A problem of optimal feedback for time-periodic patterns

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Joint work with P. Nestler and E. Schöll

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Outline

Pyragas feedback control

Control system (Schlögl, Nagumo, ...)

$$\begin{aligned} \partial_t y - \Delta y + R(y) &= u && \text{in } Q := \Omega \times (0,T) \\ y(\cdot,0) &= \Phi(\cdot), && \text{in } \Omega \\ \partial_n y &= 0 && \text{in } \Sigma := \partial\Omega \times (0,T) \end{aligned}$$

with cubic reaction term $R(y) = \rho\,(y - y_1)(y - y_2)(y - y_3)$,
$\rho > 0,\; y_1 \le y_2 \le y_3.$

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Pyragas feedback: Let $\tau > 0$ and $\Phi : \Omega \times [-\tau, 0] \to \mathbb{R}$ be given,

$$
\begin{aligned}
\partial_t y(t) - \Delta y(t) + R(y(t)) &= \kappa\,(y(t-\tau) - y(t)), && t > 0, \\
y(\cdot, s) &= \Phi(\cdot, s), && s \in [-\tau, 0].
\end{aligned}
$$

If y has period τ, then the feedback vanishes. This type of feedback is very popular in Theoretical Physics. $\longrightarrow$ Lasers

Nonlocal Pyragas feedback

Let now $\Phi : \Omega \times [-T, 0] \to \mathbb{R}$,

$$\begin{aligned} \partial_t y(t) - \Delta y(t) + R(y(t)) &= \kappa \left(\int_0^T g(\tau) y(t-\tau) d\tau - y(t) \right), \quad & t > 0, \\ y(\cdot, s) &= \Phi(\cdot, s), & s \in [-\tau, 0], \end{aligned}$$

with given feedback gain κ and a kernel $g \in L^\infty(0, T)$ that satisfies

$$0 \leq g(t) \leq \beta \quad \text{a.e. in } [0, T]$$

$$\int_0^T g(\tau)\, d\tau = 1.$$

Some references

K. Pyragas,
Continuous control of chaos by self-controlling feedback.
Phys. Rev. Lett. 1992.

J. Löber, R. Coles, J. Siebert, H. Engel, E. Schöll,
Control of chemical wave propagation in Engineering of Chemical Complexity II.
A. S. Mikhailov, G. Ertl (Eds.), World Scientific, Singapore, 2014.

E. Schöll, H.G. Schuster,
Handbook of Chaos Control.
Wiley-VCH (2008).

Many members of the optimal control community considered problems with time-delay.

Outline

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Depending on the chosen feedback kernel g, different solutions y are generated.

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$$\Omega = (0, 200),\ T = 400, \quad \Phi(x,t) := \frac{1}{2}\left(1 - \tanh\left(\frac{x - vt}{2\sqrt{2}}\right)\right)$$

"Weak gamma delay kernel" $g(t) = e^{-t}$, $y_1 = 0,\ y_3 = 1$

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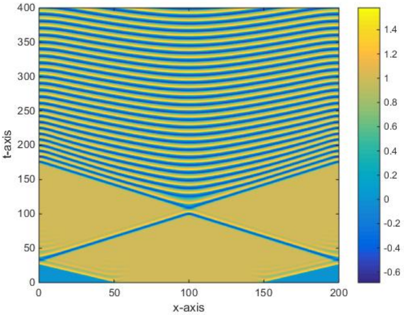
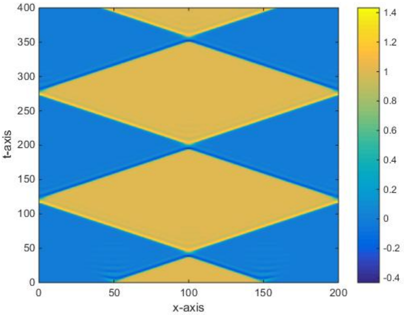


$y_2 = 0.25, \; \kappa = -1.65$



$y_2 = 0.5, \; \kappa = -1.4$

Well-posedness of the problem

Theorem (Control-to-state mapping)

For each $g \in L^\infty(0,T)$, $\Phi \in C(\bar\Omega \times [-T,0])$, the feedback equation has a unique weak solution $y_g \in C(\overline{Q}) \cap W(0,T)$. The mapping $G : g \mapsto y_g$ is of class C^∞.

