Optimal control of a Cahn–Hilliard/Navier–Stokes with variable densities

Donat Wegner (joint work with M. Hintermüller, T. Keil)

24.06.2016

OCERTO 2016 Cortona

Istituto Nazionale di Alta Matematica Francesco Severi

Outline

1. State system
   - Cahn–Hilliard/Navier–Stokes system
   - Time discretization
   - Energy estimate
   - Modified state system
   - Regularity
2. Optimization problem
   - Existence of minimizers
   - First order conditions
   - Regularization
   - Limiting system for DOP potential
3. Numerical results (single density)

CHNS system for two fluids with different densities [Dreyer], [Abels,Garcke,Grün 2012]:

$$\begin{aligned} \partial_t c + \mathbf{v} \cdot \nabla c - \operatorname{div}(m(c)\nabla w) &= 0, \\ -\Delta c + \widehat{\varphi}'(c) &= w. \end{aligned}$$

$$\begin{aligned} \partial_t(\rho(c)\mathbf{v}) - \operatorname{div}(\eta(c)\epsilon(\mathbf{v})) + \nabla\pi - w\nabla c & \\ + \operatorname{div}(\mathbf{v} \otimes (\rho(c)\mathbf{v} - \hat{\rho} m(c)\nabla w)) &= \mathbf{u}, \\ \operatorname{div} \mathbf{v} &= 0, \end{aligned} \tag{CHNS}$$

$$\begin{aligned} \mathbf{v}|_{\partial\Omega} = 0, \quad \nabla c \cdot \vec{\mathbf{n}}|_{\partial\Omega} = \nabla w \cdot \vec{\mathbf{n}}|_{\partial\Omega} &= 0, \\ (c, \mathbf{v})|_{t=0} &= (c_a, \mathbf{v}_a). \end{aligned}$$

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(c, \mathbf{v})|_{t=0} &= (c_a, \mathbf{v}_a).
\end{aligned}
$$

c	order parameter	π	pressure	$\widehat{\varphi}$	bulk free energy density
w	chemical potential	ρ	density	$\hat{\rho}_i$	fixed densities of fluids
$\mathbf{v}$	velocity	m	mobility coefficient	$\hat{\rho} := \frac{1}{2}(\hat{\rho}_2 - \hat{\rho}_1)$	
$\mathbf{u}$	distributed control	η	viscosity coefficient	$\epsilon(\mathbf{v}) := \frac{1}{2}(D\mathbf{v} + D^T\mathbf{v})$	

Density:

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Bulk free energy density $\widehat{\varphi}$:

$$\widehat{\varphi} \; = \; \widehat{\varphi}_{\mathrm{convex}} + \widehat{\varphi}_{\mathrm{concave}} \; = \varphi - \frac{\kappa}{2}|\,.\,|^2$$

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Mean value of c:

$$\mathrm{BC.} \;+\; \mathrm{div}\,\mathbf{v} = 0 \quad\Longrightarrow\quad \partial_t \int_\Omega c \;=\; 0.$$

Known results

Model:

- H. Abels, H. Garcke, G. Grün, *Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities*, Math. Models Methods Appl. Sci., 22(3), 2012.

Existence of solutions:

- H. Abels, D. Depner , H. Garcke, *Existence of weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities*, J. Math. Fluid Mech. 15, No. 3, 453-480 (2013).
- H. Abels, D. Depner , H. Garcke, *On an incompressible Navier–Stokes/Cahn–Hilliard system with degenerate mobility*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 30, No. 6, 1175-1190 (2013).
- H. Garcke, M. Hinze, C. Kahle, *A stable and linear time discretization for a thermodynamically consistent model for two-phase incompressible flow*, Applied Numerical Mathematics 99, 151–171, (2016).

Result for optimization with single densities:

- H. Hintermüller, W., *Optimal control of a semidiscrete Cahn-Hilliard-Navier-Stokes system*, SIAM J. Control Optim. 52, No. 1, 747–772 (2014).
- S.P. Frigeri, E. Roccca, J. Sprekels, *Optimal distributed control of a nonlocal Cahn–Hilliard/Navier–Stokes system in 2D*, WIAS Preprint No. 2036, (2014).
- P. Colli, G. Gilardi, J. Sprekels, *A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions*, Adv. Nonlinear Anal., 4, 311–325 (2015).
- P. Colli, G. Gilardi, E. Rocca, J. Sprekels, *Optimal distributed control of a diffuse interface model of tumor growth*, WIAS Preprint No. 2093, (2015).
- E. Rocca, J. Sprekels, *Optimal distributed control of a nonlocal convective Cahn–Hilliard equation by the velocity in three dimensions*, SIAM J. Control Optim. 53, 1654–1680 (2015).
- P. Colli, G. Gilardi, J. Sprekels, *Distributed optimal control of a nonstandard nonlocal phase field system*, WIAS Preprint No. 2266, (2016).

