Optimal controls for gradient systems associated with grain boundary motions

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0. Content of the talk

1. State system associated with grain boundary motions

- ◇ The model of parabolic PDEs proposed by [Kobayashi–Warren–Carter](1999)
- ◇ There is only few uniqueness result for the state system, e.g. [Ito–Kenmochi–Yamazaki](2008): the result in 1D-case of the spatial domain
- ◇ In 1D case, we can show an example of the no-uniqueness situation
- ◇ The 1st **Main Theorem 1** is concerned with the key-properties of the 1D-state system, and the well-posedness of the approximating systems (with the uniqueness)

2. Optimal control problems

- ◇ Notions of admissibilities: for the controls; for the solutions (cf. [Colli–F.-Shaker–Gilardi–Sprekels] (2014))
- ◇ Setting of the optimal control problems, involving the state system with no-uniqueness (cf. [Kadoya–Kenmochi–Murase] (2010))
- ◇ The 2nd **Main Theorem 2** is concerned with the existence of optimal controls
- ◇ The 3rd **Main Theorem 3** is concerned with the association between the original optimal control problem and its approximating versions

1. State systems associated with grain boundary motions

System $(\mathrm{S})_\nu$ with $\nu \geq 0$: $0 < T < \infty$, $\Omega := (0,1)$, $\Gamma := \partial\Omega = \{0,1\}$

$$\begin{cases} \partial_t(u - L\eta) - \partial_x^2 u = f(t,x), \ \ (t,x) \in \ Q := (0,T) \times \Omega, \\ \partial_t \eta - \kappa \partial_x^2 \eta + \partial I_{[0,1]}(\eta) - (\eta - \frac{1}{2}) + \alpha'(\eta)|D\theta| = -u(t,x), \ (t,x) \in Q, \\ \alpha_0(\eta)\partial_t \theta - \partial_x \left(\alpha(\eta) \dfrac{D\theta}{|D\theta|} + \nu \partial_x \theta \right) = 0, \quad \text{in } Q, \\ (-1)^{k+1} \partial_x u + n_0(u - f_\Gamma(t,k)) = 0, \ (t,k) \in \Sigma := (0,T) \times \Gamma, \\ \partial_x \eta = 0 \ \text{ on } \Sigma, \ \theta(t,0) = 0 \text{ and } \theta(t,1) = \theta_\Gamma, \\ u(0,x) = u_0(x), \ \eta(0,x) = \eta_0(x), \ \theta(0,x) = \theta_0(x), \ \ x \in \Omega. \end{cases}$$

$\diamond$ $u = u(t,x)$: relative temperature, $\diamond$ $\eta = \eta(t,x)$: orientation degree in a polycrystal,
$\diamond$ $\theta = \theta(t,x)$: orientation angle, $0 \leq \eta(t,x) \leq 1$, $(t,x) \in Q$

- $\partial I_{[0,1]}$: subdifferential of the indicator function $I_{[0,1]}$ on $[0,1]$;
- $L > 0$, $n_0 > 0$, $\kappa > 0$, $\theta_\Gamma \in (0,\pi)$: given constants;
- $u_0 \in L^2(\Omega)$, $\eta_0 \in L^2(\Omega)$, $\theta_0 \in L^2(\Omega)$: initial data.

1. State systems associated with grain boundary motions

System $(\mathbf{S})_\nu$ with $\nu \geq 0$: $0 < T < \infty$, $\Omega := (0,1)$, $\Gamma := \partial\Omega = \{0,1\}$

$$\begin{cases} \partial_t(u - L\eta) - \partial_x^2 u = f(t,x), \;\; (t,x) \in Q := (0,T) \times \Omega, \\ \partial_t \eta - \kappa \partial_x^2 \eta + \partial I_{[0,1]}(\eta) - (\eta - \frac{1}{2}) + \alpha'(\eta)|D\theta| = -u(t,x), \; (t,x) \in Q, \\ \alpha_0(\eta)\partial_t\theta - \partial_x\left(\alpha(\eta)\dfrac{D\theta}{|D\theta|} + \nu\partial_x\theta \right) = 0, \quad \text{in } Q, \\ (-1)^{k+1}\partial_x u + n_0(u - f_\Gamma(t,k)) = 0, \; (t,k) \in \Sigma := (0,T) \times \Gamma, \\ \partial_x \eta = 0 \;\text{ on } \Sigma, \; \theta(t,0) = 0 \text{ and } \theta(t,1) = \theta_\Gamma, \\ u(0,x) = u_0(x), \; \eta(0,x) = \eta_0(x), \; \theta(0,x) = \theta_0(x), \;\; x \in \Omega. \end{cases}$$

- $f \in L^2(Q)$, $f_\Gamma \in L^2(\Sigma)$: heat sources (controls).

- $\alpha_0 = \alpha_0(\eta) \geq 0$: mobility (possibly degenerate);
- $\alpha = \alpha(\eta) > 0$: mobility, with the differential α' (no-degenerate).

Typical choice (cf. [Kobayashi–Warren–Carter](1999)):

$$\alpha_0(\eta) = \alpha(\eta) = \eta^2/2, \;\; \forall \eta \in \mathbb{R}$$

Non-isothermal system $(\mathbf{S})_\nu$ (coupling system of Fix-Caginalp type):

$$\left.\begin{cases} \partial_t(u - L\eta) - \partial_x^2 u = f \ \text{ in } Q, \\ -\begin{bmatrix} \partial_t \eta \\ \alpha_0(\eta)\partial_t\theta \end{bmatrix} = \nabla \mathscr{F}_\nu(u,\eta,\theta) \ \text{ in } Q, \end{cases}\right] \text{(B.C.)+(I.C.)}$$

Free-energy ($\nu > 0$) [Warren–Kobayashi–Lobkovsky–Carter](2003)

$$\boldsymbol{v} = [u,\eta,\theta] \in L^2(\Omega)^3 \mapsto \mathscr{F}_\nu(\boldsymbol{v}) := A_0 \int_\Omega |u|^2\,dx + \Psi_{[0,1]}(\eta)$$
$$-\frac{1}{2}\int_\Omega \left(\eta - \tfrac{1}{2}\right)^2 dx + \Phi_\nu(\eta;\theta) \in (-\infty,\infty].$$

– $A_0 > 0$: const. (depending on L and n_0);

– $\eta \in H^1(\Omega) \mapsto \Psi_{[0,1]}(\eta) := \dfrac{\kappa}{2}\displaystyle\int_\Omega |\partial_x \eta|^2\,dx + \int_\Omega I_{[0,1]}(\eta)\,dx;$

– $\forall \eta \in C(\overline{\Omega})$, $\theta \in H^1(\Omega) \mapsto \Phi_\nu(\eta;\theta) := \displaystyle\int_\Omega \alpha(\eta)|\partial_x\theta|\,dx + \frac{\nu}{2}\int_\Omega |\partial_x\theta|^2\,dx,$

subject to the boundary condition $\theta(0) = 0$ and $\theta(1) = \theta_\Gamma$.