Idea of the proof:

$$\begin{aligned}
(\partial_t y - \Delta y + R(y) + \kappa\, y)(t) &= \kappa \int_0^T g(\tau) y(t-\tau)\, d\tau \\
&= \kappa \int_0^t g(\tau) y(\underbrace{t-\tau}_{s})\, d\tau + \underbrace{\kappa \int_t^T g(\tau)\Phi(t-\tau)\, d\tau}_{\Phi_g(t)} \\
&= \kappa \int_0^t g(t-s) y(x,s)\, ds + \Phi_g(t).
\end{aligned}$$

Now we proceed as in Casas, Ryll, T., CMAM 2014. $t \to \infty$?

Optimal feedback design problem (P1)

Design problem: Find a kernel g such that the solution y_g associated with g is as close as possible to a desired function $\widehat{y}$.

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$$\min_{g\in C} F(g) := \frac{1}{2}\|y_g - \widehat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|g\|^2_{L^2(0,T)} \qquad (P1)$$

where y_g solves

$$\begin{cases} \partial_t y(t) - \Delta y(t) + R(y(t)) &= \kappa \displaystyle\int_0^T g(\tau) y(t-\tau)\, d\tau - \kappa\, y(t) & \text{in } Q, \\ y(s) &= \Phi(s), \quad s \in [-T, 0], & \text{in } \Omega \\ \partial_n y(t) &= 0 & \text{in } \Sigma, \end{cases}$$

and $C = \{g \in L^\infty(0,T) : 0 \le g(\tau) \le \beta, \quad \displaystyle\int_0^T g(\tau)\, d\tau = 1\}.$

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and $C = \{ g \in L^\infty(0,T) : 0 \le g(\tau) \le \beta, \quad \displaystyle\int_0^T g(\tau)\, d\tau = 1 \}$.

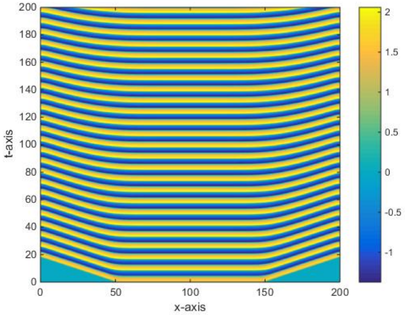
Theorem

There exists at least one optimal "control" $\bar{g}$ for this problem.

Outline

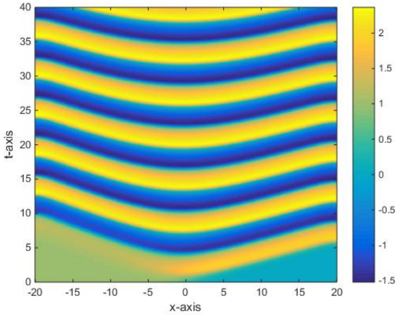
Oscillating states

For various feedback kernels g, states y_g are obtained that look like time-periodic patterns.



$y_1 = y_2 = 0,\ y_3 = 1, \kappa = -2$

$g(t) = t\, e^{-t}$



$y_1 = 0,\ y_2 = 0.25,\ y_3 = 1,\ \kappa = -2.43$

$$g(\tau) = \begin{cases} \dfrac{1}{t_2 - t_1}, & t_1 \le \tau \le t_2, \\ 0, & \text{else.} \end{cases}$$

$t_1 = 0,\ t_2 = 3$

A simple ODE with delay

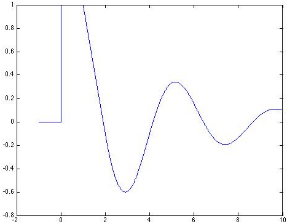
$$
\begin{aligned}
x'(t) &= {\color{red}\kappa}\, x(t-1), \quad t > 0 \\
x(s) &= \Phi(s), \qquad -1 \le s \le 0,
\end{aligned}
$$

$$\Phi(s) = 0,\ s < 0, \quad \Phi(0) = 1.$$

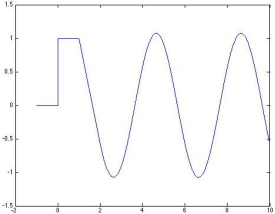
T. Erneux,

Applied delay differential equations,

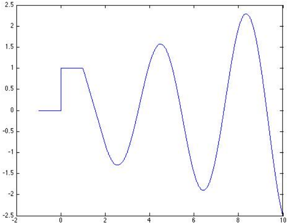
Springer, 2009



$\kappa = -1.8$, $\Phi(0) = 1$, $\Phi(s) = 0$, $s < 0$



$\kappa = -\pi/2$



$\kappa = -1.1$

A linear ODE with delay

We follow

Bellman, R.
A survey of the mathematical theory of time-lag, retarded control, and hereditary processes,
With the assistance of John M. Danskin, Jr.
The Rand Corporation,1954

