A variational approach to doubly nonlinear problems and applications

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Der Wissenschaftsfonds.

The target problem

$$\begin{aligned} \mathrm{D}\psi(\dot{u}) + \partial\phi(u) &\ni f(u), \\ u(0) &= u_0. \end{aligned} \tag{$\mathscr{P}$}$$

- ψ convex and Fréchet differentiable, ϕ convex and l.s.c., f nonmonotone and nonpotential.
- Gradient flows (parabolic equations, Cahn-Hilliard, Allen-Cahn,...), nonlocal problems (fractional heat equation,...), ODEs, systems of differential equations,...
- Nonuniqueness.

The WED approach (a simple case)

1. Dissipative target problem $(\mathscr{P})$.
2. Minimization of the so-called WED (Weighted Energy Dissipation) functional I_ε.
3. Elliptic in time regularization of the target equation $(\mathscr{P}_\varepsilon)$.
4. Passage to the limit $(\mathscr{P}_\varepsilon)\to(\mathscr{P})$.

Illustrative example

$$\begin{aligned} \dot u - \Delta u = g \text{ in } \Omega\times(0,T),\\ u = 0 \text{ on } \partial\Omega,\\ u(0) = u_0. \end{aligned} \tag{$\mathscr{P}$}$$

$$u^\varepsilon = \arg\min_{K(u_0)} I_\varepsilon, \qquad I_\varepsilon(u) = \int_0^T e^{-t/\varepsilon}\int_\Omega \frac{\varepsilon}{2}|\dot u|^2 + \frac{1}{2}|\nabla u|^2 - gu, \tag{I_ε}$$

$$K(u_0) = \{u\in H^1(0,T;L^2(\Omega)), u(0) = u_0\}$$

$$-\varepsilon\ddot u^\varepsilon + \dot u^\varepsilon - \Delta u^\varepsilon = g, \qquad \varepsilon\dot u^\varepsilon(T) = 0. \tag{$\mathscr{P}_\varepsilon$}$$

$$u^\varepsilon \to u \text{ for } \varepsilon\to 0. \tag{$(\mathscr{P}_\varepsilon)\to(\mathscr{P})$}$$

The WED approach to nonpotential/nonconvex doubly nonlinear equations

Problem. The target equation has the form

$$\mathrm{D}\psi(\dot{u}) + \partial\phi(u) \ni f(u), \tag{$\mathscr{P}$}$$

and thus $(\mathscr{P}_\varepsilon)$ reads

$$-\varepsilon\partial_t(\mathrm{D}\psi(\dot{u})) + \mathrm{D}\psi(\dot{u}) + \partial\phi(u) \ni f(u). \tag{$\mathscr{P}_\varepsilon$}$$

As f is *nonpotential*:

$$f(u) \neq \partial F(u) \quad \forall F,$$

there exists no functional such that its minimizers solve $(\mathscr{P}_\varepsilon)$.

Idea (M.-Akagi). The map S, defined by

$$S : v \mapsto w = f(v) \mapsto u = \arg \min_{u(0)=u_0} I_{\varepsilon,w},$$

$$I_{\varepsilon,w}(u) = \int_0^T e^{-t/\varepsilon}\left(\varepsilon\psi(\dot{u}) + \phi(u) - (w,u)\right), \tag{I_ε}$$

has a fixed point u_ε. Moreover, u_ε solves $(\mathscr{P}_\varepsilon)$ and $u_\varepsilon \to u$ for $\varepsilon \to 0$, where u solves $(\mathscr{P})$. Note that the minimizer of $I_{\varepsilon,w}$ is unique, but solutions to $(\mathscr{P}_\varepsilon)$ and $(\mathscr{P})$ are in general not unique.

History

Oleinik (1964), Lions (1965), Barbu (1975) Elliptic regularization.

Hirano (1994) Periodic solutions of gradient-flows.

Ilmanen (1994) Brakke mean-curvature flows of varifolds.

Mielke-Ortiz (2008) Rate-independent systems.

Mielke-Stefanelli (2011) Gradient flows (λ-convex energies).

Akagi-Stefanelli (2011-2014-2015) Doubly nonlinear equations, nonconvex gradient flows.

Serra-Tilli (2012), Stefanelli (2011) Nonlinear wave equation (solving a conjecture by De Giorgi).

Savaré et. al. (2011) Gradient flows in metric spaces.

Liero et. al. (2013) Lagrangian Mechanics.

M. (submitted, 2016) Nonpotential (nonconvex) perturbation of gradient flows.

Akagi-M. (work in progress) Nonpotential (nonconvex) perturbation of doubly nonlinear equations:

$$\mathrm{D}\psi(\dot{u}) + \partial\phi(u) \ni f(u).$$

Why WED approach to evolution equations?

- Tools and techniques from Calculus of Variations (e.g. Direct Method, Relaxation, Γ-convergence).
- Solutions to $(\mathscr{P}_\varepsilon)$ are more regular.
- Select one (or some) solution(s) to $(\mathscr{P})$ in case of nonuniqueness.
- Qualitative properties.
- Comparison principle.

Qualitative properties I

Goal.
Given R and compatibility conditions (e.g., $Ru_0 = u_0$), prove existence of a solution u to $(\mathscr{P})$ such that $Ru = u$.

Examples.

$$
\begin{array}{lcl}
Ru(x) := u(-x) & \Longrightarrow & u \text{ invariant under reflection} \\
Ru(x) := u(rx),\ r \text{ rotation} & \Longrightarrow & u \text{ invariant under rotation} \\
R := \text{ symmetric decreasing rearrangement} & \Longrightarrow & u \text{ radially symmetric and radially decreasing} \\
R := M + (u-M)^+ & \Longrightarrow & u \geq M \\
R := M - (M-u)^+ & \Longrightarrow & u \leq M
\end{array}
$$

Qualitative properties II

Assumptions.