Time discretization

$M =$ number of time steps, $\tau > 0$ time step size and $x_i := x|_{t=i\tau}$ for $x \in \{c, w, \mathbf{v}, \mathbf{u}\}$.
A triple $(c, w, \mathbf{v})$ is a solution to the time-discrete Cahn–Hilliard/Navier–Stokes system with control $\mathbf{u} \in U := L^2(\Omega; \mathbb{R}^3)^{M-1}$:

$$
\begin{aligned}
\tfrac{1}{\tau}(c_{i+1} - c_i) + \mathbf{v}_{i+1} \cdot \nabla c_i - \operatorname{div}(m(c_i)\nabla w_{i+1}) &= 0 \\
-\Delta c_{i+1} + \partial\varphi(c_{i+1}) - \kappa c_i &\ni w_{i+1} \\
\tfrac{1}{\tau}(\rho(c_i)\mathbf{v}_{i+1} - \rho(c_{i-1})\mathbf{v}_i) - \operatorname{div}(\eta(c_i)\epsilon(\mathbf{v}_{i+1})) - w_{i+1}\nabla c_i & \\
+ \operatorname{div}(\mathbf{v}_{i+1} \otimes (\rho(c_{i-1})\mathbf{v}_i - \hat{\rho} m(c_{i-1})\nabla w_i)) &= \mathbf{u}_{i+1} \qquad (\mathrm{CHNS}_\tau) \\
(c_{-1}, \mathbf{v}_0) &= (c_a, \mathbf{v}_a)
\end{aligned}
$$

$c = (c_{-1}, \ldots, c_M), \quad w = (w_0, \ldots, w_M), \quad \mathbf{v} = (\mathbf{v}_0, \ldots, \mathbf{v}_M), \quad \mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_M)$

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$$\begin{aligned}
\tfrac{1}{\tau}(c_+ - c_\circ) + \mathbf{v}_+ \cdot \nabla c_\circ - \operatorname{div}(m(c_\circ)\nabla w_+) &= 0 \\
-\Delta c_+ + \partial\varphi(c_+) - \kappa c_\circ &\ni w_+ \\
\tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) - \operatorname{div}(\eta(c_\circ)\epsilon(\mathbf{v}_+)) - w_+\nabla c_\circ & \\
+ \operatorname{div}(\mathbf{v}_+ \otimes (\rho(c_-)\mathbf{v}_\circ - \hat{\rho} m(c_-)\nabla w_\circ)) &= \mathbf{u}_+ \\
(c_{-1}, \mathbf{v}_0) &= (c_a, \mathbf{v}_a)
\end{aligned} \tag{CHNS$_\tau$}$$

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(c_{-1}, \mathbf{v}_0) &= (c_a, \mathbf{v}_a)
\end{aligned} \tag{CHNS$_\tau$}$$

$$\begin{aligned}
X_1 &:= H^1(\Omega)^{M+1}_{\mathfrak{m}} \times H^1(\Omega)^M_{\mathfrak{m}} \times H^1(\Omega;\mathbb{R}^3)^M_{0,\sigma} \\
X_2 &:= H^2(\Omega)^{M+1}_{\mathfrak{m},\vec{\mathbf{n}}} \times H^2(\Omega)^M_{\mathfrak{m},\vec{\mathbf{n}}} \times H^1(\Omega;\mathbb{R}^3)^M_{0,\sigma}
\end{aligned}$$

Solution operator $\mathcal{S}_\varphi : U \ni \mathbf{u} \mapsto \{(c, w, \mathbf{v})\}$

$c = (c_{-1}, \ldots, c_M)$, $w = (w_0, \ldots, w_M)$, $\mathbf{v} = (\mathbf{v}_0, \ldots, \mathbf{v}_M)$, $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_M)$

Optimal control problem (P_φ)

Optimal control problem

$$\min_{(c,w,\mathbf{v},\mathbf{u})} \mathcal{J}(c,w,\mathbf{v},\mathbf{u}) \quad \text{s.t.} \quad \mathbf{u} \in U_{\mathrm{ad}}, \ \ (c,w,\mathbf{v}) \in \mathcal{S}_\varphi(\mathbf{u}).$$

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Assumptions:

- $\Omega \subset \mathbb{R}^3$ bounded, sufficiently smooth
- $m,\ \eta \in C^2(\mathbb{R}),\ \ 0 < C_1 \le m(c), \eta(c) \le C_2$ for all $c \in \mathbb{R}$
- $(c_a, \mathbf{v}_a) \in H^2_{\mathfrak{m},\vec{\mathbf{n}}}(\Omega) \times H^2_{0,\sigma}(\Omega;\mathbb{R}^3), \quad \int_\Omega \varphi(c_a) < \infty$
- $U_{\mathrm{ad}} \subset U$ convex, closed, nonempty
- $\mathcal{J} \in C^1(X_1 \times U)$, bounded below, partially coercive on U_{ad} w.r.t. $\mathbf{u}$

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Next steps:

- Existence of solution to state system (for general potentials φ)
- Existence of minimizers (for general potentials φ)
- First order optimality conditions (for *smooth* potentials φ)
- Stationary system by limit process (for the DOP φ)

