0\|_{\mathbf{H}^2}$*,* $\|\mathbf{h}\|_{L^2(0,T;\mathbf{H}^{\frac{5}{2}}(\Gamma))}$*,*
$\|\partial_t \mathbf{h}\|_{L^4(0,T;\mathbf{H}^{\frac{1}{2}}(\Gamma))}$*,* Ω*, and* T*.*

Difficulties

(1) **Time-dependent boundary condition $\mathbf{h}(x,t)$:**
The total energy

$$\mathcal{E}(t) = \frac{1}{2}\|\mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\nabla \mathbf{d}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega F(\mathbf{d}(t))dx$$

DOES NOT decrease in time as in the autonomous case (different from Lin-Liu 1995).

(2) **No additional state constraint** on $|\mathbf{d}|_{\mathbb{R}^2}$:
We DO NOT make use of the weak maximum principle for $|\mathbf{d}|_{\mathbb{R}^2}$ (different from Climent-Ezquerra et al 2009, Bosia 2012, Grasselli-Wu 2013).

Idea: Suitable “lifting” functions + Some new estimates

Lifting functions

- $\mathbf{d}_E(x,t)$: **elliptic type** (Climent-Ezquerra et al 2009)

$$\begin{cases} -\Delta \mathbf{d}_E = \mathbf{0}, & \text{in } \Omega \times (0,T), \\ \mathbf{d}_E = \mathbf{h}(x,t), & \text{on } \Gamma \times (0,T). \end{cases}$$

- $\mathbf{d}_P(x,t)$: **parabolic type** (Grasselli-Wu 2013)

$$\begin{cases} \partial_t \mathbf{d}_P - \Delta \mathbf{d}_P = \mathbf{0}, & \text{in } \Omega \times (0,T), \\ \mathbf{d}_P = \mathbf{h}(x,t), & \text{on } \Gamma \times (0,T), \\ \mathbf{d}_P|_{t=0} = \mathbf{d}_{E0}, & \text{in } \Omega, \end{cases}$$

- $\mathbf{d}_{E0}$: lifting of initial datum $\mathbf{d}_0$ (Grasselli-Wu 2013)

$$\begin{cases} -\Delta \mathbf{d}_{E0} = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{d}_{E0} = \mathbf{d}_0, & \text{on } \Gamma. \end{cases}$$

Lifted system (I)

- Variable transformation

$$\widehat{\mathbf{d}} = \mathbf{d} - \mathbf{d}_E.$$

- New system for $(\mathbf{v}, \widehat{\mathbf{d}})$:

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \Delta \mathbf{v} + \nabla P = -(\nabla \mathbf{d})^{\mathrm{tr}} \Delta \widehat{\mathbf{d}}, \\ \nabla \cdot \mathbf{v} = 0, \\ \widehat{\mathbf{d}}_t + \mathbf{v} \cdot \nabla \mathbf{d} = \Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d}) - \partial_t \mathbf{d}_E(t), \end{cases}$$

subject to **homogeneous** Dirichlet boundary conditions and initial conditions

$$\begin{aligned} &\mathbf{v} = \mathbf{0}, \quad \widehat{\mathbf{d}} = \mathbf{0}, && \text{on } \Gamma \times (0, T), \\ &\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \widehat{\mathbf{d}}|_{t=0} = \mathbf{d}_0 - \mathbf{d}_{E0}, && \text{in } \Omega. \end{aligned}$$

Lower-order estimate in $\mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Omega)$

- Denote

$$\begin{aligned}
\widehat{\mathcal{E}}(t) &= \frac{1}{2}\|\mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\nabla\widehat{\mathbf{d}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega F(\mathbf{d}(t))dx, \\
\mathcal{D}_E(t) &= \|\nabla\mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\Delta\widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))\|^2_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Lemma