Hale, J. K., Verduyn Lunel, S.M.
Introduction to functional differential equations,
Springer,1993

Consider the

ODE with delay

$$
\begin{aligned}
x'(t) + a\,x(t) &= \kappa\, x(t-\tau), \quad t > 0,\\
x(s) &= \Phi(s), \quad s \in [-\tau, 0],
\end{aligned}
$$

with given a, $\kappa \in \mathbb{R}$ and $\Phi \in C^1[-\tau, 0]$.

The fundamental solution

For the variation of constants formula, we need the

Fundamental solution X

$$\begin{aligned} X'(t) + a\,X(t) &= \kappa\,X(t-\tau), \quad t > 0, \\ X(0) &= 1 \\ X(s) &= 0, \quad s < 0. \end{aligned}$$

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Theorem (Variation of constants formula)

If $\Phi \in C^1[-\tau, 0]$, then the linear delay equation above attains a unique differentiable solution x. It is given by

$$x(t) = X(t)\Phi(0) + \int_{-\tau}^{0} X(t-s-\tau)\Phi(s)\,ds$$

cf. Bellman or Hale/Verduyn Lunel.

Series expansion of X

The characteristic equation for $\lambda \in \mathbb{C}$

$$\lambda + a - \kappa\, e^{-\lambda\tau} = 0,$$

has only countably many zeros λ_j, $j \in \mathbb{N}_0$, that can be ordered with respect to decreasing real parts so that

$$Re\,\lambda_j \geq Re\,\lambda_{j+1} \quad \forall j \in \mathbb{N}_0.$$

The values λ_j appear in conjugate complex pairs.

Linear PDE case – Fourier method

For convenience, we take $\Omega = (0,1)$ and consider the case of the linear local Pyragas feedback.

Pyragas type feedback equation

$$
\begin{aligned}
\partial_t y(t) - \partial_{xx} y(t) + (a+\kappa)\, y(t) &= \kappa\, y(t-\tau), && t > 0,\\
\partial_x y(t) &= 0, && t > 0, x \in \{0,1\},\\
y(\cdot, s) &= \Phi(\cdot, s) && s \in [-\tau, 0].
\end{aligned}
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Eigenvalues μ_n, normalized eigenfunctions w_n:

$$\left\{ \begin{array}{rcll} -w_n''(x) & = & \mu_n\, w_n(x), & x \in (0,1), \\ \partial_x w_n(x) & = & 0, & x \in \{0,1\}, \end{array} \right.$$

$$\mu_n = n^2\pi^2, \quad w_n(x) = \sqrt{2}\cos(n\pi x), \quad n \in \mathbb{N}_0.$$

Fourier ansatz: $$y(x,t) = \sum_{n=0}^{\infty} y_n(t) w_n(x).$$

Family of ODEs with delay

$$\left\{ \begin{array}{rcll} y_n'(t) + a_n\, y_n(t) & = & \kappa\, y_n(t-\tau), & t > 0 \\ y_n(s) & = & \Phi_n(s), & s \in [-\tau, 0], \end{array} \right.$$

where $a_n = a + \kappa + \mu_n, \quad \Phi_n(s) = \int_0^1 \Phi(x,s)\, w_n(x)\, dx.$

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where $a_n = a + \kappa + \mu_n$, $\quad \Phi_n(s) = \int_0^1 \Phi(x,s)\, w_n(x)\, dx$.

Let $\{\, Y_n(t)\}$ denote the family of associated fundamental solutions. Then

$$Y_n(t) = \sum_{j=0}^{\infty} \frac{e^{\lambda_{n,j}\, t}}{1 + \kappa\, e^{-\lambda_{n,j}}} \qquad \text{(real function!)}$$

and

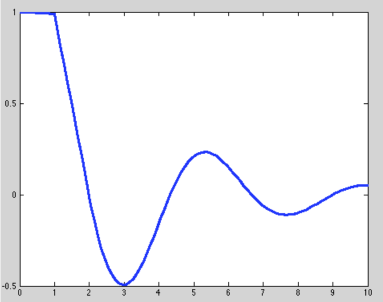
$$y_n(t) = Y_n(t)\Phi_n(0) + \int_{-\tau}^{0} Y_n(t-s-\tau)\, \Phi_n(s)\, ds.$$