Cahn–Hilliard/Navier–Stokes system:

$$\tfrac{1}{\tau}(c_+ - c_\circ) + \mathbf{v}_+ \cdot \nabla c_\circ - \operatorname{div}(m(c_\circ)\nabla w_+) = 0 \tag{1}$$

$$-w_+ - \Delta c_+ + \partial\varphi(c_+) - \kappa c_\circ \ni 0 \tag{2}$$

$$\begin{aligned}\tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) - \operatorname{div}(\eta(c_\circ)\epsilon(\mathbf{v}_+)) - w_+\nabla c_\circ & \\ + \operatorname{div}(\mathbf{v}_+ \otimes \underbrace{(\rho(c_-)\mathbf{v}_\circ - \hat{\rho} m(c_-)\nabla w_\circ)}_{:=\mathbf{z}}) &= \mathbf{u}_+\end{aligned} \tag{3}$$

$\langle\,(1), w_+\,\rangle + \langle\,(2), \tfrac{1}{\tau}(c_+ - c_\circ)\,\rangle + \langle\,(3), \mathbf{v}_+\,\rangle$:

$$\int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx$$

$$= \int_\Omega \Big[\tfrac{1}{\tau}(c_+ - c_\circ)w_+ \; + \; w_+\mathbf{v}_+ \cdot \nabla c_\circ \; + \; m(c_\circ)|\nabla w_+|^2$$

$$- \; \tfrac{1}{\tau} w_+(c_+ - c_\circ) \; + \; \tfrac{1}{\tau}\nabla c_+ \cdot \nabla(c_+ - c_\circ) \; + \; \partial\varphi(c_+)(c_+ - c_\circ) \; - \; \tfrac{\kappa}{\tau} c_\circ(c_+ - c_\circ)$$

$$+ \; \tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) \cdot \mathbf{v}_+ \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2$$

$$- \; (\mathbf{v}_+ \otimes \mathbf{z}) : D\mathbf{v}_+ \; - \; w_+\mathbf{v}_+ \cdot \nabla c_\circ \Big] \, dx$$

Energy estimate

$$
\begin{aligned}
&\int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx \\
&\quad = \int_\Omega \Big[\tfrac{1}{\tau}(c_+ - c_\circ) w_+ \; + \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \; + \; m(c_\circ) |\nabla w_+|^2 \\
&\qquad\quad - \; \tfrac{1}{\tau} w_+ (c_+ - c_\circ) \; + \; \tfrac{1}{\tau} \nabla c_+ \cdot \nabla (c_+ - c_\circ) \; + \; \partial\varphi(c_+)(c_+ - c_\circ) \; - \; \tfrac{\kappa}{\tau} c_\circ (c_+ - c_\circ) \\
&\qquad\quad + \; \tfrac{1}{\tau} (\rho(c_\circ) \mathbf{v}_+ - \rho(c_-) \mathbf{v}_\circ) \cdot \mathbf{v}_+ \; + \; \eta(c_\circ) |\epsilon(\mathbf{v}_+))|^2 \\
&\qquad\qquad - \; (\mathbf{v}_+ \otimes \mathbf{z}) : D\mathbf{v}_+ \; - \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \Big] \, dx
\end{aligned}
$$

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$$\int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx$$

$$= \int_\Omega \Big[\tfrac{1}{\tau}(c_+ - c_\circ) w_+ \; + \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \; + \; m(c_\circ) |\nabla w_+|^2$$

$$- \; \tfrac{1}{\tau} w_+ (c_+ - c_\circ) \; + \; \tfrac{1}{\tau} \nabla c_+ \cdot \nabla (c_+ - c_\circ) \; + \; \partial\varphi(c_+)(c_+ - c_\circ) \; - \; \tfrac{\kappa}{\tau} c_\circ (c_+ - c_\circ)$$

$$+ \; \tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) \cdot \mathbf{v}_+ \; + \; \eta(c_\circ) |\epsilon(\mathbf{v}_+))|^2$$

$$- \; (\mathbf{v}_+ \otimes \mathbf{z}) : D\mathbf{v}_+ \; - \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \Big] \, dx$$

$$a(a - b) = \tfrac{1}{2}[a^2 - b^2 + (a - b)^2], \quad \partial\varphi(c_+)(c_+ - c_\circ) \geq \varphi(c_+) - \varphi(c_\circ)$$

$$
\begin{aligned}
&\int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx \\
&\quad = \int_\Omega \Big[\tfrac{1}{\tau}(c_+ - c_\circ) w_+ \; + \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \; + \; m(c_\circ)|\nabla w_+|^2 \\
&\qquad - \; \tfrac{1}{\tau} w_+ (c_+ - c_\circ) \; + \; \tfrac{1}{\tau} \nabla c_+ \cdot \nabla (c_+ - c_\circ) \; + \; \partial\varphi(c_+)(c_+ - c_\circ) \; - \; \tfrac{\kappa}{\tau} c_\circ (c_+ - c_\circ) \\
&\qquad + \; \tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) \cdot \mathbf{v}_+ \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2 \\
&\qquad\quad - \; (\mathbf{v}_+ \otimes \mathbf{z}) : D\mathbf{v}_+ \; - \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \Big] \, dx
\end{aligned}
$$

$$
(\rho_\circ \mathbf{v}_+ - \rho_- \mathbf{v}_\circ)\mathbf{v}_+ \; = \; \frac{1}{2}\Big[\rho_\circ \mathbf{v}_+^2 - \rho_- \rho_\circ^2 + (\rho_\circ - \rho_-)\mathbf{v}_+^2 + \rho_-(\mathbf{v}_+ - \mathbf{v}_\circ)^2\Big]
$$

