Finite Element Methods for Parabolic Optimal Control Problems with Measures

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joint work with
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Outline

1. Motivation

2. Pointwise control: Optimality system and regularity

3. Finite element discretization in space and time

4. Conclusions and Extensions

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- Two problem classes
  (A) Pointwise control (Measures on RHS, control of the amplitude)
  (B) Sparse control problems (Control from a measure space)

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- Main tools
  - Discrete maximal parabolic regularity
  - Best approximation results in various norms

Optimal Control with Pointwise Controls

Pointwise control: model problem

$$\text{Minimize } J(q,u) = \frac{1}{2}\|u - u_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha}{2}\sum_{i=1}^{N}\|q_i\|^2_{L^2(0,T)}$$

subject to the state equation

$$\begin{aligned} \partial_t u - \Delta u &= \sum_{i=1}^{N} q_i(t)\,\delta_{x_i} && \text{in } (0,T)\times\Omega, \\ u &= 0 && \text{on } (0,T)\times\partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

and control constraints $q_a \le q_i(t) \le q_b$ for almost all $t \in (0,T)$.

- N points $x_i \in \Omega$, δ_{x_i} are the Dirac functionals
- the number N and the positions x_i are a priori known

Sparse control of parabolic PDEs

Optimal control problem

$$\text{Minimize } J(q,u) = \frac{1}{2}\|u - u_d\|^2_{L^2(I\times\Omega)} + \alpha\|q\|_X, \quad q \in X,\ u \in V$$

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with unknown N and unknown (but fixed) positions x_i of Diracs

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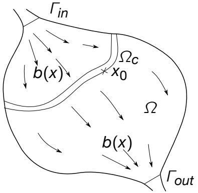
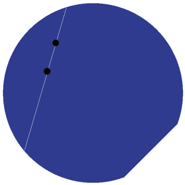
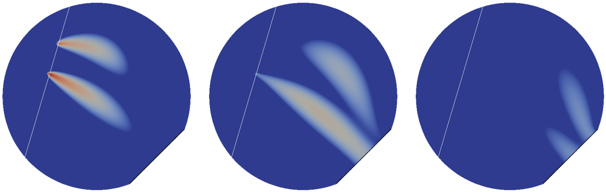
Example: Inverse point source reconstruction

- state equation: $\partial_t u - \epsilon \Delta u - b \cdot \nabla u = q$ & boundary and initial cond.







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and control constraints $q_a \le q(t) \le q_b$ for almost all $t \in (0,T)$.

- $I = (0,T)$, $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ convex and polygonal/polyhedral
- $u_d \in L^\infty(I\times\Omega)$, $q_a, q_b \in \overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$ with $q_a < q_b$

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- Existence and uniqueness of $(\bar q, \bar u) \in L^2(I)\times L^2(I\times\Omega)$

Optimality system

State equation

$$\begin{aligned} \partial_t \bar{u} - \Delta \bar{u} &= \bar{q}(t)\,\delta_{x_0} && \text{in } I \times \Omega, \\ \bar{u} &= 0 && \text{on } I \times \partial\Omega, \\ \bar{u}(0) &= 0 && \text{in } \Omega. \end{aligned}$$

Adjoint equation

$$\begin{aligned} -\partial_t \bar{z} - \Delta \bar{z} &= \bar{u} - u_d && \text{in } I \times \Omega, \\ \bar{z} &= 0 && \text{on } I \times \partial\Omega, \\ \bar{z}(T) &= 0 && \text{in } \Omega. \end{aligned}$$

Optimality condition

$$\bar{q}(t) = P_{[q_a, q_b]}\left(-\frac{1}{\alpha}\bar{z}(t, x_0)\right) \quad \text{for almost all } t \in (0, T)$$

Regularity

two dimensional case

- $\bar{u} \in L^r(I; W_0^{1,s}(\Omega)) \cap W^{1,r}(I; W^{-1,s}(\Omega)), \quad r < \infty, s < 2$

three dimensional case

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  $\bar{z} \in L^r(I; W^{2,p}(B_d)) \cap W^{1,r}(I; L^p(B_d)), p < \infty, B_d = B_d(x_0), \overline{B_d} \subset \Omega$

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Outline

Finite element discretization in space and time

Discretization concept

- state/adjoint variable: dG(0) in time (implicit Euler), P_1 in space
- control variable: dG(0) in time
- maximal time step k, maximal spacial mesh size h

