Exponential stability of abstract evolution equations with time delay feedback

Cristina Pignotti

University of L'Aquila

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The problem

Let $\mathcal{H}$ be a fixed Hilbert space with norm $\|\cdot\|$, and consider an operator $\mathcal{A}$ from $\mathcal{H}$ into itself that generates a C_0-semigroup $(S(t))_{t\geq 0}$ that is exponentially stable, i.e., there exist two positive constants M and ω such that

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\omega t}, \ \forall t \geq 0.$$

We consider the evolution equation

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t), U(t-\tau)) \quad \text{in } (0, +\infty) \\ U(0) = U_0, \ U(t-\tau) = f(t), \quad \forall t \in (0, \tau), \end{cases} \qquad \textbf{(P)} \qquad (1$$

where $\tau > 0$ is the time delay, the nonlinear term $F : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ satisfies some Lipschitz conditions, the initial datum U_0 belongs to $\mathcal{H}$ and $f \in C([0, \tau]; \mathcal{H})$.

The problem

Time delay effects often appear in many applications and physical problems. On the other hand, it is well-known (cfr. Datko, Lagnese & Polis (1986), Datko (1988), Bátkai & Piazzera (2005), Xu, Yung & Li (2006), Nicaise & P. (2006)) that they can induce some instability.

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Then, we are interested in giving an exponential stability result for such a problem under a suitable smallness condition on the delay τ .

The problem

In Nicaise & P. (2015) we have studied the case of linear dependency on the delayed state $U(t-\tau)$, i.e. we have considered the model

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + G(U(t)) + k\mathcal{B}U(t-\tau) \quad \text{in } (0,+\infty) \\ U(0) = U_0, \ \mathcal{B}U(t-\tau) = g(t), \quad \forall t \in (0,\tau), \end{cases} \qquad \textbf{(PL)}$$

where $\mathcal{B}$ is a fixed bounded operator from $\mathcal{H}$ into itself, $G : \mathcal{H} \to \mathcal{H}$ satisfies some Lipschitz conditions, the initial datum U_0 belongs to $\mathcal{H}$ and $g \in C([0,\tau];\mathcal{H})$.

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where $\mathcal{B}$ is a fixed bounded operator from $\mathcal{H}$ into itself, $G : \mathcal{H} \to \mathcal{H}$ satisfies some Lipschitz conditions, the initial datum U_0 belongs to $\mathcal{H}$ and $g \in C([0,\tau];\mathcal{H})$.

We have proved an exponential stability result for such a problem under a suitable condition between the constant k and the constants M, ω, τ, the norm of $\mathcal{B}$ and the nonlinear term G.

Particular results

For some particular examples (see e.g. Bátkai & Piazzera (2005), Ammari, Nicaise & P. (2010), P. (2012), Said-Houari & Soufyane (2012), Guesmia (2013), Liu (2013), Alabau-Boussouira, Nicaise & P. (2014), Chentouf (2015)) we know that the above problem, under certain *smallness* conditions on the delay feedback $k\mathcal{B}$, is exponentially stable,

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Our proof is simpler with respect to the ones used so far for particular models. Moreover, we emphasize its generality. Indeed, it applies to every model in the previous abstract form when the operator $\mathcal{A}$ generates an exponentially stable semigroup.

Well-posedness

Now, we assume that F is Lipschitz continuous, namely

$$\exists \gamma > 0 \quad \text{such that}$$
$$\|F(U_1, U_2) - F(U_1^*, U_2^*)\|_{\mathcal{H}} \leq \gamma(\|U_1 - U_1^*\|_{\mathcal{H}} + \|U_2 - U_2^*\|_{\mathcal{H}}),$$
$$\forall\ (U_1, U_2), (U_1^*, U_2^*) \in \mathcal{H} \times \mathcal{H}\,.$$

Moreover, we assume that $F(0, 0) = 0$.

The following well–posedness result holds.

PROPOSITION

For any initial datum $U_0 \in \mathcal{H}$ and $f \in C([0, \tau]; \mathcal{H})$, there exists a unique (mild) solution $U \in C([0, +\infty), \mathcal{H})$ of problem (P). Moreover,

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(U(s), U(s-\tau))\, ds. \qquad (F)$$

Proof.