Energy estimate

$$\int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx$$

$$= \int_\Omega \Big[\tfrac{1}{\tau}(c_+ - c_\circ) w_+ \; + \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \; + \; m(c_\circ)|\nabla w_+|^2$$

$$- \; \tfrac{1}{\tau} w_+ (c_+ - c_\circ) \; + \; \tfrac{1}{\tau} \nabla c_+ \cdot \nabla (c_+ - c_\circ) \; + \; \partial\varphi(c_+)(c_+ - c_\circ) \; - \; \tfrac{\kappa}{\tau} c_\circ (c_+ - c_\circ)$$

$$+ \; \tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) \cdot \mathbf{v}_+ \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2$$

$$- \; (\mathbf{v}_+ \otimes \mathbf{z}) : D\mathbf{v}_+ \; - \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \Big] \, dx$$

$$\int_\Omega (\mathbf{v} \otimes \mathbf{z}) : D\mathbf{v} \, dx \; = \; -\frac{1}{2} \int_\Omega |\mathbf{v}|^2 \, \mathrm{div}\, \mathbf{z} \, dx$$

Energy estimate

$$
\begin{aligned}
&\int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx \\
&= \int_\Omega \Big[\tfrac{1}{\tau}(c_+ - c_\circ)w_+ \; + \; w_+\mathbf{v}_+ \cdot \nabla c_\circ \; + \; m(c_\circ)|\nabla w_+|^2 \\
&\qquad - \; \tfrac{1}{\tau} w_+(c_+ - c_\circ) \; + \; \tfrac{1}{\tau}\nabla c_+ \cdot \nabla(c_+ - c_\circ) \; + \; \partial\varphi(c_+)(c_+ - c_\circ) \; - \; \tfrac{\kappa}{\tau} c_\circ(c_+ - c_\circ) \\
&\qquad + \; \tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) \cdot \mathbf{v}_+ \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2 \\
&\qquad\quad - \; (\mathbf{v}_+ \otimes \mathbf{z}) : D\mathbf{v}_+ \; - \; w_+\mathbf{v}_+ \cdot \nabla c_\circ \Big] \, dx \\
&\geq \int_Q \Big[m(c_\circ)|\nabla w_+|^2 \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2 \; + \; \tfrac{1}{2\tau}\big[|\nabla c_+ - \nabla c_\circ|^2 + \kappa(c_+ - c_\circ)^2\big] \\
&\qquad + \; \frac{1}{\tau}\Big(\frac{1}{2}\rho(c_\circ)|\mathbf{v}_+|^2 + \frac{1}{2}|\nabla c_+|^2 + \widehat{\varphi}(c_+)\Big) \; - \; \frac{1}{\tau}\Big(\frac{1}{2}\rho(c_-)|\mathbf{v}_\circ|^2 + \frac{1}{2}|\nabla c_\circ|^2 + \widehat{\varphi}(c_\circ)\Big) \\
&\qquad + \; \frac{|\mathbf{v}_+|^2}{2}\Big[\operatorname{div}\mathbf{z} + \tfrac{1}{\tau}(\rho(c_\circ) - \rho(c_-))\Big] \; + \; \rho(c_-)|\mathbf{v}_+ - \mathbf{v}_\circ|^2 \Big]
\end{aligned}
$$

Energy estimate

$$
\begin{aligned}
&\int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx \\
&= \int_\Omega \Big[\tfrac{1}{\tau}(c_+ - c_\circ) w_+ \; + \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \; + \; m(c_\circ)|\nabla w_+|^2 \\
&\quad - \tfrac{1}{\tau} w_+ (c_+ - c_\circ) \; + \; \tfrac{1}{\tau} \nabla c_+ \cdot \nabla (c_+ - c_\circ) \; + \; \partial\varphi(c_+)(c_+ - c_\circ) \; - \; \tfrac{\kappa}{\tau} c_\circ (c_+ - c_\circ) \\
&\quad + \tfrac{1}{\tau} (\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) \cdot \mathbf{v}_+ \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2 \\
&\qquad - (\mathbf{v}_+ \otimes \mathbf{z}) : D\mathbf{v}_+ \; - \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \Big] \, dx \\
&\geq \int_Q \Big[m(c_\circ)|\nabla w_+|^2 \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2 \; + \; \tfrac{1}{2\tau} \big[|\nabla c_+ - \nabla c_\circ|^2 + \kappa (c_+ - c_\circ)^2 \big] \\
&\quad + \frac{1}{\tau}\Big(\frac{1}{2}\rho(c_\circ)|\mathbf{v}_+|^2 + \frac{1}{2}|\nabla c_+|^2 + \widehat{\varphi}(c_+) \Big) \; - \; \frac{1}{\tau}\Big(\frac{1}{2}\rho(c_-)|\mathbf{v}_\circ|^2 + \frac{1}{2}|\nabla c_\circ|^2 + \widehat{\varphi}(c_\circ) \Big) \\
&\quad + \frac{|\mathbf{v}_+|^2}{2} \Big[\operatorname{div} \mathbf{z} + \tfrac{1}{\tau}(\rho(c_\circ) - \rho(c_-)) \Big] \; + \; \rho(c_-)|\mathbf{v}_+ - \mathbf{v}_\circ|^2 \Big]
\end{aligned}
$$