Characteristic equations: $\quad \lambda + a_n - \kappa\, e^{-\lambda\, \tau} = 0.$

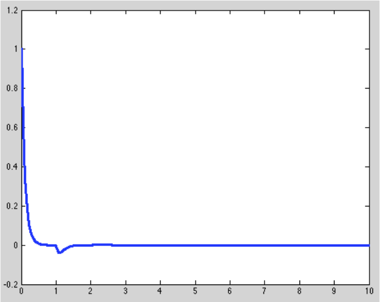
Roots: $\quad \lambda_{n,j} \in \mathbb{C}, \quad j \in \mathbb{N}_0.$

Examples of fundamental solutions

We take $a = 1 + 0.01, \kappa = -1$, i.e. $a_n = 0.01 + n^2\pi^2$, $n \in \mathbb{N}_0$.



Fundamental solution Y_0



Fundamental solution Y_1.

For $t \geq t_0 > 0$, the functions Y_n decay rapidly as $n \to \infty$.

Zeros of the characteristic equation

For each $n \in \mathbb{N}_0$, the zeros $\lambda_{n,j},\ j \in \mathbb{N}_0$, appear in pairs of conjugate complex numbers and can be ordered with respect to decreasing real parts, we have w.l.o.g

$$\lambda_{n,j} = \begin{cases} r_{n,j}, & j = 2k+1 \\ \overline{r_{n,j}}, & j = 2k, \end{cases} \qquad k \in \mathbb{N}_0.$$

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Lemma

Assume that $\kappa < 0$. Then, for each $n \in \mathbb{N}$, the zeros of the characteristic equations obey

$$\operatorname{Re} r_{n,j} > \operatorname{Re} r_{n,j+1} \quad \forall j \in \mathbb{N}_0.$$

Proof: Follows from the characteristic equations by a careful discussion of the zeros v of the equation

$$0 = v + c\, e^{v \cot v} \sin v.$$

Growth Assumption: $\operatorname{Re} r_{n,j} > \operatorname{Re} r_{n+1,j} \quad \forall n \in \mathbb{N}_0.$

Fourier expansion of $y(\cdot, t)$

Fourier expansion for $y(x,t)$:

$$y(x,t) = \sum_{n=0}^{\infty} \left(Y_n(t)\Phi_n(0) + \int_{-\tau}^{0} Y_n(t-s-\tau)\Phi_n(s)\,ds \right) w_n(x), \quad t \geq \tau.$$

By the Lemma ($j = 0$ dominates) with $u_n = \text{Re}\ \lambda_{n,0}$, $v_n = \text{Im}\ \lambda_{n,0}$

$$Y_n(t) \sim 2\, e^{u_n t} \frac{\cos(v_n\, t) + \kappa e^{-u_n} \cos(v_n\,(t+1))}{1 + 2\kappa e^{-u_n} cos(v_n) + \kappa^2\, e^{-2u_n}} =: \eta_n(t), \quad t \to \infty.$$

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Next, the growth assumption yields ($n = 0$ dominates)

$$\begin{aligned} y(x,t) &\sim \sum_{n=0}^{\infty} \left(\eta_n(t)\Phi_n(0) + \int_{-\tau}^{0} \eta_n(t-s-\tau)\Phi_n(s)\,ds \right) w_n(x) \\ &\sim \underbrace{\left(\eta_0(t)\Phi_0(0) + \int_{-\tau}^{0} \eta_0(t-s-\tau)\Phi_0(s)\,ds \right)}_{\textit{oscillating if not identically vanishing}} w_0(x) \end{aligned}$$

Oscillatory behavior

Let Φ be in $C^1([-\tau, 0], L^\infty(\Omega))$. If $\kappa < 0$, the growth assumption is fulfilled and

$$t \mapsto \eta_0(t)\Phi_0(0) + \int_{-\tau}^{0} \eta_0(t - s - \tau)\Phi_0(s)\, ds$$

is not identically vanishing, then $y(x, t)$ is oscillatory as $t \to \infty$.

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is not identically vanishing, then $y(x, t)$ is oscillatory as $t \to \infty$.

- Our case of "integral pyragas" can be discussed similarly, but it seems to be even more difficult to estimate the behavior of the $\lambda_{n,j}$. Plots of the eigenvalues indicate an encouraging behavior.
- The nonlinear equation stabilizes patterns that, in the linear case, would be unstable. The discussion is open.
- Our result for the local Pyragas complies with a result of Bainov and Mishev, who show that the solution is oscillating according to their definition.