Non-isothermal system $(\mathrm{S})_\nu$ (coupling system of Fix-Caginalp type):

$$\left\{\begin{array}{l} \partial_t(u - L\eta) - \partial_x^2 u = f \ \text{ in } Q, \\ -\begin{bmatrix} \partial_t \eta \\ \alpha_0(\eta)\partial_t\theta \end{bmatrix} = \nabla \mathscr{F}_\nu(u, \eta, \theta) \ \text{ in } Q, \end{array}\right] \quad \text{(B.C.)+(I.C.)}$$

Free-energy $(\nu = 0)$ [Warren–Kobayashi–Lobkovsky–Carter](2003)

$$\boldsymbol{v} = [u, \eta, \theta] \in L^2(\Omega)^3 \mapsto \mathscr{F}_0(\boldsymbol{v}) := A_0 \int_\Omega |u|^2 \, dx + \Psi_{[0,1]}(\eta)$$
$$-\frac{1}{2}\int_\Omega \left(\eta - \tfrac{1}{2}\right)^2 dx + \Phi_0(\eta; \theta) \in (-\infty, \infty].$$

- $A_0 > 0$: const. (depending on L and n_0);
- $\eta \in H^1(\Omega) \mapsto \Psi_{[0,1]}(\eta) := \dfrac{\kappa}{2}\displaystyle\int_\Omega |\partial_x \eta|^2 \, dx + \int_\Omega I_{[0,1]}(\eta) \, dx;$
- $\forall \eta \in C(\overline{\Omega})$, $\theta \in BV(\Omega) \mapsto \Phi_0(\eta; \theta) := \displaystyle\int_{\overline{\Omega}} \alpha(\eta)|D[\theta]^{\mathrm{ex}}|$, with the extension $[\theta]^{\mathrm{ex}} \in BV_{\mathrm{loc}}(\mathbb{R})$ s.t. $[\theta]^{\mathrm{ex}} \equiv 0$ on $(-\infty, 0]$ and $[\theta]^{\mathrm{ex}} \equiv \theta_\Gamma$ on $[1, \infty)$.

Approximating system $(\mathbf{S})_\varepsilon^\nu$ with $\nu \geq 0$ and $\varepsilon \in (0, \frac{1}{2}]$:

$$\begin{cases} \partial_t(u - L\eta) - \partial_x^2 u = f(t,x), \ \ (t,x) \in \ Q, \\ \partial_t \eta - \kappa \partial_x^2 \eta \ + \beta_\varepsilon(\eta) - (\eta - \frac{1}{2}) + \alpha'(\eta)\sqrt{|\partial_x\theta|^2 + \varepsilon^2} = -u(t,x), \ (t,x) \in Q, \\ \alpha_\varepsilon(\eta)\partial_t\theta - \partial_x \left(\alpha(\eta) \dfrac{\partial_x \theta}{\sqrt{|\partial_x\theta|^2 + \varepsilon^2}} + (\nu + \varepsilon)\partial_x\theta \right) = 0, \quad \text{in } Q, \\ \text{(B.C.) + (I.C.)} \end{cases}$$

- $\alpha_\varepsilon = \alpha_0 + \varepsilon$; • β_ε: Yosida resularization of $\partial I_{[0,1]}$;

Approximating free-energy:

$$\boldsymbol{v} = [u, \eta, \theta] \in L^2(\Omega)^3 \mapsto \mathscr{F}_\varepsilon^\nu(\boldsymbol{v}) := A_0 \int_\Omega |u|^2 \, dx + \Psi_{[0,1]}(\eta)$$
$$-\frac{1}{2} \int_\Omega \left(\eta - \tfrac{1}{2}\right)^2 dx + \Phi_\varepsilon^\nu(\eta; \theta) \in (-\infty, \infty].$$

– $\theta \in H^1(\Omega) \mapsto \Phi_\varepsilon^\nu(\eta;\theta) := \displaystyle\int_\Omega \alpha(\eta)\sqrt{|\partial_x\theta|^2 + \varepsilon^2}\, dx + \frac{\nu + \varepsilon}{2} \int_\Omega |\partial_x\theta|^2 \, dx$,
subject to the boundary condition $\theta(0) = 0$ and $\theta(1) = \theta_\Gamma$, $\forall \eta \in C(\overline{\Omega})$.

Assumptions.

(A1) $0 \leq \alpha_0 \in C^1(\mathbb{R}^2)$, s.t. $\alpha_0^{-1}(0) = \{0\}$.

(A2) $0 < \alpha \in C^2(\mathbb{R})$: convex, $\alpha'(0) = 0$, and $\delta_* := \inf \alpha(\mathbb{R}) > 0$.

◇ Notations

- $V := H^1(\Omega)$: Hilbert space, endowed with the inner product:

$$(v, z)_V := (\partial_x v, \partial_x z)_{L^2(\Omega)} + n_0(v, z)_{L^2(\Gamma)}, \ \forall [v, z] \in V^2.$$

- V^*: dual space of V, $F : V \longrightarrow V^*$ duality map.