We use an iterative argument. Namely in the interval $(0,\tau)$, problem (P) can be seen as a standard evolution problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F_1(U(t)) \quad \text{in } (0,\tau) \\ U(0) = U_0, \end{cases}$$

where $F_1(U(t)) = F(U(t), f(t))$. This problem has a unique solution $U \in C([0,\tau],\mathcal{H})$ (see [Th. 1.2, Ch. 6 of Pazy (1983)]) satisfying

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F_1(U(s))\,ds.$$

This yields $U(t)$, for $t \in [0,\tau]$.

Therefore on $(\tau, 2\tau)$, problem (P) can be seen as the evolution problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F_2(U(t)) \quad \text{in } (\tau, 2\tau) \\ U(\tau) = U(\tau-), \end{cases}$$

where $F_2(U(t)) = F(U(t), U(t-\tau))$, since $U(t-\tau)$ can be regarded as a datum being known from the first step. Hence, this problem has a unique solution $U \in C([\tau, 2\tau], \mathcal{H})$ given by

$$U(t) = S(t-\tau)U(\tau-) + \int_\tau^t S(t-s)F_2(U(s))\,ds, \forall t \in [\tau, 2\tau].$$

By iterating this procedure, we obtain a global solution U satisfying (F). $\square$

THEOREM [Nicaise & P. (2016)]

Let us assume $\gamma < \frac{\omega}{2M}$. *There exists* $\tau_0 := \tau_0(\gamma, \omega, M)$ *such that the time delay* τ *satisfies* $\tau < \tau_0$, *then there are* $\omega' > 0$, $M' > 0$, *such that the solution* $U \in C([0, +\infty), \mathcal{H})$ *of problem* (P), *with* $U_0 \in \mathcal{H}$ *and* $f \in C([0, \tau]; \mathcal{H})$, *satisfies*

$$\|U(t)\|_{\mathcal{H}} \leq M' e^{-\omega' t}\Big(\|U_0\|_{\mathcal{H}} + \gamma \int_0^\tau e^{\omega s}\|f(s)\|_{\mathcal{H}}\, ds\Big), \quad \forall t \geq 0\,.$$

Remarks

• Note that our stability theorem is very general. Indeed, it furnishes stability results for previously studied models for wave type equations (cfr. Ammari, Nicaise & P. (2010), P. (2012), Nicaise & P. (2014), Chentouf (2015), Han, Li & Liu (2016))), Timoshenko models (cfr. Said-Houari & Soufyane (2012), Apalara & Messaoudi (2014)).

• Also, it includes recent stability results for problems with viscoelastic damping and time delay (cfr. Guesmia (2013), Alabau-Boussouira, Nicaise & P. (2014), Dai & Yang (2014), (2016)).

Concrete models

Wave equation with internal dampings

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a boundary $\partial\Omega$ of class C^2. We denote by $\nu(x)$ the outer unit normal vector at a point $x \in \partial\Omega$. Let m be the standard multiplier, $m(x) = x - x_0$, $x_0 \in \mathbb{R}^n$, and let ω be the intersection between an open neighborhood of the set

$$\Gamma_0 = \{\, x \in \partial\Omega \ : \ m(x) \cdot \nu(x) > 0 \,\}$$

and Ω. Let us consider the system

$$\begin{array}{l} u_{tt}(x,t) - \Delta u(x,t) + a\chi_\omega u_t(x,t) + k u_t(x,t-\tau) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \ \ \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad \text{on} \ \ \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in} \ \ \Omega, \\ u_t(x,t) = f(x,t), \quad \text{in} \ \ \Omega \times (-\tau, 0), \end{array}$$

where a, k are real numbers, $a > 0$, and the initial data are taken in suitable spaces.

For simplicity we consider the delay term acting in the whole domain Ω.

Wave equation with internal dampings

Let us denote

$$U := (u, u_t)^T.$$

Then, the above problem for $k = 0$, that is without time delay, may be rewritten as

$$\begin{cases} U' = \mathcal{A}U\,, \\ U(0) = (u_0, u_1)^T\,, \end{cases}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} v \\ \Delta u - a\chi_\omega v \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega).$$

Wave equation with internal dampings

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega),$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\rangle := \int_\Omega \{\nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\} dx$$

It is well–known that $\mathcal{A}$ generates a C_0- semigroup exponentially stable (see e.g. Chen (1979)).

Then, from our abstract theorem, an exponential stability estimate still holds in presence of a time delay feedback with $|k|$ sufficiently small (cfr. P. (2012)).