Bainov, D.D., Mishev, D.P.
Oscillation theory for neutral differential equations with delay,
A. Hilger, 1991

Outline

Optimal feedback design for periodic target functions

We recall the

Optimal control problem (P1):

$$\min_{g\in C} F(g) := \frac{1}{2}\|y_g - \widehat{y}\|^2_{L^2(Q)} + \frac{\nu}{2}\|g\|^2_{L^2(0,T)}$$

$$\left\{\begin{array}{rcll} \partial_t y(t) - \Delta y(t) + R(y(t)) & = & \kappa \displaystyle\int_0^T g(\tau) y(t-\tau) d\tau - \kappa\, y(t) & \text{in } Q, \\ y(s) & = & \Phi(s), \quad s \in [-T, 0], & \text{in } \Omega \\ \partial_n y(t) & = & 0 & \text{in } \Sigma, \end{array}\right.$$

where $C = \{g \in L^\infty(0,T) : 0 \le g(\tau) \le \beta, \quad \int_0^T g(\tau)\, d\tau = 1\}.$

Adjoint equation

The reduced functional F is differentiable. The necessary condition for an optimal $\bar{g}$ is

$$F'(\bar{g})(g - \bar{g}) \geq 0 \quad \forall g \in C.$$

Adjoint equation

For given $g \in C$, the associated adjoint state φ_g is defined as the solution to

$$\begin{aligned} -\partial_t\varphi(t) - \Delta\varphi(t) + R'(y_g(t))\,\varphi(t) \\ = \kappa \int_0^T g(\tau)\,\varphi(t+\tau)\,d\tau - \kappa\,\varphi(t) + y_g(x,t) - \widehat{y}(t), \end{aligned}$$

$$\varphi(s) = 0, \quad s \in [T, 2T],$$

$$\partial_n\varphi(t) = 0, \quad t \in (0, T).$$

First order necessary optimality conditions

There holds

$$F'(g)h = \int_0^T \left[\int_0^T \int_\Omega \varphi_g(x,\tau) y_g(x,\tau - t) dx d\tau + \nu\, g(t) \right] h(t)\, dt.$$

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The integral constraint $\int_0^T g(t)\, dt = 1$ is a linear and regular constraint.

Theory of Lagrange multiplier rules in Banach spaces $\Longrightarrow$

Theorem (Necessary optimality conditions)

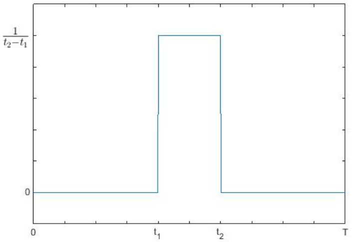
Let $\bar{g} \in C$ be the optimal kernel for the problem of optimal feedback design. Then there exists a Lagrange multiplier $\bar{\mu} \in \mathbb{R}$ such that

$$\int_0^T \left[\int_0^T \int_\Omega \varphi_{\bar{g}}(x,\tau) y_{\bar{g}}(x,\tau - t)\, dxd\tau + \bar{\mu} + \nu\, \bar{g}(t) \right] (g(t) - \bar{g}(t))\, dt \geq 0 \ \forall g \in C.$$

Class of step functions g

Select $0 \leq t_1 < t_2 \leq T$; $t_2 - t_1 \geq \delta > 0$

$$g(\tau) = \begin{cases} \dfrac{1}{t_2 - t_1}, & t_1 \leq \tau \leq t_2, \\ 0, & \text{else.} \end{cases}$$

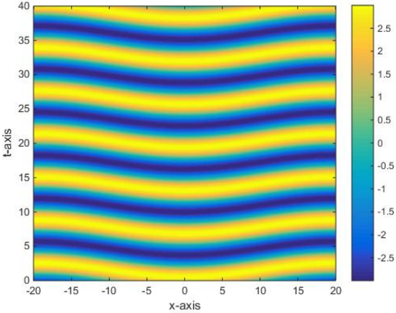


Here, κ, t_1, t_2 are our control parameters to be optimized.

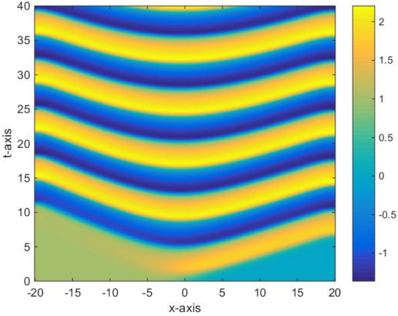
The integral restriction is automatically fulfilled.