Let $(\mathbf{v}, \mathbf{d})$ be a smooth solution to problem (1)–(3) *on $[0, T]$. The following inequality holds for $t \in [0, T]$:*

$$\frac{d}{dt}\widehat{\mathcal{E}}(t) + \mathcal{D}_E(t) \le \frac{3}{2}\|\partial_t\mathbf{d}_E(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{4}\|\partial_t\mathbf{d}_E(t)\|^4_{\mathbf{L}^4(\Omega)} + \frac{13}{3}\int_\Omega F(\mathbf{d}(t))dx + \frac{1}{4}|\Omega|.$$

Lifted system (II)

- Variable transformation

$$\widetilde{\mathbf{d}} = \mathbf{d} - \mathbf{d}_P.$$

- New system for $(\mathbf{v}, \widetilde{\mathbf{d}})$:

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} - \Delta\mathbf{v} + \nabla P = -(\nabla\mathbf{d})^{\mathrm{tr}}\Delta\mathbf{d}, \\ \nabla\cdot\mathbf{v} = 0, \\ \widetilde{\mathbf{d}}_t + \mathbf{v}\cdot\nabla\mathbf{d} = \Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}), \end{cases}$$

subject to **homogeneous** Dirichlet boundary conditions and initial conditions

$$\begin{aligned} &\mathbf{v} = \mathbf{0}, \quad \widetilde{\mathbf{d}} = \mathbf{0}, && \text{on } \Gamma\times(0,T), \\ &\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \widetilde{\mathbf{d}}|_{t=0} = \mathbf{d}_0 - \mathbf{d}_{E0}, && \text{in } \Omega. \end{aligned}$$

Higher-order estimate in $\mathbf{H}^1(\Omega) \times \mathbf{H}^2(\Omega)$

- Denote

$$\begin{aligned}
\mathcal{A}_P(t) &= \|\nabla \mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\Delta \widetilde{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))\|^2_{\mathbf{L}^2(\Omega)}, \\
\mathcal{B}_P(t) &= \|S\mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla(\Delta \widetilde{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t)))\|^2_{\mathbf{L}^2(\Omega)}, \\
R(t) &= \|\mathbf{d}_P(t)\|^4_{\mathbf{H}^2(\Omega)} + \|\mathbf{d}_P(t)\|^2_{\mathbf{H}^3(\Omega)} + 1.
\end{aligned}$$

Lemma

Let $n = 2$. If $(\mathbf{v}, \mathbf{d})$ is a smooth solution to problem (1)–(3), *then*

$$\frac{d}{dt}\mathcal{A}_P(t) + \mathcal{B}_P(t) \le C_T\Big(\mathcal{A}_P^2(t) + R(t)\Big),$$

where C_T is a positive constant depending on $\|\mathbf{v}_0\|_{\mathbf{L}^2}$, $\|\mathbf{d}_0\|_{\mathbf{H}^1}$, $\|\mathbf{h}\|_{L^2(0,T;\mathbf{H}^{\frac{5}{2}}(\Gamma))}$, $\|\partial_t \mathbf{h}\|_{L^4(0,T;\mathbf{H}^{\frac{1}{2}}(\Gamma))}$, Ω and T.

Existence and continuous dependence

- Existence: semi-Galerkin scheme + *a priori* estimates
- Continuous dependence in $\mathbf{H}^1(\Omega) \times \mathbf{H}^2(\Omega)$:

Lemma

Let $(\mathbf{v}^{(i)}, \mathbf{d}^{(i)})$ $(i = 1, 2)$ *be two strong solutions corresponding to the initial data* $(\mathbf{v}_0^{(i)}, \mathbf{d}_0^{(i)})$ *and the boundary data* $\mathbf{h}^{(i)}$ *on* $[0, T]$*. Denote* $\bar{\mathbf{v}} = \mathbf{v}^{(1)} - \mathbf{v}^{(2)}$, $\bar{\mathbf{d}} = \mathbf{d}^{(1)} - \mathbf{d}^{(2)}$, $\bar{\mathbf{v}}_0 = \mathbf{v}_0^{(1)} - \mathbf{v}_0^{(2)}$, $\bar{\mathbf{d}}_0 = \mathbf{d}_0^{(1)} - \mathbf{d}_0^{(2)}$, $\bar{\mathbf{h}} = \mathbf{h}^{(1)} - \mathbf{h}^{(2)}$*, then for* $t \in [0, T]$