Range for controls:

$$\mathcal{H} := L^2(Q) \times L^2(\Sigma)$$

†. We regard any $\boldsymbol{f} = [f, f_\Gamma] \in \mathcal{H}$, as $\boldsymbol{f} \in V^*$, via the identification:

$$\langle \boldsymbol{f}, z \rangle = (f, z)_{L^2(\Omega)} + n_0(f_\Gamma, z)_{L^2(\Gamma)}, \ \forall z \in V.$$

Range for solutions:

$$D_\nu := \left\{ \ \tilde{\boldsymbol{v}} = [\tilde{u}, \tilde{\eta}, \tilde{\theta}] \in L^2(\Omega)^3 \ \middle| \ \begin{array}{l} \tilde{\eta} \in H^1(\Omega), \ \tilde{\theta} \in D(\Phi_\nu(\tilde{\eta}; \cdot\,)) \\ 0 \leq \tilde{\eta} \leq 1 \text{ and } 0 \leq \tilde{\theta} \leq \theta_\Gamma \text{ a.e. in } \Omega \end{array} \right\}$$

Definition of solution to (S)$_\nu$: $\forall \nu \geq 0, \forall \boldsymbol{f} = [f, f_\Gamma] \in \mathcal{H}, \forall \boldsymbol{v}_0 = [u_0, \eta_0, \theta_0] \in D_\nu$, $\boldsymbol{v} = [u, \eta, \theta] \in L^2(Q)^3$ is called a solution to (S)$_\nu$, iff. the following conditions hold.

(S0) $u \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$, $u(0) = u_0$ in $L^2(\Omega)$, $\eta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \subset C(\overline{Q})$, $\eta(0) = \eta_0$ in $L^2(\Omega)$, $\alpha_0(\eta)\theta \in W^{1,2}(0,T;L^2(\Omega))$, $\Phi_\nu(\eta;\theta) \in L^\infty(0,T)$, $\alpha_0(\eta)\theta(0) = \alpha(\eta_0)\theta_0$ in $L^2(\Omega)$.

$$\theta^* := \partial_t[\alpha_0(\eta)\theta] - \partial_t[\alpha_0(\eta)]\theta \in L^2(Q), \quad \theta^* = \alpha_0(\eta)\partial_t\theta \text{ in } Q \setminus \eta^{-1}(0).$$

(S1) u solves the following evolution equation:

$$\partial_t\bigl(u - L\eta\bigr)(t) + Fu(t) = \boldsymbol{f}(t) \text{ in } V^*, \text{ a.e. } t \in (0,T).$$

(S2) η solves the following variational inequality:

$$\left(\partial_t\eta(t) - (\eta(t) - \tfrac{1}{2}), \eta(t) - \varphi\right)_{L^2(\Omega)} + \Psi_{[0,1]}(\eta(t)) - \Psi_{[0,1]}(\varphi) + \int_{\overline{\Omega}} (\eta(t) - \varphi)\alpha'(\eta(t))|D[\theta(t)]^{\mathrm{ex}}| \leq 0, \quad \forall \varphi \in H^1(\Omega), \text{ a.e. } t \in (0,T).$$

(S3) θ solves the following evolution equation:

$$\theta^*(t) + \partial\Phi_\nu(\eta(t);\theta(t)) \ni 0, \text{ in } L^2(\Omega), \text{ a.e. } t \in (0,T),$$

where $\forall \tilde{\eta} \in H^1(\Omega)$, $\partial\Phi_\nu(\tilde{\eta};\cdot)$ denotes L^2-subdifferential of $\Phi_\nu(\tilde{\eta};\cdot)$.

Definition of solution to $(\mathbf{S})_\varepsilon^\nu$: $\forall \nu \geq 0$, $\forall \varepsilon \in (0, \frac{1}{2}]$, $\forall \boldsymbol{f} = [f, f_\Gamma] \in \mathcal{H}$, $\forall \boldsymbol{v}_0 = [u_0, \eta_0, \theta_0] \in D_\nu$, $\boldsymbol{v} = [u, \eta, \theta] \in L^2(Q)^3$ is called a solution to $(\mathrm{S})_\varepsilon^\nu$, iff. the following conditions hold.

$(\mathrm{S0})_\varepsilon$ $u \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$, $u(0) = u_0$ in $L^2(\Omega)$,

$\eta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \subset C(\overline{Q})$, $\eta(0) = \eta_0$ in $L^2(\Omega)$,

$\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \subset C(\overline{Q})$, $\theta(0) = \theta_0$ in $L^2(\Omega)$.

$(\mathrm{S1})_\varepsilon$ u solves the following evolution equation:

$$\partial_t \big(u - L\eta\big)(t) + Fu(t) = \boldsymbol{f}(t) \ \text{ in } V^*, \text{ a.e. } t \in (0,T).$$

$(\mathrm{S2})_\varepsilon$ η solves the following variational identity:

$$\begin{aligned}&(\partial_t\eta(t) - (\eta(t) - \tfrac{1}{2}), \varphi)_{L^2(\Omega)} + \kappa(\partial_x\eta(t), \partial_x\varphi)_{L^2(\Omega)} + (\beta_\varepsilon(\eta(t)), \varphi)_{L^2(\Omega)}\\ &+(\alpha'(\eta(t))\sqrt{|\partial_x\theta(t)|^2 + \varepsilon^2}, \varphi)_{L^2(\Omega)} = 0, \ \forall \varphi \in H^1(\Omega), \text{ a.e. } t \in (0,T).\end{aligned}$$

$(\mathrm{S3})_\varepsilon$ θ solves the following evolution equation:

$$\alpha_\varepsilon(\eta(t))\partial_t\theta(t) + \partial\Phi_\varepsilon^\nu(\eta(t);\theta(t)) \ni 0, \text{ in } L^2(\Omega), \text{ a.e. } t \in (0,T),$$

where $\forall \tilde{\eta} \in H^1(\Omega)$, $\partial\Phi_\varepsilon^\nu(\tilde{\eta};\cdot)$ denotes L^2-subdifferential of $\Phi_\nu(\tilde{\eta};\cdot)$.