$\Omega = (-20, 20)$, Time interval of observation: $[20, 40]$

$$\widehat{y}(x,t) = 3\,\sin(t - \cos(\tfrac{\pi}{20}(x+20))),$$



Desired pattern $\widehat{y}$

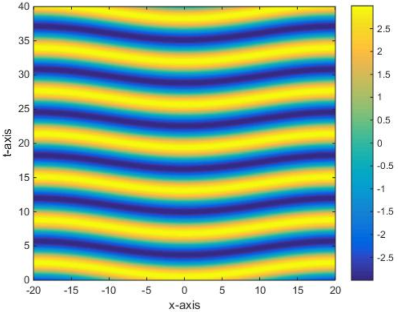


Optimal pattern

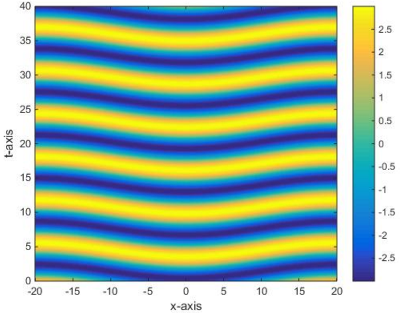
Computed optimal value: $F = 1.320e+03$, $|\nabla F(\kappa, t_1, t_2)| = 4.5e-02$

Two equivalent periodic patterns

Periodic patterns should be considered as equivalent, if they just differ by a time shift.



$$\widehat{y} = 3\sin\left(t - \cos\left(\frac{\pi}{20}(x+20)\right)\right)$$



$$w = 3\sin\left(t{\color{red}+3} - \cos\left(\frac{\pi}{20}(x+20)\right)\right)$$

The right pattern is a simple time shift of the left, but

$$\frac{1}{2}\iint_Q (\widehat{y} - w)^2\,dxdt \approx {\color{red}1.4374\mathrm{e}+04}$$

Re-definition of the objective function

Measure the difference of y to the closest time shift of $\widehat{y} : \Omega \times \mathbb{R} \to \mathbb{R}$:

Shifted objective function

$$f(g) := \min_{s} \iint_Q (y_g(x,t) - \widehat{y}(x, t+s))^2 \, dxdt.$$

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$$f(g) := \min_{s} \iint_Q (y_g(x,t) - \widehat{y}(x, t+s))^2 \, dxdt.$$

Assume that $\widehat{y}$ is $T/2$-periodic, define $\tilde{Q} := \Omega \times [T/2, T]$. For constant $\|y_g\|_{L^2(\tilde{Q})}$ and periodic $\widehat{y}$, the minimization of f is equivalent to

Optimal control problem (P2)

$$\min_{g \in C} f_{\mathrm{corr}}(g) := \min_{s \in [0, T/2]} \left(1 - \frac{\left(y_g\,,\, \widehat{y}(\cdot + s)\right)_{L^2(\tilde{Q})}}{\|y_g\|_{L^2(\tilde{Q})} \, \|\widehat{y}(\cdot + s)\|_{L^2(\tilde{Q})}} \right) \tag{P2}$$

Re-definition of the objective function

Measure the difference of y to the closest time shift of $\widehat{y} : \Omega \times \mathbb{R} \to \mathbb{R}$:

Shifted objective function

$$f(g) := \min_{s} \iint_{Q} (y_g(x,t) - \widehat{y}(x,t+s))^2 \, dxdt.$$

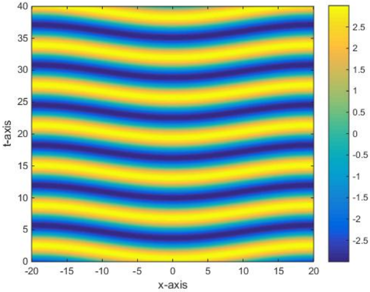
Assume that $\widehat{y}$ is $T/2$-periodic, define $\tilde{Q} := \Omega \times [T/2, T]$. For constant $\|y_g\|_{L^2(\tilde{Q})}$ and periodic $\widehat{y}$, the minimization of f is equivalent to

Optimal control problem (P2)

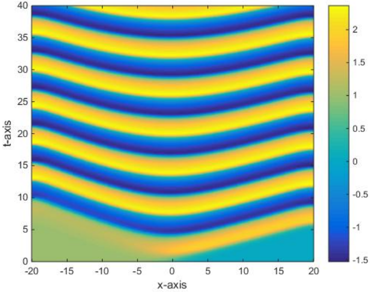
$$\min_{g \in C} f_{\mathrm{corr}}(g) := \min_{s \in [0, T/2]} \left(1 - \frac{(y_g \, , \, \widehat{y}(\cdot + s))_{L^2(\tilde{Q})}}{\|y_g\|_{L^2(\tilde{Q})} \, \|\widehat{y}(\cdot + s)\|_{L^2(\tilde{Q})}} \right) \tag{P2}$$

This idea of using the normalized cross correlation is a well-known tool in signal processing. It is also used in quantum control.