Total energy: $E(\mathbf{v}, c, \rho) := \int_\Omega \frac{\rho|\mathbf{v}|^2}{2} + \frac{|\nabla c|^2}{2} + \widehat{\varphi}(c) \, dx$

$$\int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx$$

$$= \int_\Omega \Big[\tfrac{1}{\tau}(c_+ - c_\circ) w_+ \; + \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \; + \; m(c_\circ)|\nabla w_+|^2$$

$$- \; \tfrac{1}{\tau} w_+ (c_+ - c_\circ) \; + \; \tfrac{1}{\tau} \nabla c_+ \cdot \nabla(c_+ - c_\circ) \; + \; \partial\varphi(c_+)(c_+ - c_\circ) \; - \; \tfrac{\kappa}{\tau} c_\circ (c_+ - c_\circ)$$

$$+ \; \tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) \cdot \mathbf{v}_+ \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2$$

$$- \; (\mathbf{v}_+ \otimes \mathbf{z}) : D\mathbf{v}_+ \; - \; w_+ \mathbf{v}_+ \cdot \nabla c_\circ \Big] \, dx$$

$$\geq \int_Q \Big[m(c_\circ)|\nabla w_+|^2 \; + \; \eta(c_\circ)|\epsilon(\mathbf{v}_+))|^2 \; + \; \tfrac{1}{2\tau}\big[|\nabla c_+ - \nabla c_\circ|^2 + \kappa(c_+ - c_\circ)^2\big]$$

$$+ \; \frac{1}{\tau}\Big(E(\mathbf{v}_+, c_+, \rho(c_\circ)) - E(\mathbf{v}_\circ, c_\circ, \rho(c_-)) \Big)$$

$$+ \; \frac{|\mathbf{v}_+|^2}{2} \Big[\operatorname{div} \mathbf{z} + \tfrac{1}{\tau}(\rho(c_\circ) - \rho(c_-)) \Big] \; + \; \rho(c_-)|\mathbf{v}_+ - \mathbf{v}_\circ|^2 \Big]$$

$$\operatorname{div} \mathbf{z} \; = \; \hat\rho[\mathbf{v}_\circ \cdot \nabla c_- + \operatorname{div}(m(c_-)\nabla w_\circ)] \; = \; \tfrac{1}{\tau}(\rho(c_\circ) - \rho(c_-))$$

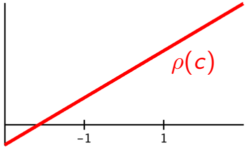
Total energy: $E(\mathbf{v}, c, \rho) := \int_\Omega \frac{\rho|\mathbf{v}|^2}{2} + \frac{|\nabla c|^2}{2} + \widehat{\varphi}(c)\, dx$

Lemma 1 (Energy estimate)

If $(c, w, \mathbf{v})$ *is a solution to the one time step CHNS system with control* $\mathbf{u}_+$, *then*

$$E(\mathbf{v}_+, c_+, \rho(c_\circ)) + \int_\Omega \Big[\frac{\rho(c_-)|\mathbf{v}_+ - \mathbf{v}_\circ|^2}{2} + \frac{|\nabla c_+ - \nabla c_\circ|^2}{2} + \frac{\kappa|c_+ - c_\circ|^2}{2} + \tau\eta(c_\circ)|\epsilon(\mathbf{v}_+)|^2 + \tau m(c_\circ)|w_+|^2 \Big] dx$$
$$\leq E(\mathbf{v}_\circ, c_\circ, \rho(c_-)) + \int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx.$$

for $\mathbf{z} = \rho(c_-)\mathbf{v}_\circ - \hat{\rho} m(c_-)\nabla w_\circ$.



Energy estimate

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Lemma 1 (Energy estimate)

If $(c, w, \mathbf{v})$ is a solution to the one time step CHNS system with control $\mathbf{u}_+$, then

$$E(\mathbf{v}_+, c_+, \tilde{\rho}(c_\circ)) + \int_\Omega \Big[\frac{\tilde{\rho}(c_-)|\mathbf{v}_+ - \mathbf{v}_\circ|^2}{2} + \frac{|\nabla c_+ - \nabla c_\circ|^2}{2} + \frac{\kappa|c_+ - c_\circ|^2}{2} + \tau\eta(c_\circ)|\epsilon(\mathbf{v}_+)|^2 + \tau m(c_\circ)|w_+|^2\Big] dx \leq E(\mathbf{v}_\circ, c_\circ, \tilde{\rho}(c_-)) + \int_\Omega \mathbf{u}_+ \cdot \mathbf{v}_+ \, dx.$$