Main Theorem 1 (Key-properties of solutions). (I) $\forall \nu \geq 0$, let us define:

$$S_\nu : \mathcal{H} \times D_\nu \ni [\boldsymbol{f}, \boldsymbol{v}_0] \mapsto \left\{ \boldsymbol{v} = [u, \eta, \theta] : \text{ solution to } (\mathrm{S})_\nu \right\} \subset L^2(Q)^3.$$

Then, $\forall [\boldsymbol{f}, \boldsymbol{v}_0] \in \mathcal{H} \times D_\nu$, $S_\nu(\boldsymbol{f}, \boldsymbol{v}_0) \neq \emptyset$, NOT singleton in general, and moreover:

$$\begin{cases} \nu_n \to \nu,\ \boldsymbol{f}_n \to \boldsymbol{f} \text{ in } \mathcal{H},\ \boldsymbol{v}_{0,n} \to \boldsymbol{v}_0 \text{ in } L^2(\Omega)^3, \text{ as } n \to \infty, \\ \{\mathscr{F}_{\nu_n}(\boldsymbol{v}_{0,n})\}: \text{b.d.d., and } \boldsymbol{v}_n = [u_n, \eta_n, \theta_n] \in S_{\nu_n}(\boldsymbol{f}_n, \boldsymbol{v}_{0,n}), \forall n \in \mathbb{N}, \end{cases}$$

$$\Longrightarrow \exists \{n_k\} \subset \{n\}, \quad \exists \boldsymbol{v} = [u, \eta, \theta] \in S_\nu(\boldsymbol{f}, \boldsymbol{v}_0), \text{ s.t.}$$

$$\begin{cases} \boldsymbol{v}_{n_k} = [u_{n_k}, \eta_{n_k}, \theta_{n_k}] \to \boldsymbol{v} = [u, \eta, \theta] \text{ weakly-}* \text{ in } L^\infty(0, T; L^2(\Omega)), \\ [u_{n_k}, \eta_{n_k}, \alpha_0(\eta_{n_k})\theta_{n_k}] \to [u, \eta, \alpha_0(\eta)\theta] \text{ in } L^2(Q) \times C(\overline{Q}) \times C([0, T]; L^2(\Omega)), \\ \sqrt{\nu_{n_k}}\theta_{n_k} \to \sqrt{\nu}\theta \text{ in } L^2(0, T; H^1(\Omega)), \text{ as } k \to \infty. \end{cases}$$

(II) $\forall \nu \geq 0, \forall \varepsilon \in (0, \frac{1}{2}]$, let us define:

$$S_\varepsilon^\nu : \mathcal{H} \times D_\nu \ni [\boldsymbol{f}, \boldsymbol{v}_0] \mapsto \left\{ \boldsymbol{v} = [u, \eta, \theta] : \text{ solution to } (\mathrm{S})_\varepsilon^\nu \right\} \subset L^2(Q)^3.$$

Then, $\forall [\boldsymbol{f}, \boldsymbol{v}_0] \in \mathcal{H} \times D_\nu$, $S_\varepsilon^\nu(\boldsymbol{f}, \boldsymbol{v}_0)$ be a singleton and continuous w.r.t. the strong topologies from $\mathcal{H} \times L^2(\Omega)^3$ into $L^2(Q) \times C(\overline{Q}) \times C([0, T]; L^2(\Omega))$.

†. (I) is obtained by the observation of approximating limit $\varepsilon \to 0$, based on (II).

Main Theorem 1 (Key-properties of solutions). (I) $\forall \nu \geq 0$, let us define:

$$S_\nu : \mathcal{H} \times D_\nu \ni [\boldsymbol{f}, \boldsymbol{v}_0] \mapsto \big\{ \boldsymbol{v} = [u, \eta, \theta] : \text{ solution to } (\mathrm{S})_\nu \big\} \subset L^2(Q)^3.$$

Then, $\forall [\boldsymbol{f}, \boldsymbol{v}_0] \in \mathcal{H} \times D_\nu$, $S_\nu(\boldsymbol{f}, \boldsymbol{v}_0) \neq \emptyset$, NOT singleton in general, and moreover:

$$\begin{cases} \nu_n \to \nu, \ \boldsymbol{f}_n \to \boldsymbol{f} \text{ in } \mathcal{H}, \ \boldsymbol{v}_{0,n} \to \boldsymbol{v}_0 \text{ in } L^2(\Omega)^3, \text{ as } n \to \infty, \\ \{\mathscr{F}_{\nu_n}(\boldsymbol{v}_{0,n})\}: \text{b.d.d., and } \ \boldsymbol{v}_n = [u_n, \eta_n, \theta_n] \in S_{\nu_n}(\boldsymbol{f}_n, \boldsymbol{v}_{0,n}), \forall n \in \mathbb{N}, \end{cases}$$

$$\Longrightarrow \ \exists \{n_k\} \subset \{n\}, \quad \exists \boldsymbol{v} = [u, \eta, \theta] \in S_\nu(\boldsymbol{f}, \boldsymbol{v}_0), \ \text{ s.t.}$$

$$\begin{cases} \boldsymbol{v}_{n_k} = [u_{n_k}, \eta_{n_k}, \theta_{n_k}] \to \boldsymbol{v} = [u, \eta, \theta] \text{ weakly-}* \text{ in } L^\infty(0, T; L^2(\Omega)), \\ [u_{n_k}, \eta_{n_k}, \alpha_0(\eta_{n_k})\theta_{n_k}] \to [u, \eta, \alpha_0(\eta)\theta] \text{ in } L^2(Q) \times C(\overline{Q}) \times C([0, T]; L^2(\Omega)), \\ \sqrt{\nu_{n_k}}\theta_{n_k} \to \sqrt{\nu}\theta \text{ in } L^2(0, T; H^1(\Omega)), \ \text{ as } k \to \infty. \end{cases}$$

(II) $\forall \nu \geq 0, \forall \varepsilon \in (0, \frac{1}{2}]$, let us define:

$$S_\varepsilon^\nu : \mathcal{H} \times D_\nu \ni [\boldsymbol{f}, \boldsymbol{v}_0] \mapsto \big\{ \boldsymbol{v} = [u, \eta, \theta] : \text{ solution to } (\mathrm{S})_\varepsilon^\nu \big\} \subset L^2(Q)^3.$$

Then, $\forall [\boldsymbol{f}, \boldsymbol{v}_0] \in \mathcal{H} \times D_\nu$, $S_\varepsilon^\nu(\boldsymbol{f}, \boldsymbol{v}_0)$ be a singleton and continuous w.r.t. the strong topologies from $\mathcal{H} \times L^2(\Omega)^3$ into $L^2(Q) \times C(\overline{Q}) \times C([0, T]; L^2(\Omega))$.