Outline



Desired pattern $\widehat{y}$



Optimal pattern

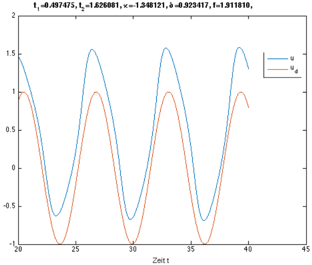
Computed optimal values:

$$f_{\text{corr}} = 0.1229,\ [t_1 = 0], t_2 = 3.0031,\ \kappa = -2.4318$$

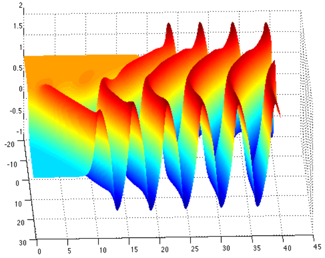
Example 3: Varying parameters κ, t_1, t_2, s

$\Omega = (-20, 20)$, time interval: $[20, 40]$, $\widehat{y}(0,t) = \sin(t-1)$,

$\Phi(x,t) := \frac{1}{2}\left(1 - \tanh\left(\frac{x - vt}{2\sqrt{2}}\right)\right)$, $\qquad f = \min_s \frac{1}{2}\int_{T/2}^{T} (y(0,t) - \widehat{y}(0,t+s))^2 dt$



$\widehat{y}(0,t)$ *and optimal* $y(0,t)$



Optimal y

Optimal solution:

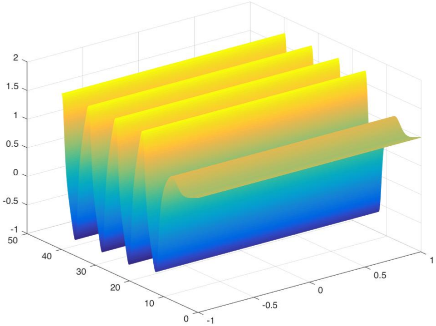
$\kappa = -1.348$, $t_1 = 0.4974$, $t_2 = 1.626$, $f = 1.0118$, shift $s = 0.9234$

Example 4 – reconstruct a given kernel g

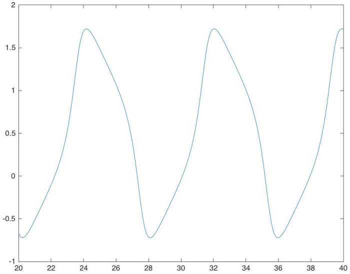
Data: Delay-time $T = 10$, $T_{end} = 40$, $\Omega = (-1, 1)$,

$y_1 = 0$, $y_2 = 0.25$, $y_3 = 1$, $\rho = 1$, $\kappa = -2$

$\Phi(x, t) = 1$; We define $\widehat{y}$ as the state associated with $g(t) = e^{-t}$.