Wave equation with boundary feedback and internal time delay

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. We denote by $\partial\Omega$ the boundary of Ω and we assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0,\ \Gamma_1$ are closed subsets of $\partial\Omega$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. Moreover we assume $meas\Gamma_0 > 0$. Let us consider the system

$$
\begin{array}{l}
u_{tt}(x,t) - \Delta u(x,t) + k u_t(x,t-\tau) = 0, \quad x \in \Omega,\ t > 0,\\
u(x,t) = 0, \quad x \in \Gamma_0,\ t > 0\\
\frac{\partial u}{\partial \nu}(x,t) = -a u_t(x,t), \quad x \in \Gamma_1,\ t > 0\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,\\
u_t(x,t) = g(x,t), \quad x \in \Omega,\ t \in (-\tau, 0),
\end{array}
$$

where ν stands for the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω and $\frac{\partial u}{\partial \nu}$ is the normal derivative; a and k are two real numbers, $a > 0$, and the initial data are taken in suitable spaces. Denoting by m the standard multiplier, that is $m(x) = x - x_0$, we assume

$$
m(x) \cdot \nu(x) \leq 0, \quad x \in \Gamma_0, \quad \text{and} \quad m(x) \cdot \nu(x) \geq \delta > 0, \quad x \in \Gamma_1.
$$

Wave equation with boundary feedback

Let $A = -\Delta$ be the unbounded operator in $H = L^2(\Omega)$ with domain

$$H_1 = \mathcal{D}(A) = \left\{ u \in H^2(\Omega),\ u_{|\Gamma_0} = 0,\ \frac{\partial u}{\partial \nu}_{|\Gamma_1} = 0 \right\}.$$

We define

$$C \in \mathcal{L}(L^2(\Gamma_1); H_{-1}),\ Cu = \sqrt{a}\, A_{-1} N u,\ \forall\, u \in L^2(\Gamma_1),$$

$$C^* w = \sqrt{a} w_{|\Gamma_1},\ \forall\, w \in \mathcal{D}(A^{\frac{1}{2}}) = H_{\frac{1}{2}},$$

where $H_{-1} = (\mathcal{D}(A))'$ (the duality is in the sense of H), A_{-1} is the extension of A to H, namely for all $h \in H$ and $\varphi \in \mathcal{D}(A)$, $A_{-1} h$ is the unique element in H_{-1} such that

$$\langle A_{-1} h; \varphi \rangle_{H_{-1} - H_1} = \int_\Omega h A \varphi \, dx.$$

Wave equation with boundary feedback

Here and below $N \in \mathcal{L}(L^2(\Gamma_1); L^2(\Omega))$, $\forall\, v \in L^2(\Gamma_1)$, Nv is the unique solution (transposition solution) of

$$\Delta w = 0,\ w_{|\Gamma_0} = 0,\ \frac{\partial w}{\partial \nu}_{|\Gamma_1} = v.$$

We can rewrite the problem without delay term as

$$\begin{array}{l} u_{tt}(t) + Au(t) + CC^* u_t(t) = 0, \quad t \geq 0, \\ u(0) \, = \, u^0,\ u_t(0) = u^1, \end{array}$$

and then as an abstract Cauchy problem in a product Banach space. For this take the Hilbert space $\mathcal{H} := H_{\frac{1}{2}} \times H$ and the unbounded linear operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H},$$

$$\mathcal{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -Au_1 - CC^* u_2 \end{pmatrix},$$

where

$$\mathcal{D}(\mathcal{A}) := \left\{ (u_1, u_2) \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \; : \; Au_1 + CC^* u_2 \in H \right\}.$$

The operator $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{H}$.

The exponential stability of the previous model with $k = 0$ it is well-known (see Lasiecka & Triggiani (1987), Komornik & Zuazua (1990)).

Then, our abstract theorem result allows to extend the exponential stability result also in presence of time delay, for $|k|$ small (cfr. Ammari, Nicaise & P. (2010)).