$$\|\bar{\mathbf{v}}(t)\|^2_{\mathbf{H}^1(\Omega)} + \|\bar{\mathbf{d}}(t)\|^2_{\mathbf{H}^2(\Omega)} + \int_0^t (\|\bar{\mathbf{v}}(\tau)\|^2_{\mathbf{H}^2(\Omega)} + \|\bar{\mathbf{d}}(\tau)\|^2_{\mathbf{H}^3(\Omega)}) d\tau$$

$$\leq C_T \left[\|\bar{\mathbf{v}}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\bar{\mathbf{d}}_0\|^2_{\mathbf{H}^2(\Omega)} + \int_0^t \left(\|\bar{\mathbf{h}}(\tau)\|^2_{\mathbf{H}^{\frac{5}{2}}(\Gamma)} + \|\partial_t \bar{\mathbf{h}}(\tau)\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \right) d\tau \right],$$

where $C_T > 0$ *is a constant depending on* $\|\mathbf{v}_0^{(i)}\|_{\mathbf{H}^1(\Omega)}$, $\|\mathbf{d}_0^{(i)}\|_{\mathbf{H}^2(\Omega)}$, $\|\mathbf{h}^{(i)}\|_{L^2(0,T;\mathbf{H}^{\frac{5}{2}}(\Gamma))}$, $\|\partial_t \mathbf{h}^{(i)}\|_{L^4(0,T;\mathbf{H}^{\frac{1}{2}}(\Gamma))}$, Ω *and* T*.*

Outline

Main results

(1) Existence of an optimal boundary control

(2) First-order necessary optimality condition

Admissible set of controls

- Let $T \in (0, +\infty)$ and $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$ be given.
- Denote

$$\mathcal{U} := \left\{ \mathbf{h} \mid \mathbf{h} \in L^2(0, T; \mathbf{H}^{\frac{5}{2}}(\Gamma)),\ \partial_t \mathbf{h} \in L^4(0, T; \mathbf{H}^{\frac{1}{2}}(\Gamma)) \right\},$$

and the *affine space of control functions* $\widetilde{\mathcal{U}}$ (associated with $(\mathbf{v}_0, \mathbf{d}_0)$)

$$\widetilde{\mathcal{U}} := \left\{ \mathbf{h} \mid \mathbf{h} \in \mathcal{U} \text{ with } \mathbf{h}(x, t)|_{t=0} = \mathbf{d}_0(x)|_\Gamma \right\}.$$

- **Admissible set of control functions**:

$$\widetilde{\mathcal{U}}_{\mathrm{ad}}^M := \left\{ \mathbf{h} \in \widetilde{\mathcal{U}} \mid \|\mathbf{h}\|_{L^2(0,T;\mathbf{H}^{\frac{5}{2}}(\Gamma))} \leq M, \quad \|\partial_t \mathbf{h}\|_{L^4(0,T;\mathbf{H}^{\frac{1}{2}}(\Gamma))} \leq M \right\}$$

for suitably large $M > 0$.