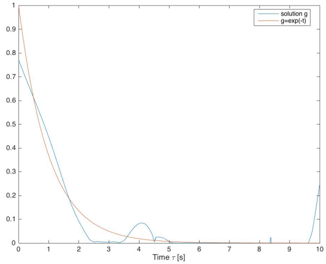
Desired $\widehat{y}$



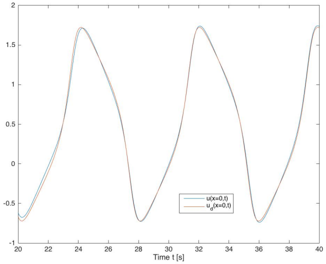
$\widehat{y}(0, t)$

Reconstructed g

Without regularization



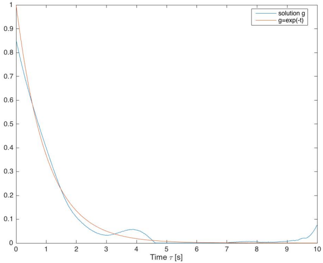
Optimal g



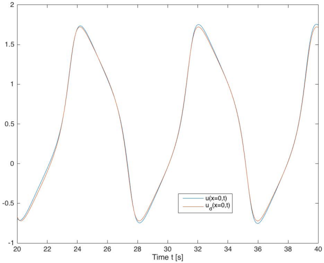
Optimal state

Example 5 – Optimal kernel with regularization

$\nu = 10^{-3}$



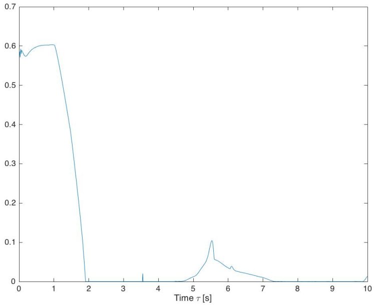
Optimal g



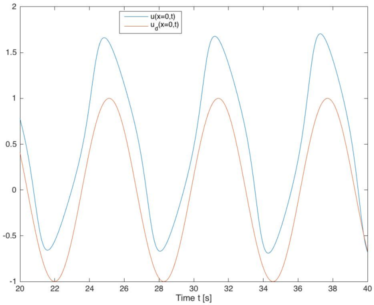
Optimal state

Example 6 – synthetic desired state $\widehat{y}$

Data as above Example 4, but $\widehat{y}(x,t) = \sin(t + \pi/2)$, $\nu = 0$.



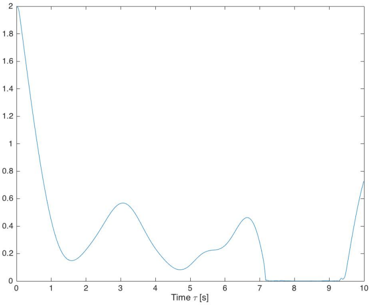
Optimal g



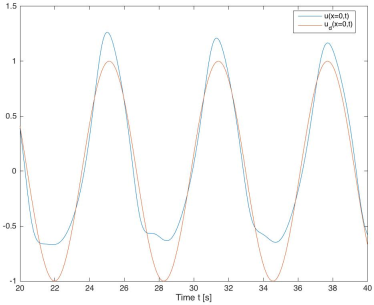
Optimal state

Example 7 – without integral constraint

Data as in the last example, but we skip the integral constraint $\int_0^T g(t)dt = 1$.



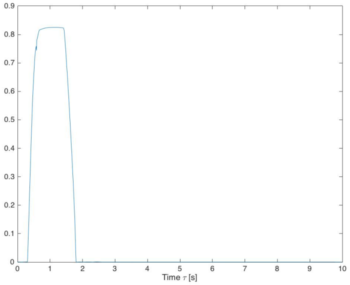
Optimal g



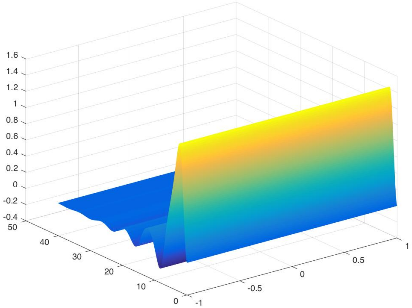
Optimal state

$\widehat{y} = 0$

$\Phi(x, t) = 1$, $\kappa = -1$, no regularization



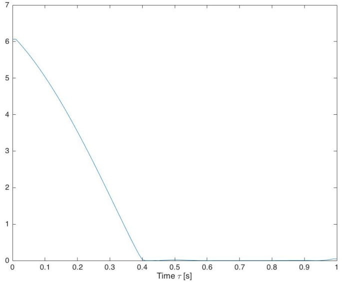
Optimal g



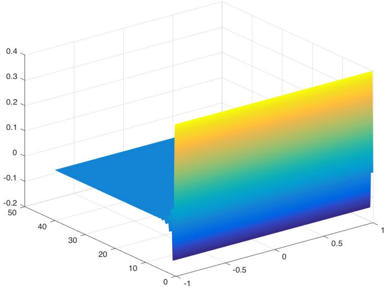
Optimal state

Example 9 – extinction of an incoming wave

$\Phi(x, t) = sin(3t)$, $\kappa = -5$, no integral constraint



Optimal g



Optimal state

Reference:

P. Nestler, E. Schöll, F.T.,
Optimization of nonlocal time-delayed feedback controllers
COAP 2015, doi 10.1007/s10589-015-9809-6.

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Optimization of nonlocal time-delayed feedback controllers
COAP 2015, doi 10.1007/s10589-015-9809-6.

Thank you