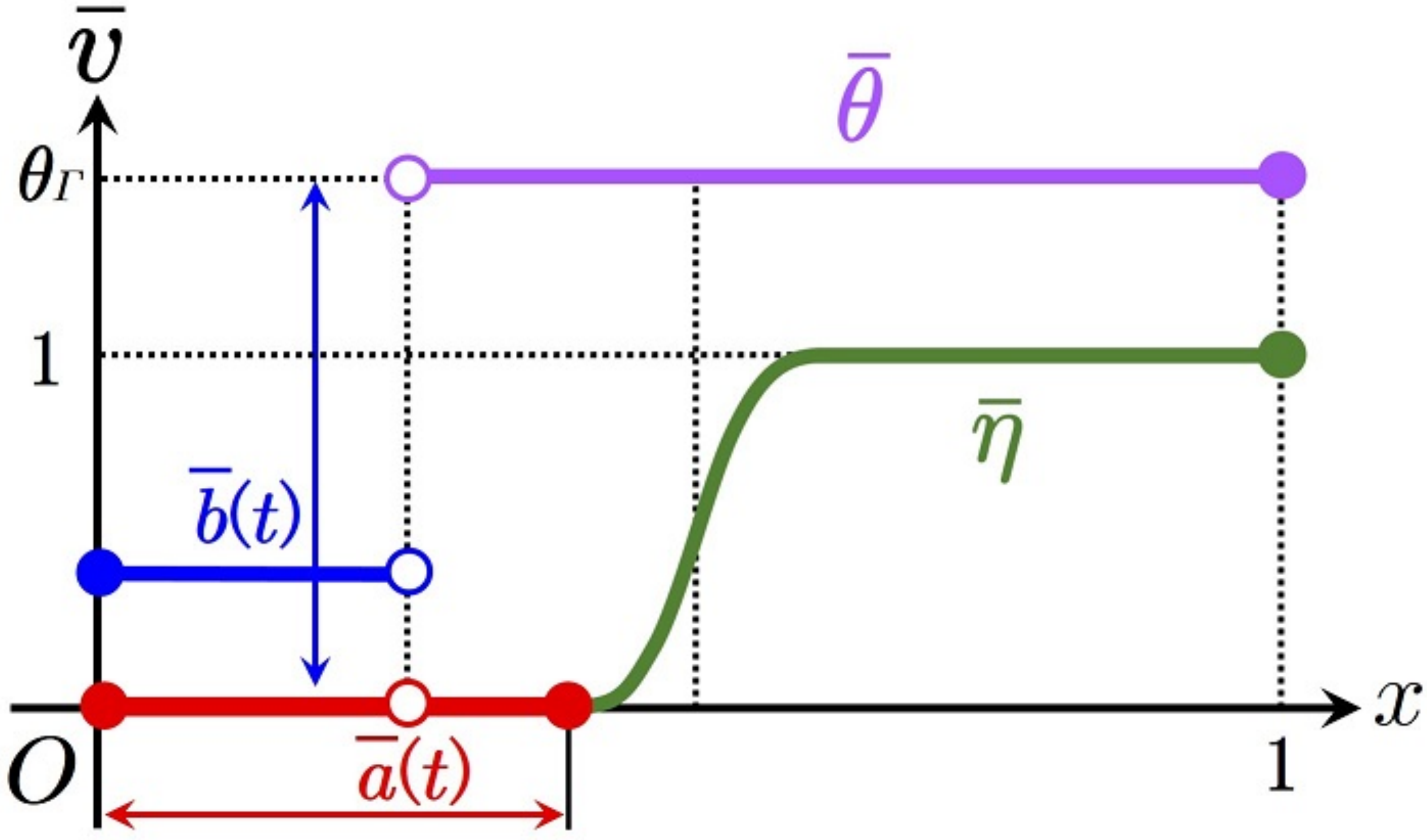
‡. (II) is deduced from [Kenmochi–Niezgódka](1994), [Ito–Kenmochi–Yamazaki(2008)].

◇ **Example of no-uniqueness situation:** when $\nu = 0$ and $\boldsymbol{f} = [f, f_\Gamma] \equiv [0, 0]$

Multiple solutions $\bar{\boldsymbol{v}} = [\bar{u}, \bar{\eta}, \bar{\theta}]$ with common initial data:

$\bar{u}(t,x) = 0$ and $\bar{\eta}(t,x) = \frac{1}{2}\left(\sin\left(\frac{2x-1}{2\sqrt{\kappa}}\right) + 1\right)\chi_{\left(\frac{1-\pi\sqrt{\kappa}}{2}, \frac{1+\pi\sqrt{\kappa}}{2}\right)}(x) + \chi_{\left(\frac{1+\pi\sqrt{\kappa}}{2}, 1\right)}(x)$,

$\forall (t,x) \in Q$ steady-state in Fix–Caginalp system [Chen–Elliott](1994)