The control-to-state operator $\mathcal{S}$

Definition

$$\mathcal{H} := \left[C([0,T];\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{H})\right]$$
$$\times \left[C([0,T];\mathbf{H}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^3(\Omega)) \cap H^1(0,T;\mathbf{H}^1(\Omega))\right]$$

The *control-to-state* mapping $\mathcal{S}$ (associated with $(\mathbf{v}_0, \mathbf{d}_0)$):

$$\mathcal{S} : \widetilde{\mathcal{U}} \to \mathcal{H}, \quad \mathbf{h} \in \widetilde{\mathcal{U}} \mapsto \mathcal{S}(\mathbf{h}) := (\mathbf{v}, \mathbf{d}) \in \mathcal{H},$$

where $(\mathbf{v}, \mathbf{d})$ is the global strong solution to problem (1)–(3) on $[0, T]$ subject to $(\mathbf{v}_0, \mathbf{d}_0)$.

Lemma

The control-to-state mapping $\mathcal{S}$ is Lipschitz continuous *from $\widetilde{\mathcal{U}}$ into the space*

$$\mathcal{W} := \left[C([0,T];\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega))\right] \times \left[C([0,T];\mathbf{H}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^3(\Omega))\right].$$

Existence

Theorem (Cavaterra-Rocca-Wu 2016)

Suppose that the assumption (A1) is satisfied and $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$*. The optimal control problem (**CP**) admits at least one solution* $((\mathbf{v}^\sharp, \mathbf{d}^\sharp), \mathbf{h}^\sharp)$ *such that*

$$\mathbf{h}^\sharp \in \widetilde{\mathcal{U}}_{\mathrm{ad}}^{M}$$

and $(\mathbf{v}^\sharp, \mathbf{d}^\sharp) = \mathcal{S}(\mathbf{h}^\sharp)$ *is the unique global strong solution to problem* (1)–(3) *subject to the initial data* $(\mathbf{v}_0, \mathbf{d}_0)$*.*

Proof: finding a minimizing sequence + passing to the limit

Differentiability of $\mathcal{S}$ (I)

Definition

Consider the control-to-state operator $\mathcal{S}$ associated with a given initial data $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$. Let

$$\mathcal{W}_1 := \left[C([0,T];\mathbf{H}) \cap L^2(0,T;\mathbf{V})\right] \times \left[C([0,T];\mathbf{H}^1(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega))\right]$$

We say that $\mathcal{S} : \widetilde{\mathcal{U}} \to \mathcal{H} \subset \mathcal{W}_1$ is **Fréchet differentiable**, if for any $\mathbf{h}^* \in \widetilde{\mathcal{U}}$, there exists a linear operator denoted by $\mathcal{S}'(\mathbf{h}^*) : \widetilde{\mathcal{U}} - \{\mathbf{h}^*\} \to \mathcal{W}_1$ such that

$$\lim_{\|\boldsymbol{\xi}\|_{\mathcal{U}} \to 0} \frac{\|\mathcal{S}(\mathbf{h}^* + \boldsymbol{\xi}) - \mathcal{S}(\mathbf{h}^*) - \mathcal{S}'(\mathbf{h}^*)(\boldsymbol{\xi})\|_{\mathcal{W}_1}}{\|\boldsymbol{\xi}\|_{\mathcal{U}}} = 0,$$

where $\boldsymbol{\xi} \in \widetilde{\mathcal{U}} - \{\mathbf{h}^*\}$ is an arbitrary (small) perturbation of $\mathbf{h}^*$.

Key: the linearized system

Lemma

Let $\mathbf{h}^* \in \widetilde{\mathcal{U}}$ *be given and* $(\mathbf{v}^*, \mathbf{d}^*) = \mathcal{S}(\mathbf{h}^*)$*. For any* $\boldsymbol{\xi} \in \widetilde{\mathcal{U}} - \{\mathbf{h}^*\}$*, the* ***linearized problem***