Wave equation with memory and delay

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary. Let us consider the following problem:

$$\begin{aligned} & u_{tt}(x,t) - \Delta u(x,t) + \int_0^\infty \mu(s)\Delta u(x,t-s)ds \\ & \qquad\qquad\qquad + k u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty) \\ & u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \\ & u(x,t) = u_0(x,t) \quad \text{in } \Omega \times (-\infty,0] \end{aligned}$$

where the initial datum u_0 belongs to a suitable space, k is a real number and the memory kernel $\mu : [0,+\infty) \to [0,+\infty)$ is a locally absolutely continuous function satisfying

i) $\mu(0) = \mu_0 > 0$;
ii) $\int_0^{+\infty} \mu(t)dt = \tilde{\mu} < 1$;
iii) $\mu'(t) \leq -\alpha\mu(t)$, for some $\alpha > 0$.

We know that the above problem is exponentially stable for $k = 0$ (see e.g. Giorgi, Munõz Rivera & Pata (2001)).

Wave equation with memory and delay

As in Dafermos (1970), let us introduce the new variable

$$\eta^t(x,s) := u(x,t) - u(x,t-s).$$

Then, we can rewrite the problem, without delay term, as

$$\begin{array}{l}
u_{tt}(x,t) = (1-\tilde{\mu})\Delta u(x,t) + \int_0^\infty \mu(s)\Delta\eta^t(x,s)ds = 0 \quad \text{in } \Omega \times (0,+\infty) \\
\eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t) \quad \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \\
u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \\
\eta^t(x,s) = 0 \quad \text{in } \partial\Omega \times (0,+\infty),\ t \geq 0, \\
u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\
\eta^0(x,s) = \eta_0(x,s) \quad \text{in } \Omega \times (0,+\infty),
\end{array}$$

with initial conditions

$$\begin{array}{l}
u_0(x) = u_0(x,0), \quad x \in \Omega, \\
u_1(x) = \frac{\partial u_0}{\partial t}(x,t)|_{t=0}, \quad x \in \Omega, \\
\eta_0(x,s) = u_0(x,0) - u_0(x,-s), \quad x \in \Omega,\ s \in (0,+\infty).
\end{array}$$

Wave equation with memory and delay

We denote $\mathcal{U} := (u, u_t, \eta^t)^T$. Then we can rewrite the problem in the abstract form

$$\begin{cases} \mathcal{U}' = \mathcal{A}\mathcal{U}, \\ \mathcal{U}(0) = (u_0, u_1, \eta_0)^T, \end{cases} \qquad (PA1)$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} := \begin{pmatrix} v \\ (1-\tilde{\mu})\Delta u + \int_0^\infty \mu(s)\Delta w(s)ds \\ -w_s + v \end{pmatrix},$$

with domain (cf. Giorgi, Munõz Rivera & Pata (2001), Pata (2010))

$$\mathcal{D}(\mathcal{A}) := \Big\{(u, v, \eta)^T \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2_\mu((0,+\infty); H_0^1(\Omega)) \ : \\ (1-\tilde{\mu})u + \int_0^\infty \mu(s)\eta(s)ds \in H^2(\Omega) \cap H_0^1(\Omega), \\ \eta' \in L^2_\mu((0,+\infty); H_0^1(\Omega))\Big\},$$

Wave equation with memory and delay

where $L^2_\mu((0,\infty);H^1_0(\Omega))$ is the Hilbert space of H^1_0- valued functions on $(0,+\infty)$, endowed with the inner product

$$\langle \varphi,\psi\rangle_{L^2_\mu((0,\infty);H^1_0(\Omega))} = \int_\Omega \left(\int_0^\infty \mu(s)\nabla\varphi(x,s)\nabla\psi(x,s)ds\right)dx.$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} := H^1_0(\Omega)\times L^2(\Omega)\times L^2_\mu((0,\infty);H^1_0(\Omega)),$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u\\ v\\ w\end{pmatrix}, \begin{pmatrix} \tilde u\\ \tilde v\\ \tilde w\end{pmatrix}\right\rangle_{\mathcal{H}} := (1-\tilde\mu)\int_\Omega \nabla u\nabla\tilde u dx + \int_\Omega v\tilde v dx$$
$$+\int_\Omega\int_0^\infty \mu(s)\nabla w\nabla\tilde w dsdx.$$

Wave equation with memory and delay

It is well–known that $\mathcal{A}$ generates a C_0-semigroup exponentially stable. Then our exponentially stability result applies for $|k|$ small to the model with time delay (cfr. Guesmia (2013), Alabau–Boussouira, Nicaise & P. (2014), Dai & Yang (2014), (2016)).