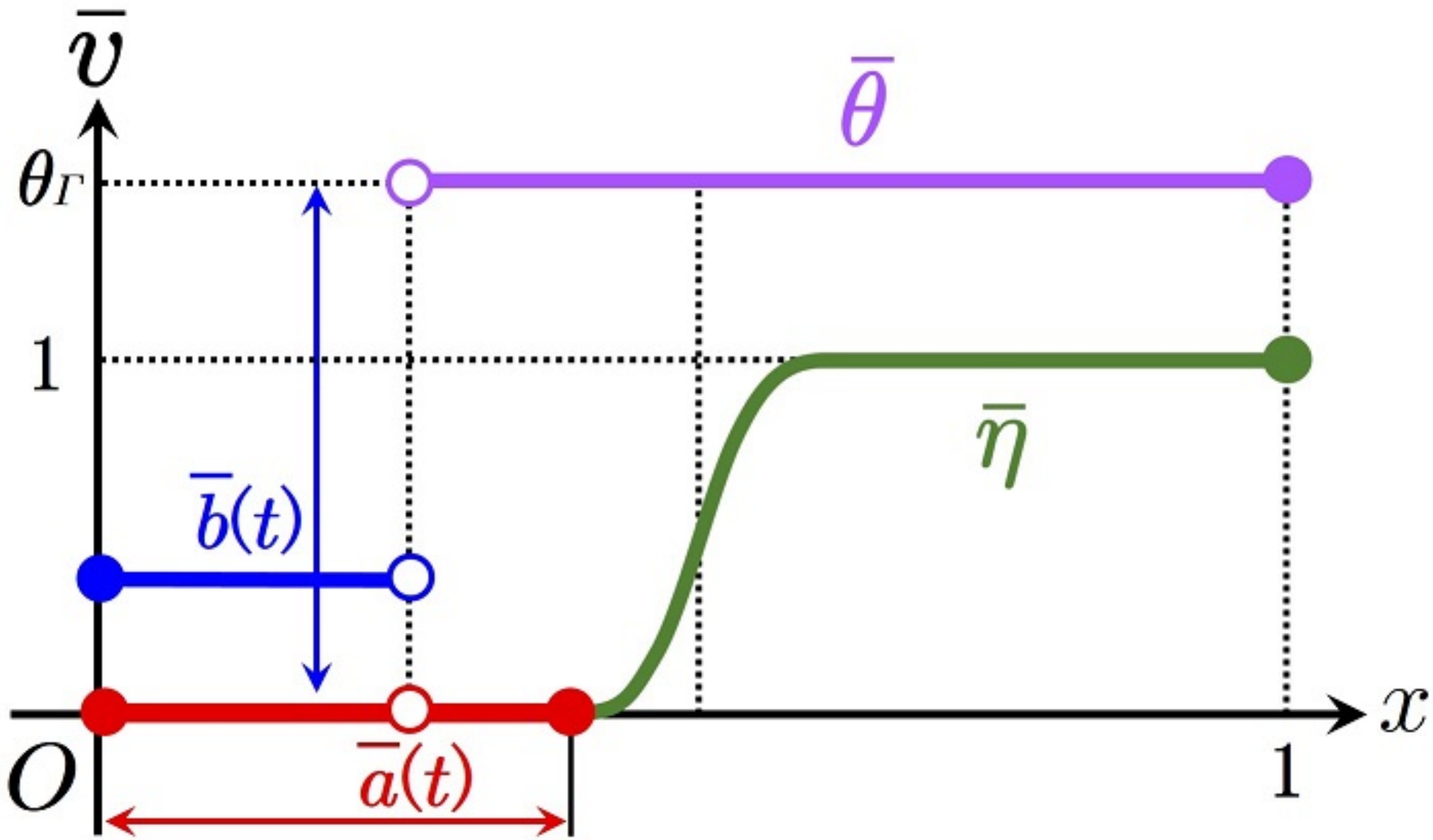


◇ **Example of no-uniqueness situation:** when $\nu = 0$ and $\boldsymbol{f} = [f, f_\Gamma] \equiv [0, 0]$

Multiple solutions $\bar{\boldsymbol{v}} = [\bar{u}, \bar{\eta}, \bar{\theta}]$ with common initial data:

$$\bar{\theta}(t,x) = \bar{b}(t)\chi_{(0,\bar{a}(t))}(x) + \theta_\Gamma \chi_{(\bar{a}(t),1)}(x),\ \bar{a}(t) \in \bar{\eta}^{-1}(0),\ \bar{b}(t) \in [0, \theta_\Gamma],\ \forall (t,x) \in Q$$

$$\Longrightarrow\ \alpha_0(\bar{\eta})\partial_t \bar{\theta}(t) + \partial\Phi_0(\bar{\eta}(t); \bar{\theta}(t)) \ni 0 \ \text{ in } L^2(\Omega),\ \text{ a.e. } t \in (0,T)$$



2. Optimal controls problems

Keypoint: admissible classes $\forall \nu \geq 0$, $\forall \boldsymbol{v}_0 = [u_0, \eta_0, \theta_0] \in D_\nu$: fixed

Class of admissible controls:

$$\forall M > 0,\ \mathcal{H}_M := \left\{ \ \boldsymbol{f} = [f, f_\Gamma] \in \mathcal{H} \ \middle| \ |f|_{H^1(Q)} \leq M \text{ and } |f_\Gamma|_{H^1(\Sigma)} \leq M \ \right\}$$

†. $\mathcal{H}_M$ is to give a constraint for the controls, to get the compactness for in $\mathcal{H}$ (cf. [Colli–F.-Shaker–Gilardi–Sprekels] (2014)).

Class of admissible solutions: $\forall M > 0, \forall \boldsymbol{f} \in \mathcal{H}_M$,

$$A_M^\nu(\boldsymbol{f}, \boldsymbol{v}_0) := \left\{ \boldsymbol{v} \in S_\nu(\boldsymbol{f}, \boldsymbol{v}_0) \ \middle| \ \begin{array}{l} \exists \{\boldsymbol{v}_n\} = \{S_{\varepsilon_n}^\nu(\boldsymbol{f}_n, \boldsymbol{v}_{0,n})\} \subset L^2(Q)^3 \text{ with} \\ \{\varepsilon_n < 2^{-n}\} \subset (0, \tfrac{1}{2}],\quad \{\boldsymbol{f}_n\} \subset \mathcal{H}_M, \\ \{\boldsymbol{v}_{0,n} \in D_{\nu+\varepsilon_n}\} \subset L^2(\Omega)^3, \text{ s.t.} \\ \quad \boldsymbol{v}_n \to \boldsymbol{v} \text{ weakly in } L^2(Q)^3, \\ \quad \boldsymbol{f}_n \to \boldsymbol{f} \text{ in } \mathcal{H},\quad \boldsymbol{v}_{0,n} \to \boldsymbol{v}_0 \text{ in } L^2(\Omega)^3, \\ \quad \mathscr{F}_{\varepsilon_n}^\nu(\boldsymbol{v}_{0,n}) \to \mathscr{F}_\nu(\boldsymbol{v}_0), \text{ as } n \to \infty \end{array} \right\}$$

‡. In the light of Main Theorem 1, the weak convergence for $\{\boldsymbol{v}_n\}$ can be changed by:
$[u_n, \eta_n, \alpha_{\varepsilon_n}(\eta_n)\theta_n] \to [u, \eta, \alpha_0(\eta)\theta]$ in $L^2(Q) \times C(\overline{Q}) \times C([0,T]; L^2(\Omega))$, as $n \to \infty$.