$$
\begin{aligned}
&\partial_t \boldsymbol{\omega} - \Delta \boldsymbol{\omega} + \nabla \hat{P} + (\mathbf{v}^* \cdot \nabla)\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}^* \\
&\qquad = -\nabla \cdot (\nabla \boldsymbol{\phi} \odot \nabla \mathbf{d}^*) - \nabla \cdot (\nabla \mathbf{d}^* \odot \nabla \boldsymbol{\phi}), && \text{in } \Omega \times (0,T), \\
&\nabla \cdot \boldsymbol{\omega} = 0, && \text{in } \Omega \times (0,T), \\
&\partial_t \boldsymbol{\phi} - \Delta \boldsymbol{\phi} + (\mathbf{v}^* \cdot \nabla)\boldsymbol{\phi} + (\boldsymbol{\omega} \cdot \nabla)\mathbf{d}^* = -\mathbf{f}'(\mathbf{d}^*)\boldsymbol{\phi}, && \text{in } \Omega \times (0,T), \\
&\qquad \text{with } \mathbf{f}'(\mathbf{d}^*)\boldsymbol{\phi} := 2(\mathbf{d}^* \cdot \boldsymbol{\phi})\mathbf{d}^* + |\mathbf{d}^*|^2 \boldsymbol{\phi} - \boldsymbol{\phi}, \\
&\boldsymbol{\omega} = \mathbf{0}, \quad \boldsymbol{\phi} = \boldsymbol{\xi}, && \text{on } \Gamma \times (0,T), \\
&\boldsymbol{\omega}|_{t=0} = \mathbf{0}, \quad \boldsymbol{\phi}|_{t=0} = \mathbf{0}, && \text{in } \Omega.
\end{aligned}
$$

admits a unique ***weak solution*** $(\boldsymbol{\omega}, \boldsymbol{\phi})$ *such that*

$$
\begin{aligned}
&\boldsymbol{\omega} \in C([0,T]; \mathbf{H}) \cap L^2(0,T; \mathbf{V}) \cap H^1(0,T; \mathbf{V}'), \\
&\boldsymbol{\phi} \in C([0,T]; \mathbf{H}^1(\Omega)) \cap L^2(0,T; \mathbf{H}^2(\Omega)) \cap H^1(0,T; \mathbf{L}^2(\Omega)).
\end{aligned}
$$

Differentiability of $\mathcal{S}$ (II)

Theorem (Cavaterra-Rocca-Wu 2016)

Given $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$, the corresponding state-to-control operator $\mathcal{S}$ is Fréchet differentiable.
For any $\mathbf{h}^* \in \widetilde{\mathcal{U}}$, the Fréchet derivative $\mathcal{S}'(\mathbf{h}^*)$ is given by

$$\mathcal{S}'(\mathbf{h}^*)\boldsymbol{\xi} = (\boldsymbol{\omega}, \boldsymbol{\phi}), \quad \forall \boldsymbol{\xi} \in \widetilde{\mathcal{U}} - \{\mathbf{h}^*\},$$

where $(\boldsymbol{\omega}, \boldsymbol{\phi})$ is the unique global weak solution to the linearized problem.

Proof: Consider

$$\mathbf{h} = \mathbf{h}^* + \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \widetilde{\mathcal{U}} - \{\mathbf{h}^*\}, \quad (\mathbf{v}, \mathbf{d}) = \mathcal{S}(\mathbf{h}), \quad (\mathbf{v}^*, \mathbf{d}^*) = \mathcal{S}(\mathbf{h}^*).$$

For the difference

$$(\mathbf{w}, \mathbf{e}) := (\mathbf{v} - \mathbf{v}^* - \boldsymbol{\omega},\ \mathbf{d} - \mathbf{d}^* - \boldsymbol{\phi}),$$

we obtain

$$\|\mathbf{w}\|_{C([0,T];\mathbf{L}^2)} + \|\mathbf{e}\|_{C([0,T];\mathbf{H}^1)} + \|\mathbf{w}\|_{L^2(0,T;\mathbf{H}^1)} + \|\mathbf{e}\|_{L^2(0,T;\mathbf{H}^2)} \le C_T \|\boldsymbol{\xi}\|_{\mathcal{U}}^2.$$

The first-order necessary optimality condition (I)