More general nonlinearities

Now we consider a more general class of nonlinearities. More precisely, we assume that for every constant c there exists a positive constant $L(c)$ such that

$$\|F(U_1, U_2) - F(U_1^*, U_2^*)\|_{\mathcal{H}} \leq L(c)(\|U_1 - U_2\|_{\mathcal{H}} + \|U_1^* - U_2^*\|_{\mathcal{H}}),$$

for all $(U_1, U_2), (U_1^*, U_2^*) \in \mathcal{H} \times \mathcal{H}$ with $\|U_i\|_{\mathcal{H}}, \|U_i^*\|_{\mathcal{H}} \leq c, \ i = 1, 2.$

Moreover, we assume that there exists an increasing continuous function $\chi : [0, +\infty) \to [0, +\infty)$, with $\chi(0) = 0$, such that $\forall \ (U_1, U_2) \in \mathcal{H} \times \mathcal{H}$,

$$\|F(U_1, U_2)\|_{\mathcal{H}} \leq \chi(\max\{\|U_1\|_{\mathcal{H}}, \|U_2\|_{\mathcal{H}}\})(\|U_1\|_{\mathcal{H}} + \|U_2\|_{\mathcal{H}}).$$

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for all $(U_1,U_2),(U_1^*,U_2^*)\in\mathcal{H}\times\mathcal{H}$ with $\|U_i\|_{\mathcal{H}},\|U_i^*\|_{\mathcal{H}}\le c,\ i=1,2.$

Moreover, we assume that there exists an increasing continuous function $\chi:[0,+\infty)\to[0,+\infty)$, with $\chi(0)=0$, such that $\forall\ (U_1,U_2)\in\mathcal{H}\times\mathcal{H}$,

$$\|F(U_1,U_2)\|_{\mathcal{H}}\le\chi(\max\{\|U_1\|_{\mathcal{H}},\|U_2\|_{\mathcal{H}}\})(\|U_1\|_{\mathcal{H}}+\|U_2\|_{\mathcal{H}})\,.$$

We can give an exponential stability result under a well–posedness assumption for *small* initial data.

The case F locally Lipschitz

THEOREM [Nicaise & P. (2016)]
Suppose that

$\exists\ \rho_0 > 0$ and $C_{\rho_0} > 0$ such that $\forall\ U_0 \in \mathcal{H},\ f \in C([0,\tau];\mathcal{H})$
with $(\|U_0\|^2_{\mathcal{H}} + \int_0^\tau e^{\omega s}\|f(s)\|^2_{\mathcal{H}} ds)^{1/2} < \rho_0$,

there exists a unique global solution
$U \in C([0,+\infty,\mathcal{H})$ to (P) with $\|U(t)\|_{\mathcal{H}} \le C_{\rho_0} < \chi^{-1}\left(\frac{\omega}{2M}\right),\ \forall\ t > 0$.

Then, there exists $\tilde{\tau}_0 := \tilde{\tau}_0(\gamma,\omega,M)$ *such that for every* $U_0 \in \mathcal{H}$ *and* $f \in C([0,\tau];\mathcal{H})$ *satisfying the assumption from* (H)*, if the time delay verifies* $\tau < \tilde{\tau}_0$ *, the solution* U *of problem* (P) *satisfies the exponential decay estimate*

$$\|U(t)\|_{\mathcal{H}} \le \tilde{M}e^{-\tilde{\omega}t}(\|U_0\|_{\mathcal{H}} + \chi(C_{\rho_0})\int_0^\tau e^{\omega s}\|f(s)\|_{\mathcal{H}}\, ds), \quad \forall t \ge 0\,,$$

for suitable positive constants $\tilde{M}, \tilde{\omega}$.

Examples

We now give a class of examples for wich assumption (H) is satisfied.
Let us consider the second order evolution equation

$$\begin{aligned} &u_{tt} + A_1 u + CC^* u_t = \nabla G(u) + kBB^* u_t(t-\tau), \quad t > 0, \qquad (II) \\ &u(0) = u_0,\ u_t(0) = u_1,\ B^* u_t(t-\tau) = g(t),\ t \in (0,\tau), \end{aligned}$$

with $(u_0, u_1) \in V \times H$, where H is a real Hilbert space, with norm $\|\cdot\|_H$,