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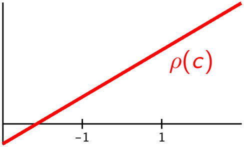

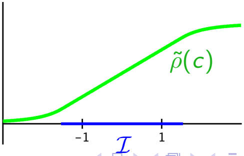

Take

$$\begin{aligned}\mathbf{z} &:= G(c_\circ, c_-, w_\circ, \mathbf{v}_\circ)\\ &:= \begin{cases}\rho(c_-)\mathbf{v}_\circ - \hat{\rho} m(c_-)\nabla w_\circ & \text{if } c_\circ, c_- \in \mathcal{I} \text{ a.e. on } \Omega\\ (\text{Bogovskii operator}) & \text{otherwise.}\end{cases}\end{aligned}$$

Modified Cahn–Hilliard/Navier–Stokes system

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Modified Cahn–Hilliard/Navier–Stokes system:

$$\begin{aligned} \tfrac{1}{\tau}(c_+ - c_\circ) + \mathbf{v}_+ \cdot \nabla c_\circ - \operatorname{div}(m(c_\circ)\nabla w_+) &= 0 \\ -\Delta c_+ + \partial\varphi(c_+) - \kappa c_\circ &\ni w_+ \\ \tfrac{1}{\tau}(\tilde{\rho}(c_\circ)\mathbf{v}_+ - \tilde{\rho}(c_-)\mathbf{v}_\circ) - \operatorname{div}(\eta(c_\circ)\epsilon(\mathbf{v}_+)) - w_+\nabla c_\circ & \\ + \operatorname{div}(\mathbf{v}_+ \otimes G(c_\circ, c_-, w_\circ, \mathbf{v}_\circ)) &= \mathbf{u}_+ \end{aligned} \qquad (\overline{\text{CHNS}}_\tau)$$

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Theorem 2 (Existence of solution to $(\overline{\text{CHNS}}_\tau)$)

Let $M = 1$, $\varphi : H^1_{\mathfrak{m}} \to \overline{\mathbb{R}}$ convex, lsc and $c_-, c_\circ \in H^1_{\mathfrak{m}}$, $c_\circ \in D(\partial\varphi)$, $\mathbf{v}_\circ \in H^1_{0,\sigma}$, $\mathbf{u}_+ \in L^2$. Then $(\overline{\text{CHNS}}_\tau)$ has a solution $(c_+, w_+, \mathbf{v}_+) \in \overline{\mathcal{S}}_\varphi(\mathbf{u}_+) \subset H^1_{\mathfrak{m}} \times H^1_{\mathfrak{m}} \times H^1_{0,\sigma}$.

Sketch of the proof: Rewrite ($\overline{\text{CHNS}}_\tau$) as

$G(c_+, w_+, \mathbf{v}_+) = F(c_+, w_+, \mathbf{v}_+)$, with $G(c_+, w_+, \mathbf{v}_+) := \begin{pmatrix} -\operatorname{div}(m(c_\circ \nabla w_+)) \\ -\Delta c_+ + \partial\varphi(c_+) \\ -\operatorname{div}(\eta(c_\circ)\epsilon(\mathbf{v}_+)) \end{pmatrix}$

Show: G invertible, F, G^{-1} continuous,

$F \circ G^{-1} : L^{3/2} \to L^{3/2}$ continuous + compact

Apply Leray-Schauder principle

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Theorem 3 (Regularity of solution to $(\overline{\text{CHNS}}_\tau)$)

Every solution $(c, w, \mathbf{v})$ of $(\overline{\text{CHNS}}_\tau)$ belongs to X_2, if $c_a, \mathbf{v}_a \in H^2$ and $\mathbf{u} \in U$. Moreover, the mapping $U \ni \mathbf{u} \mapsto \overline{\mathcal{S}}_\varphi(\mathbf{u}) \subset X_2$ is bounded.

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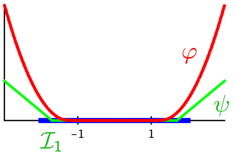
Based on a result by [Hintermüller,Schiela,Wollner'14] we obtain:

Theorem 4 (Existence of solutions to (CHNS_τ))

If $S \subset U$ bnd, $\mathcal{I}_1 \subset \mathcal{I}^\circ$, $c_a \in [-1, 1]$ a.e., there exists $\beta > 0$:

$$\varphi(x) \;\geq\; \psi(x) := \beta|x - \operatorname{Proj}_{\mathcal{I}_1}(x)| \qquad \forall x \in \mathbb{R}$$

$$\Longrightarrow \quad \overline{\mathcal{S}}_\varphi(\mathbf{u}) = \mathcal{S}_\varphi(\mathbf{u}) \qquad \forall \mathbf{u} \in S.$$



Theorem 5 (Existence of minimizers to (P_φ))

Under the assumptions of Theorem 4, (P_φ) admits a minimizer.