Target profile: $\boldsymbol{v}^\dagger = [u^\dagger, \eta^\dagger, \theta^\dagger] \in L^2(Q)^3$

Optimal control problem $(\mathrm{OP}; \boldsymbol{v}_0)^\nu_M$ with $\nu \geq 0$, $M > 0$, $\boldsymbol{v}_0 \in D_\nu$: to find $\boldsymbol{f}^* = [f^*, f^*_\Gamma] \in \mathcal{H}_M$ (optimal control), which minimize the following cost functional $\mathscr{J}_M$ on $\mathcal{H}_M$:

$$\boldsymbol{f} = [f, f_\Gamma] \in \mathcal{H}_M \mapsto \mathscr{J}_M(\boldsymbol{f}) := \inf \left\{ \varpi_{\boldsymbol{f}}(\boldsymbol{v}) \;\middle|\; \boldsymbol{v} \in A^\nu_M(\boldsymbol{f}, \boldsymbol{v}_0) \right\},$$

with $\boldsymbol{v} = [u, \eta, \theta] \in L^2(Q) \times L^\infty(Q)^2 \mapsto \varpi_{\boldsymbol{f}}(\boldsymbol{v}) := \dfrac{1}{2} \displaystyle\int_0^T \Big(|(u - u^\dagger)(t)|^2_{L^2(\Omega)} +$

$$+ |(\eta - \eta^\dagger)(t)|^2_{L^2(\Omega)} + \left| \left[\alpha_0(\eta)\theta - \alpha_0(\eta^\dagger)\theta^\dagger \right](t) \right|^2_{L^2(\Omega)} \Big)\, dt$$

$$+ \frac{1}{2} \int_0^T \left(|f(t)|^2_{L^2(\Omega)} + |f_\Gamma(t)|^2_{L^2(\Gamma)} \right) dt$$

†. This setting is refer to [Kadoya–Kenmochi–Murase] (2010).

Main Theorem 2 (Existence of optimal controls). $\forall \nu \geq 0, \forall M > 0, \forall \boldsymbol{v}_0 \in D_\nu$, $\exists [\boldsymbol{f}^*, \boldsymbol{v}^*]$ (optimal pair), s.t.:

$$d^*_M(\boldsymbol{v}_0) := \inf_{\boldsymbol{f} \in \mathcal{H}_M} \mathscr{J}_M(\boldsymbol{f}) = \varpi_{\boldsymbol{f}^*}(\boldsymbol{v}^*)$$

Target profile: $\boldsymbol{v}^\dagger = [u^\dagger, \eta^\dagger, \theta^\dagger] \in L^2(Q)^3$

Approximating control problem $(\mathbf{OP}; \boldsymbol{v}_0)^{\nu}_{M,\varepsilon}$ **with** $\nu, M, \varepsilon \in (0, \frac{1}{2}], \boldsymbol{v}_{0,\varepsilon} \in D_{\nu+\varepsilon}$**:** to find $\boldsymbol{f}^*_\varepsilon = [f^*_\varepsilon, f^*_{\Gamma,\varepsilon}] \in \mathcal{H}_M$ (approximating optimal control), which minimize the following approximating cost functional $\mathscr{J}_{M,\varepsilon}$ on $\mathcal{H}_M$:

$$\boldsymbol{f} \in \mathcal{H}_M \mapsto \mathscr{J}_{M,\varepsilon}(\boldsymbol{f}) := \frac{1}{2}\int_0^T \Big(|(u - u^\dagger)(t)|^2_{L^2(\Omega)} + $$

$$+ |(\eta - \eta^\dagger)(t)|^2_{L^2(\Omega)} + \Big| \big[\alpha_\varepsilon(\eta)\theta - \alpha_\varepsilon(\eta^\dagger)\theta^\dagger \big](t) \Big|^2_{L^2(\Omega)} \Big)\, dt$$

$$+ \frac{1}{2}\int_0^T \Big(|f(t)|^2_{L^2(\Omega)} + |f_\Gamma(t)|^2_{L^2(\Gamma)} \Big)\, dt, \ \text{with} \ \ \boldsymbol{v} = [u, \eta, \theta] := S^\nu_\varepsilon(\boldsymbol{f}, \boldsymbol{v}_0)$$

Proposition 1 (Existence of approximating optimal controls). $\forall \nu \geq 0$, $\forall M > 0$, $\forall \varepsilon \in (0, \frac{1}{2}]$, $\forall \boldsymbol{v}_{0,\varepsilon} \in D_{\nu+\varepsilon}$, $\exists [\boldsymbol{f}^*_\varepsilon, \boldsymbol{v}^*_\varepsilon]$ (approximating optimal pair), s.t.:

$$d^*_{M,\varepsilon}(\boldsymbol{v}_{0,\varepsilon}) := \inf_{\boldsymbol{f} \in \mathcal{H}_M} \mathscr{J}_{M,\varepsilon}(\boldsymbol{f})$$

†. This proposition is obtained by means of the standard argument of minimizing sequence.