and $A_1 : \mathcal{D}(A_1) \to H$, is a positive self–adjoint operator with a compact inverse in H. Moreover, we denote by $V := \mathcal{D}(A_1^{\frac{1}{2}})$ the domain of $A_1^{\frac{1}{2}}$.
Further, for i=1,2, let W_i be a real Hilbert space (which will be identified to its dual space) and let $C \in \mathcal{L}(W_1, H)$, $B \in \mathcal{L}(W_2, H)$.
Assume that, for some constant $\mu > 0$

$$\|B^* u\|_{W_2}^2 \le \mu \|C^* u\|_{W_1}^2, \quad \forall\ u \in V\,.$$

Let be given a functional $G : V \to \mathbb{R}$ such that G is Gâteaux differentiable at any $x \in V$.

Examples

We further assume (cfr. Alabau-Boussouira, Cannarsa & Sforza (2008)) that

a) For any $u \in V$ there exists a positive constant $c(u)$ such that

$$|DG(u)(v)| \leq c(u)\|v\|_H, \ \forall v \in V,$$

where $DG(u)$ is the Gâteaux derivative of G at u.

Consequently $DG(u)$ can be extended in the whole H and we will denote by $\nabla G(u)$ the unique element in H such that

$$(\nabla G(u), v)_H = DG(u)(v), \ \forall v \in H.$$

b) For all $c > 0$, there exists $L(c) > 0$ such that

$$\|\nabla G(u) - \nabla G(v)\|_H \leq L(c)\|A_1^{\frac{1}{2}}(u-v)\|_H$$

for all $u, v \in V$ such that $\|A_1^{\frac{1}{2}}u\|_H \leq c$ and $\|A_1^{\frac{1}{2}}v\|_H \leq c$.

Examples

c) There exists a suitable increasing continuous function ψ satisfying $\psi(0) = 0$ such that

$$\|\nabla G(u)\|_H \leq \psi(\|A_1^{\frac{1}{2}} u\|)\|A_1^{\frac{1}{2}} u\|_H^2, \ \forall u \in V.$$

Denoting $v := u_t$ and $U := (u, v)^T$, this problem may be rewritten in the form (P) with

$$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ -A_1 & -\,CC^* \end{pmatrix},$$

$$F(U) := (0, \nabla G(u))^T, \quad \mathcal{B}U := (0, BB^* v)^T.$$

The assumptions on G imply that F satisfies previous assumptions in $\mathcal{H} := V \times H$, with $\chi = \psi$.

Examples

We define the energy of solutions of problem (II) as

$$E(t) := E(t, u(\cdot)) = \frac{1}{2}\|u_t\|_H^2 + \frac{1}{2}\|A_1^{\frac{1}{2}} u\|_H^2 - G(u)$$
$$+\frac{1}{2}\int_{t-\tau}^{t} |k| \|B^* u_t(s)\|_{W_2}^2 ds\,.$$

We will show that for the above model an exponential stability result holds, for $|k|$ small, if $\mathcal{A}$ generates an exponentially stable semigroup.
First of all note that

$$\begin{aligned} E'(t) &= -\|C^* u_t(t)\|_{W_1}^2 + k\langle B^* u_t(t), B^* u_t(t-\tau)\rangle \\ &+\frac{|k|}{2}\|B^* u_t(t)\|_{W_2}^2 - \frac{|k|}{2}\|B^* u_t(t-\tau)\|_{W_2}^2 \\ &\le -\|C^* u_t(t)\|_{W_1}^2 + |k|\,\|B^* u_t(t)\|_{W_2}^2 \end{aligned}$$

Then, if $|k| < \frac{1}{\mu}$, the energy is not increasing.
We can prove the following well-posedness result for sufficiently small data

Examples

PROPOSITION

The assumption (H) *is satisfied for* $|k| < 1/\mu$.

$\Longrightarrow$ If $\mathcal{A}$ generates an exponentially stable continuous semigroup on $\mathcal{H}$, then the exponential estimate holds for small initial data, for τ small enough.

• The abstract model (II) includes semilinear versions of previously analyzed concrete models for wave-type equations (cfr. Nicaise & P. (2006)). Of course, due to the presence of the nonlinearity we obtain the stability result (for small initial data) under a more restrictive assumption on the delay feedback. Observe also that models with viscoelastic damping could be considered but with also an extra not–delayed damping necessary to avoid blow-up of solutions, at least for small data.

Thank you for your attention!