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Discretized optimal control problem

$$\text{Minimize } J(q_{kh}, u_{kh}) = \frac{1}{2}\|u_{kh} - u_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha}{2}\|q_{kh}\|^2_{L^2(0,T)}$$

subject to $u_{kh} \in X_{kh}$, $q_{kh} \in Q_{kh,\mathrm{ad}} = Q_{kh} \cap Q_{\mathrm{ad}}$

$$B(u_{kh}, \varphi_{kh}) = \int_0^T q_{kh}(t)\varphi_{kh}(t, x_0) \quad \text{for all } \varphi_{kh} \in X_{kh}.$$

$\to$ $B(v,\varphi) = \sum_{m=1}^{M} \langle v_t, \varphi\rangle_{I_m\times\Omega} + (\nabla v, \nabla\varphi)_{I\times\Omega} + \sum_{m=2}^{M}([v]_{m-1}, \varphi^+_{m-1})_\Omega + (v^+_0, \varphi^+_0)_\Omega$

A priori error estimates

Error estimates by Gong, Hinze, and Zhou

- two dimension: with $k \leq ch^2$

$$\|\bar{q} - \bar{q}_{kh}\| \leq C(k^{\frac{1}{2}} + h)$$

- three dimensions: with $k \leq ch^3$

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Our error estimates

- two dimension: with no k–h coupling

$$\|\bar{q} - \bar{q}_{kh}\| \leq C\, l_h(k + h^2), \quad l_h = |\ln h|^{\frac{7}{2}}$$

- three dimensions: with no k–h coupling

$$\|\bar{q} - \bar{q}_{kh}\| \leq \, C l_{kh}(k^{\frac{1}{2}} + h), \quad l_{kh} = |\ln k|^6 |\ln h|^2$$

Required estimates (three dimensional case)

From coercivity and optimality systems . . .

$$\alpha\|\bar{q}-\bar{q}_h\| \leq \left(\int_0^T \big(\bar{z}(t,x_0)-\hat{z}_{kh}(t,x_0)\big)^2\,dt\right)^{\frac{1}{2}} \quad \text{with} \quad \hat{z}_{kh} = z_{kh}(u_{kh}(\bar{q})).$$

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- Decompose $\bar{z} - \hat{z}_{kh} = \bar{z} - \tilde{z}_{kh} + \tilde{z}_{kh} - \hat{z}_{kh}$ with $B(\varphi_{kh}, \bar{z} - \tilde{z}_{kh}) = 0$

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$$\alpha\|\bar q-\bar q_h\| \le \left(\int_0^T \left(\bar z(t,x_0)-\hat z_{kh}(t,x_0)\right)^2 dt\right)^{\frac12} \quad \text{with} \quad \hat z_{kh} = z_{kh}(u_{kh}(\bar q)).$$

- Decompose $\bar z - \hat z_{kh} = \bar z - \tilde z_{kh} + \tilde z_{kh} - \hat z_{kh}$ with $B(\varphi_{kh}, \bar z - \tilde z_{kh}) = 0$
- Require local error estimates for $\bar z - \tilde z_{kh}$ in $L^\infty(\Omega; L^2(I))$ norm

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- Require stability estimate for $\tilde{z}_{kh}-\hat{z}_{kh}$ leading to

$$\|u-u_{kh}\|_{L^2(I;L^{3/2}(\Omega))}$$

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- Require stability estimate for $\tilde z_{kh} - \hat z_{kh}$ leading to

$$\|u - u_{kh}\|_{L^2(I;L^{3/2}(\Omega))}$$

- Require error estimates in $L^2(I; L^{3/2}(\Omega))$ norm

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- Poisson equation

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- Local / interior best approximation (Schatz / Wahlbin)

$$|(v - v_h)(x_0)| \le c|\ln h| \inf_{\chi \in V_h} \|v - \chi\|_{L^\infty(B_d(x_0))} + cd^{-\frac{N}{2}} \|v - v_h\|_{L^2(\Omega)}$$

Global best approximation type results

- Heat equation

$$\begin{aligned} \partial_t v - \Delta v &= f && \text{in } (0,T) \times \Omega, \\ v &= 0 && \text{on } (0,T) \times \partial\Omega, \\ v(0) &= 0 && \text{in } \Omega \end{aligned}$$