Main Theorem 3 (Association between the control problems) Let $\nu \geq 0$, $M > 0$ be fixed. Then, $\forall \boldsymbol{v}_0 \in D_\nu$, $\exists [\boldsymbol{f}^*, \boldsymbol{v}^*] \in \mathcal{H}_M \times A_M^\nu(\boldsymbol{f}; \boldsymbol{v}_0)$ (optimal pair), $\exists \{\varepsilon_n < 2^{-n}\}$, $\exists \{\boldsymbol{v}_{0,n} \in D_{\nu+\varepsilon_n}\}$, $\exists \{[\boldsymbol{f}_n^*, \boldsymbol{v}_n^*]\} \subset \mathcal{H}_M \times S_{\varepsilon_n}^\nu(\boldsymbol{f}_n^*; \boldsymbol{v}_{0,n})$ (sequence of approximating optimal pair) s.t. :

$$d^*_{M,\varepsilon_n}(\boldsymbol{v}_{0,n}) = \mathscr{J}_{M.\varepsilon_n}(\boldsymbol{f}_n^*) \to d^*_M(\boldsymbol{v}_0) = \varpi_{\boldsymbol{f}^*}(\boldsymbol{v}^*) \ \text{ as } n \to \infty$$

◇ Keypoints of the proofs

Main Theorem 2: diagonal argument

Let us take: a minimizing sequence $\{\boldsymbol{f}_n\} \subset \mathcal{H}_M$ for the cost functional $\mathscr{J}_M$,
with a sequence of admissible solutions $\boldsymbol{v}_n \in A_M^\nu(\boldsymbol{f}_n, \boldsymbol{v}_0)$, $\forall n \in \mathbb{N}$

$\Longrightarrow$ $\exists \{\varepsilon_m < 2^{-m}\}$, $\exists \{\boldsymbol{v}_{0,m} \in D_{\nu+\varepsilon_m}\}$, $\exists \{\boldsymbol{f}_m^n\} \subset \mathcal{H}_M$ with $\{\boldsymbol{v}_m^n = S_{\varepsilon_m}^\nu(\boldsymbol{f}_n, \boldsymbol{v}_0)\}$, s.t. :

$\boldsymbol{v}_{0,m} \to \boldsymbol{v}_0$ in $L^2(\Omega)^3$, $\mathscr{F}_{\varepsilon_n}^\nu(\boldsymbol{v}_{0,m}) \to \mathscr{F}_\nu(\boldsymbol{v}_0)$,
$\boldsymbol{f}_m^n \to \boldsymbol{f}_n$ in $\mathcal{H}$, as $m \to \infty$, $\forall n \in \mathbb{N}$

$\longleftarrow$ [**Admissibility of solutions**]

$\Longrightarrow$ Applying the diagonal argument:

$\exists [\boldsymbol{f}^*, \boldsymbol{v}^*] \in \mathcal{H}_M \times A_M^\nu(\boldsymbol{f}^*, \boldsymbol{v}_0)$ (optimal pair)

$\longleftarrow$ [**Admissibility of controls**]

□

Main Theorem 3: a consequence from Main Theorem 1 & the admissibilities

Let us take: any optimal pair $[\bar{\boldsymbol{f}}, \bar{\boldsymbol{v}}] \in \mathcal{H}_M \times A^{\nu}_M(\bar{\boldsymbol{f}}, \boldsymbol{v}_0)$

$\Longrightarrow$ $\exists\{\bar{\varepsilon}_n < 2^{-n}\}$, $\exists\{\bar{\boldsymbol{v}}_{0,n} \in D_{\nu+\bar{\varepsilon}_n}\}$, $\exists\{\bar{\boldsymbol{f}}_n\} \subset \mathcal{H}_M$ with $\{\bar{\boldsymbol{v}}_n = S^{\nu}_{\bar{\varepsilon}_n}(\bar{\boldsymbol{f}}_n, \bar{\boldsymbol{v}}_{0,n})\}$, s.t. :

$$d^*_M(\boldsymbol{v}_0) = \lim_{n\to\infty} \mathscr{J}_{M,\bar{\varepsilon}_n}(\bar{\boldsymbol{f}}_n) \geq \varlimsup_{n\to\infty} d^*_{M,\bar{\varepsilon}_n}(\bar{\boldsymbol{v}}_{0,n})$$

$\longleftarrow$ [**Main Theorem 1 & Admissibility of solutions**]

On the other hand: taking approximating optimal pairs $[\bar{\boldsymbol{f}}^*_n, \bar{\boldsymbol{v}}^*_n] \in \mathcal{H}_M \times S^{\nu}_{\bar{\varepsilon}_n}(\bar{\boldsymbol{f}}^*_n, \bar{\boldsymbol{v}}_{0,n})$,

$$\varlimsup_{n\to\infty} d^*_{M,\bar{\varepsilon}_n}(\bar{\boldsymbol{v}}_{0,n}) = \lim_{n\to\infty} \varpi_{\bar{\boldsymbol{f}}^*_n}(\bar{\boldsymbol{v}}^*_n) \geq d^*_M(\boldsymbol{v}_0)$$

$\longleftarrow$ [**Main Theorem 1 & Admissibility of controls**]

$\square$

Remark. If $\nu > 0$, then the conclusions as in Main Theorems 2–3 can be obtained by adopting the following standard type functional:

$$[\boldsymbol{f}, \boldsymbol{v}] \in \mathcal{H} \times L^2(Q)^3 \mapsto \frac{1}{2}|\boldsymbol{v} - \boldsymbol{v}^\dagger|^2_{L^2(Q)} + \frac{1}{2}|\boldsymbol{f}|^2_{\mathcal{H}}, \quad \text{with the target } \boldsymbol{v}^\dagger \in L^2(Q)^3$$

as a constitute conponent of the cost functional.

3. Problems in future

(I) Necessary conditions for the optimal pairs

Strategy: "Necessary condition for the approximating optimal pairs"
$\Rightarrow$ "Observation for the original optimal pairs, via the approximating limit"

(II) Expansion of the results to the general higher-dimenaional cases

Keypoints: to find an appropriate approximating state systems, which have the both of "uniqueness" and "smoothness"

(III) Expansion of the results to the cases with anisotropies

Strategy: continuation of the recent study [Moll–S.–Watanabe](2010) concerned with the anisotropic versions of the state systems

(IV) Structural observations for solutions

Keypoints: previous works of total-variation flows, e.g. [Andreu–Caselles–Mazón](2004), [Bellettini–Caselles–Novaga](2002), [Kobayashi–Giga](1999), [Giga–Giga](2010), [Moll](2005–), [Rybka–Mucha](2000–), [S.](2000–), e.t.c.

Strategy: to obtain some additional properties of solutions, including the concrete profiles of optimal pairs

(V) Uniqueness of the state systems

Status: there is no advance, yet