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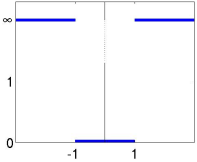
Theorem 6 (First order necessary optimality conditions for *smooth* φ)

Assume
- $\varphi \in C^1(H^1_{\mathfrak{m}}(\Omega))$, $\varphi' \in \mathcal{L}(H^2_{\mathfrak{m},\vec{\mathfrak{n}}}(\Omega); L^2(\Omega))$
- $z = (c, w, \mathbf{v}, \mathbf{u}) \in X_2 \times U_{\mathrm{ad}}$ *is a minimizer for* (P_φ).

Then there exists $(p, r, \mathbf{q}) \in H^1(\Omega)^M_{\mathfrak{m}} \times H^1(\Omega)^M_{\mathfrak{m}} \times H^1(\Omega; \mathbb{R}^N)^{M-1}_{0,\sigma}$:

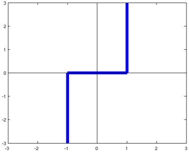
$$
\begin{aligned}
\tfrac{1}{\tau}(p_- - p_\circ) - \Delta r_- + \varphi''(c_\circ)^* r_- - \kappa r_+ &= \mathcal{J}_{c_\circ}(z) + F_1(z, p_\circ, \mathbf{q}_\circ, \mathbf{q}_+)\\
-r_- - \operatorname{div}(m_- \nabla p_-) - \mathbf{q}_- \cdot \nabla c_- &= \mathcal{J}_{w_\circ}(z) + F_2(z, \mathbf{q}_\circ)\\
\tfrac{1}{\tau}\rho_-(\mathbf{q}_- - \mathbf{q}_\circ) - (D\mathbf{q}_-)(\rho_{--}\mathbf{v}_- - \hat{\rho} m_{--} \nabla w_-) &\\
- \operatorname{div}(\eta_- \epsilon(\mathbf{q}_-)) + p_- \nabla c_- &= \mathcal{J}_{\mathbf{v}_\circ}(z) + F_3(z, \mathbf{q}_\circ)\\
(\mathcal{J}_{\mathbf{u}_\circ}(z) - \mathbf{q}_-)_1^{M-1} &\in [\mathbb{R}_+(U_{\mathrm{ad}} - \mathbf{u})]^+.
\end{aligned}
$$

Regularization of the double obstacle potential φ:



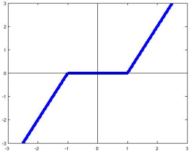
φ

$\rightarrow$



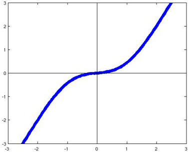
$\partial\varphi$

$\rightarrow$



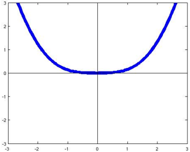
$(\partial\varphi)_\alpha$

$\rightarrow$



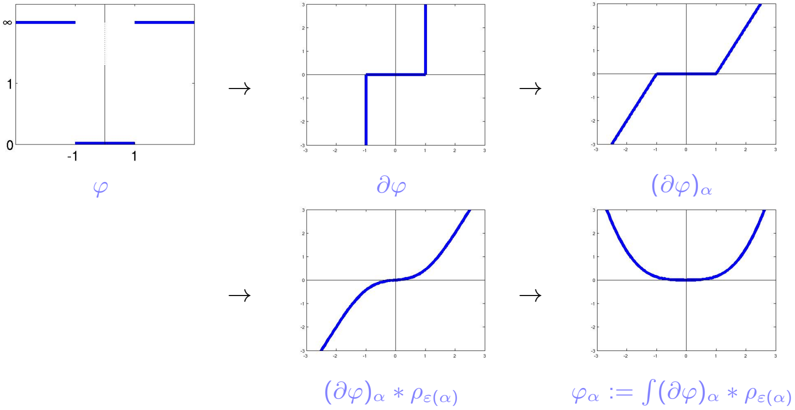
$(\partial\varphi)_\alpha * \rho_{\varepsilon(\alpha)}$

$\rightarrow$



$\varphi_\alpha := \int (\partial\varphi)_\alpha * \rho_{\varepsilon(\alpha)}$

Regularization of the double obstacle potential φ:



Theorem 7 (Consistency $\alpha \to 0$)

Let $\left((c^{(\alpha)}, w^{(\alpha)}, \mathbf{v}^{(\alpha)}, \mathbf{u}^{(\alpha)})\right)_{\alpha>0}$ *be a sequence of minimizers for* $(\mathrm{P}_{\varphi_\alpha})$.
Then a subsequence converges weakly in X_2 *to a minimizer* $(c, w, \mathbf{v}, \mathbf{u})$ *for* (P_φ).