- Galerkin approximtion $v_{kh} \in X_{kh}$

$$B(v - v_{kh}, \varphi_{kh}) = 0 \quad \forall \varphi_{kh} \in X_{kh}$$

Theorem

$$\|v - v_{kh}\|_{L^\infty(\Omega;L^2(I))} \le c l_{kh} \inf_{\chi \in X_{kh}} \left(\|v - \chi\|_{L^2(I;L^\infty(\Omega))} + \|v - \pi_k v\|_{L^2(I;L^\infty(\Omega))} \right)$$

Interior best approximation type results

- Let v be the solution of the heat equation
- Let $v_{kh} \in X_{kh}$ be its Galerkin approximation
- $x_0 \in \Omega$, $B_d = B_d(x_0)$ with $\overline{B_d} \subset \Omega$

Theorem

$$\left(\int_0^T \big(v(t,x_0) - v_{kh}(t,x_0)\big)^2 \, dt\right)^{\frac{1}{2}} \leq c l_{kh} \inf_{\chi \in X_{kh}} \Big\{ \|v - \chi\|_{L^2(I;L^\infty(B_d))}$$

$$+ d^{-\frac{N}{2}} \|v - \chi\|_{L^2(I;L^2(\Omega))} + d^{-\frac{N}{2}} h \|\nabla(v - \chi)\|_{L^2(I;L^2(\Omega))} + \dots \Big\}.$$

Tools to prove best approximation results

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Maximal parabolic regularity results

Continuous maximal parabolic regularity

Let $f \in L^r(I; L^p(\Omega))$ with $1 < r, p < \infty$ and let

$$\partial_t v - \Delta v = f, \quad v(0) = 0.$$

Then $\partial_t v, \Delta v \in L^r(I; L^p(\Omega))$ and

$$\|\partial_t v\|_{L^r(I;L^p(\Omega))} + \|\Delta v\|_{L^r(I;L^p(\Omega))} \leq c\|f\|_{L^r(I;L^p(\Omega))}.$$

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Discrete maximal parabolic regularity

Let $f \in L^r(I; L^p(\Omega))$ with $1 \leq r, p \leq \infty$ and let

$$B(v_{kh}, \varphi_{kh}) = (f, \varphi_{kh})_{I\times\Omega} \quad \text{for all} \quad \varphi_{kh} \in X_{kh}. \quad \text{Then,}$$

$$\left(\sum_{m=1}^{M} k_m \|k_m^{-1}[v_{kh}]_{m-1}\|_{L^p(\Omega)}^r\right)^{\frac{1}{r}} + \|\Delta_h v_{kh}\|_{L^r(I;L^p(\Omega))} \leq c|\ln k|\|f\|_{L^r(I;L^p(\Omega))}.$$

Maximal parabolic regularity in general norms

- Require a resolvent estimate for a norm $|||\cdot|||$ on V_h

$$\left|\left|\left|(z+\Delta_h)^{-1}\chi\right|\right|\right| \leq \frac{M_h}{|z|}|||\chi|||, \quad \text{for } z \in \mathbb{C} \setminus \{\, z \in \mathbb{C} \mid |\arg(z)| \leq \gamma \,\}$$

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Discrete maximal parabolic regularity

$$\left(\sum_{m=1}^{M} k_m \left|\left|\left| k_m^{-1}[v_{kh}]_{m-1} \right|\right|\right|^s + \int_I |||\Delta_h v_{kh}|||^s dt\right)^{\frac{1}{s}} \le C M_h |\ln k| \left(\int_I |||f|||^s dt\right)^{\frac{1}{s}}$$

Maximal parabolic regularity: tools for proof

- Smoothing estimates for homogeneous equation ($f = 0$)

$$\sup_{t \in I_m} \left(|||\partial_t u_{kh}(t)||| + |||\Delta_h u_{kh}(t)||| \right) + k_m^{-1} |||[u_{kh}]_{m-1}||| \leq \frac{CM_h}{t_m} |||P_h u_0|||,$$

- dG(r) discretization as subdiagonal Pade approximation
- Dunford-Taylor formula
- resolvent estimates
- generalization of Eriksson, Johnson, Larsson (Part VI) 1998

Outline

Conclusions and Extensions

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  - Discrete maximal parabolic results
  - Global and local (almost) best approximations in $L^\infty(\Omega; L^2(I))$
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Extensions

- Numerical analysis for sparse control problem in $\mathcal{M}(\Omega; L^2(0,T))$
- Global and local (almost) best approximations in $L^\infty(I \times \Omega)$
  - Application to parabolic control problems with state constraints
- Global and local (almost) best approximations in $L^\infty(I; W^{1,\infty}(\Omega))$
  - Application to parabolic control problems with gradient constraints