Theorem (Cavaterra-Rocca-Wu 2016)

Given $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$. *Assume that* $\mathbf{h}^\sharp$ *is an optimal control for problem (**CP**) in the admissible set* $\widetilde{\mathcal{U}}_{\mathrm{ad}}^M$ *with the associate state* $(\mathbf{v}^\sharp, \mathbf{d}^\sharp) = \mathcal{S}(\mathbf{h}^\sharp)$. *Then for any* $\mathbf{h} \in \widetilde{\mathcal{U}}_{\mathrm{ad}}^M$, *denoting by* $(\boldsymbol{\omega}, \boldsymbol{\phi})$ *the unique global weak solution to the linearized problem corresponding to* $\boldsymbol{\xi} = \mathbf{h} - \mathbf{h}^\sharp$, *we have*

$$
\begin{aligned}
&\beta_1 \int_0^T \int_\Omega (\mathbf{v}^\sharp - \mathbf{v}_Q) \cdot \boldsymbol{\omega}\, dxdt + \beta_2 \int_0^T \int_\Omega (\mathbf{d}^\sharp - \mathbf{d}_Q) \cdot \boldsymbol{\phi}\, dxdt \\
&\quad + \beta_3 \int_\Omega (\mathbf{v}^\sharp(T) - \mathbf{v}_\Omega) \cdot \boldsymbol{\omega}(T) dx + \beta_4 \int_\Omega (\mathbf{d}^\sharp(T) - \mathbf{d}_\Omega) \cdot \boldsymbol{\phi}(T) dx \\
&\quad + \gamma \int_0^T \int_\Gamma \mathbf{h}^\sharp \cdot (\mathbf{h} - \mathbf{h}^\sharp) dSdt \geq 0, \quad \forall\, \mathbf{h} \in \widetilde{\mathcal{U}}_{\mathrm{ad}}^M.
\end{aligned}
$$

The adjoint system and adjoint state (I)

- Let $\mathbf{h}^\sharp$ be an optimal control for problem (**CP**) in the admissible set $\widetilde{\mathcal{U}}_{\mathrm{ad}}^M$ with the associate state $(\mathbf{v}^\sharp, \mathbf{d}^\sharp) = \mathcal{S}(\mathbf{h}^\sharp)$.
- The associated **adjoint state** $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ satisfies

$$\partial_t \tilde{\mathbf{p}} + \Delta \tilde{\mathbf{p}} + \nabla \tilde{P} + (\mathbf{v}^\sharp \cdot \nabla)\tilde{\mathbf{p}} - (\tilde{\mathbf{p}} \cdot \nabla^{\mathrm{tr}})\mathbf{v}^\sharp - (\tilde{\mathbf{q}} \cdot \nabla^{\mathrm{tr}})\mathbf{d}^\sharp + \beta_1(\mathbf{v}^\sharp - \mathbf{v}_Q) = \mathbf{0},$$
$$\nabla \cdot \tilde{\mathbf{p}} = 0,$$
$$\partial_t \tilde{\mathbf{q}} + \Delta \tilde{\mathbf{q}} + (\mathbf{v}^\sharp \cdot \nabla)\tilde{\mathbf{q}} - \mathbf{f}'(\mathbf{d}^\sharp)\tilde{\mathbf{q}} - \tilde{\mathbf{r}}(\mathbf{d}^\sharp, \tilde{\mathbf{p}}) + \beta_2(\mathbf{d}^\sharp - \mathbf{d}_Q) = \mathbf{0},$$

where

$$\tilde{\mathbf{r}}(\mathbf{d}^\sharp, \tilde{\mathbf{p}}) = \nabla \cdot [\nabla^{\mathrm{tr}}\mathbf{d}^\sharp \odot (\nabla \tilde{\mathbf{p}} + \nabla^{\mathrm{tr}}\tilde{\mathbf{p}})],$$

and on the boundary Γ and at time $t = T$,

$$\tilde{\mathbf{p}}|_\Gamma = \mathbf{0}, \quad \tilde{\mathbf{q}}|_\Gamma = \mathbf{0},$$
$$\tilde{\mathbf{p}}|_{t=T} = \beta_3(\mathbf{v}^\sharp(T) - \mathbf{v}_\Omega), \quad \tilde{\mathbf{q}}|_{t=T} = \beta_4(\mathbf{d}^\sharp(T) - \mathbf{d}_\Omega).$$