Theorem 8 (Limiting ε-almost C-stationarity conditions)

$\exists$ subseq. $(p^{(\alpha)}, r^{(\alpha)}, \mathbf{q}^{(\alpha)}) \underset{X_1}{\rightharpoonup} (p, r, \mathbf{q})$, $(\varphi_\alpha)''(c_+^{(\alpha)})^* r_\circ^{(\alpha)} \underset{(H^1_{\mathrm{m}})^*}{\rightharpoonup} \lambda_\circ$, $a_\circ^{(\alpha)} := \varphi'_\alpha(c_\circ^{(\alpha)}) \underset{L^2_{\mathrm{m}}}{\rightharpoonup} a_\circ$:

$$\begin{aligned}
\tfrac{1}{\tau}(c_+ - c_\circ) + \mathbf{v}_+ \cdot \nabla c_\circ - \operatorname{div}(m(c_\circ)\nabla w_+) &= 0\\
-\Delta c_+ + a_+ - \kappa c_\circ &= w_+\\
\tfrac{1}{\tau}(\rho(c_\circ)\mathbf{v}_+ - \rho(c_-)\mathbf{v}_\circ) - \operatorname{div}(\eta(c_\circ)\epsilon(\mathbf{v}_+)) - w_+\nabla c_\circ &\\
+\operatorname{div}(\mathbf{v}_+ \otimes (\rho(c_-)\mathbf{v}_\circ - \hat\rho m(c_-)\nabla w_\circ)) &= \mathbf{u}_+
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{\tau}(p_- - p_\circ) - \Delta r_- + \lambda_- - \kappa r_+ &= \mathcal{J}_{c_\circ}(z) + F_1(z, p_\circ, \mathbf{q}_\circ, \mathbf{q}_+)\\
-r_- - \operatorname{div}(m_-\nabla p_-) - \mathbf{q}_- \cdot \nabla c_- &= \mathcal{J}_{w_\circ}(z) + F_2(z, \mathbf{q}_\circ)\\
\tfrac{1}{\tau}\rho_-(\mathbf{q}_- - \mathbf{q}_\circ) - (D\mathbf{q}_-)(\rho_{-}{}_-\mathbf{v}_- - \hat\rho m_{-}{}_-\nabla w_-) &\\
-\operatorname{div}(\eta_-\epsilon(\mathbf{q}_-)) + p_-\nabla c_- &= \mathcal{J}_{\mathbf{v}_\circ}(z) + F_3(z, \mathbf{q}_\circ)\\
(\mathcal{J}_{\mathbf{u}_\circ}(z) - \mathbf{q}_-)_1^{M-} &\in [\mathbb{R}_+(U_{\mathrm{ad}} - \mathbf{u})]^+.
\end{aligned}$$

Moreover, for all Lipschitz functions $\Lambda : \mathbb{R} \to \mathbb{R}$ *with* $\Lambda(-1) = \Lambda(1) = 0$:

$$(a_\circ, \Lambda(c_\circ))_{L^2_{\mathrm{m}}} = 0 \qquad \langle \lambda_\circ, \Lambda(c_\circ)\rangle_{H^1_{\mathrm{m}}} = 0$$

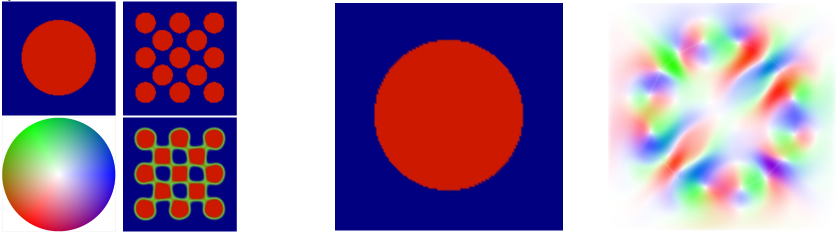
$$(a_\circ, r_\circ)_{L^2_{\mathrm{m}}} = 0 \qquad \liminf(\lambda_\circ^{(\alpha)}, r_\circ^{(\alpha)})_{L^2_{\mathrm{m}}} \geq 0$$

For all $\varepsilon > 0$ *there exists* $M_\circ^\varepsilon \subset M_\circ := \{-1 < c_\circ < 1\}$, $|M_\circ \setminus M_\circ^\varepsilon| < \varepsilon$:

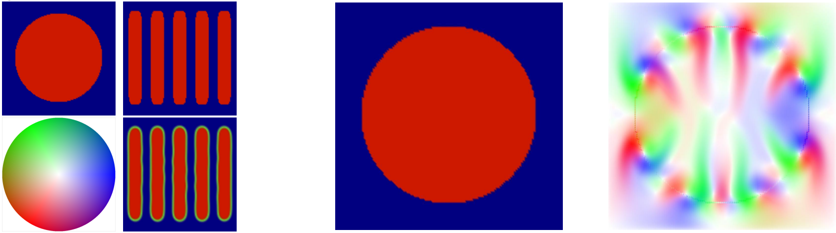
$$\langle \lambda_\circ, v\rangle = 0 \quad \forall v \in H^1_{\mathrm{m}}(\Omega),\ \ v|_{\Omega\setminus M_\circ^\varepsilon} = 0$$

Numerical examples (with single density)

Example 1:



Example 2:



Conclusion

Summary:

- Energy-stable time-discretization
- Existence of solutions to the (modified) state equations
- Existence of minimizers
- First order conditions for smooth potentials
- Regularization of the potential
- Limiting system for the double-obstacle potential

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Thank You for Your Attention!