The adjoint system and adjoint state (II)

Lemma

Assume

$$\begin{aligned} &\mathbf{v}_\Omega \in \mathbf{V}, \quad \text{if } \beta_3 > 0, \\ &\mathbf{d}_\Omega \in \mathbf{H}^1(\Omega) \quad \textit{with } (\mathbf{d}^\sharp(T) - \mathbf{d}_\Omega)|_\Gamma = \mathbf{0}, \quad \text{if } \beta_4 > 0. \end{aligned}$$

Then the adjoint problem admits a unique weak solution $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ *such that*

$$\begin{aligned} &\tilde{\mathbf{p}} \in C([0,T];\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega) \cap \mathbf{V}) \cap H^1(0,T;\mathbf{H}), \\ &\tilde{\mathbf{q}} \in C([0,T];\mathbf{H}^s(\Omega)) \cap L^2(0,T;\mathbf{H}^{1+s}(\Omega)) \cap H^1(0,T;\mathbf{H}^{s-1}(\Omega)), \end{aligned}$$

for some $s \in (\frac{1}{2}, 1)$.

Proof: change of variable

$$\mathbf{p}(t) = \tilde{\mathbf{p}}(T-t), \quad \mathbf{q}(t) = \tilde{\mathbf{q}}(T-t).$$

The first-order necessary optimality condition (II)

Theorem (Cavaterra-Rocca-Wu 2016)

Assume

$$\mathbf{v}_\Omega \in \mathbf{V}, \quad \text{if } \beta_3 > 0,$$
$$\mathbf{d}_\Omega \in \mathbf{H}^1(\Omega) \quad \textit{with } (\mathbf{d}^\sharp(T) - \mathbf{d}_\Omega)|_\Gamma = \mathbf{0}, \quad \text{if } \beta_4 > 0.$$

Suppose that $\mathbf{h}^\sharp$ *is an optimal control for problem (**CP**) in* $\widetilde{\mathcal{U}}_{\mathrm{ad}}^M$ *with the associate state* $(\mathbf{v}^\sharp, \mathbf{d}^\sharp) = \mathcal{S}(\mathbf{h}^\sharp)$ *and* $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ *is the adjoint state. Then*

$$\begin{aligned} &\int_0^T \int_\Gamma (\nabla \tilde{\mathbf{q}} \cdot \mathbf{n}) \cdot (\mathbf{h} - \mathbf{h}^\sharp) dSdt \\ &\quad - \int_0^T \int_\Gamma (\nabla \cdot [\nabla^{\mathrm{tr}} \mathbf{d}^\sharp \odot (\nabla \tilde{\mathbf{p}} + \nabla^{\mathrm{tr}} \tilde{\mathbf{p}})] \cdot \mathbf{n}) \cdot (\mathbf{h} - \mathbf{h}^\sharp) dSdt \\ &\quad + \gamma \int_0^T \int_\Gamma \mathbf{h}^\sharp \cdot (\mathbf{h} - \mathbf{h}^\sharp) dSdt \geq 0, \quad \forall\, \mathbf{h} \in \widetilde{\mathcal{U}}_{\mathrm{ad}}^M. \end{aligned}$$

Open problems

- Optimal boundary control under additional state constraint $|\mathbf{d}| = 1$
- Optimal boundary control under additional state constraint $|\mathbf{d}| = 1$
- General E-L system with rotation/stretching effects

The End

Thank You !