- $RD(I_{\varepsilon,w}) \subset D(I_{\varepsilon,w})$ for all $w \in \{f(v) : v = Rv\}$. In particular, $Ru_0 = u_0$.
- $I_{\varepsilon,w}(Ru) \leq I_{\varepsilon,w}(u)$ for all u and all $w \in \{f(v) : v = Rv\}$.
- Convergence $\varepsilon \to 0$ preserves invariance.

Idea. Let $v = Rv$. Then,

$$S : v \mapsto w = f(v) \mapsto u = \arg \min_{u(0)=u_0=Ru_0} I_{\varepsilon,w}(u) = \int_0^T e^{-t/\varepsilon} (\varepsilon\psi(\dot{u}) + \phi(u) - (w,u)).$$

Thus,

$$I_{\varepsilon,w}(Ru) \leq I_{\varepsilon,w}(u).$$

By uniqueness of the minimizer $Ru = u$. Thus, S preserves invariance. Hence, there exists a fixed point u^ε of S satisfies

$$Ru^\varepsilon = u^\varepsilon.$$

Finally,

$$Ru = \lim_{\varepsilon\to 0} Ru^\varepsilon = \lim_{\varepsilon\to 0} u^\varepsilon = u.$$

Qualitative properties III, applications

Example 1. Consider equation

$$\alpha(\dot{u}) - \nabla\cdot(B(x)|\nabla u|^{m-2}\nabla u) + C|u|^{m-2}u - D(x)|u|^{q-2}u = h(x) \text{ in } \Omega\times(0,T), \qquad (\mathscr{P})$$

$$au + b\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\times(0,T),$$

$$u(0) = u_0 \text{ in } \Omega.$$

Let R be given by one of the following

- symmetric decreasing, monotone decreasing rearrangements,
- $Ru(x) = u(rx), \qquad r:\Omega\to\Omega$ linear and $|\det r| = 1$,
- $R(u) = M + (u-M)^+, \qquad M\in\mathbb{R}$,
- $R(u) = M - (M-u)^+, \qquad M\in\mathbb{R}$.

Then, if $Ru_0 = u_0$ and some compatibility conditions are satisfied, there exists a solution u to $(\mathscr{P})$ such that $Ru = u$.

Qualitative properties IV, applications

Example 2. Consider the *nonlocal heat equation*:

$$\begin{aligned} \dot{u} + (-\Delta)^s u + \gamma u = g \text{ in } \Omega \times (0,T), \\ u = 0 \text{ in } (\mathbb{R}^d \setminus \Omega) \times (0,T), \\ u(0) = u_0 \text{ in } \Omega. \end{aligned} \tag{$\mathscr{P}$}$$

Let R be given by one of the following

- symmetric decreasing, monotone decreasing rearrangements,
- $Ru(x) = u(rx)$, $\qquad r : \Omega \to \Omega$ linear and $|\det r| = 1$,
- $R(u) = u^+$,
- $R(u) = -u^-$.

Then, if $Ru_0 = u_0$ and some compatibility conditions are satisfied, there exists a solution u to $(\mathscr{P})$ such that $Ru = u$.

Qualitative properties V, applications

Example 3. For all $0 \le u_0 \le K$, $v_0 \ge 0$ a.e. in Ω, there exists a solution (u, v) to the *diffusive Lokta-Volterra prey-predator system*

$$\begin{aligned} &u_t - D_1 \Delta u = Au\left(1 - \frac{u}{K}\right) - \frac{Buv}{1+Eu} \qquad \text{in } \Omega \times (0,T), \\ &v_t - D_2 \Delta v = \frac{Cuv}{1+Eu} - Dv \qquad \text{in } \Omega \times (0,T), \\ &\partial_n u = 0 = \partial_n v \qquad \text{on } \partial\Omega \times (0,T), \\ &u(0) = u_0, \quad v(0) = v_0 \qquad \text{in } \Omega, \end{aligned} \tag{$\mathscr{P}$}$$

such that

$$0 \le u(t) \le K, \quad v(t) \ge 0 \text{ a.e. in } \Omega.$$

Here

$$R(u,v) = ((u \wedge K)^+, v^+).$$

Qualitative properties, advantages

- Large number of applications (($\mathscr{P}$) and R).
- Low regularity is needed (cf. Sliding methods).
- Nonuniqueness of solutions.
- Noninvertible maps R (cf. rearrangements).

Comparison principle

Idea. Let $f = \mathrm{D}\phi_2(u)$. Thus, the WED functional $I_\varepsilon(u) = \int_0^T e^{-t/\varepsilon}(\varepsilon\psi(\dot{u}) + \phi(u) - \phi_2(u))$ admits a minimizer u^ε. Furthermore, $u_\varepsilon \to u$, where u solves $(\mathscr{P})$.
Let $u_0 \le v_0$. Assume

$$I_\varepsilon(u \vee v) + I_\varepsilon(u \wedge v) \le I_\varepsilon(u) + I_\varepsilon(v) \quad \forall u, v.$$

Let

$$u^\varepsilon \in \arg \min_{w(0)=u_0} I_\varepsilon(w)$$

$$v^\varepsilon \in \arg \min_{w(0)=v_0} I_\varepsilon(w)$$

Then,

$$\begin{aligned}
&(u^\varepsilon \wedge v^\varepsilon)(0) = u_0,\\
&(u^\varepsilon \vee v^\varepsilon)(0) = v_0,\\
&I_\varepsilon(u^\varepsilon \wedge v^\varepsilon) \le I_\varepsilon(u^\varepsilon) + I_\varepsilon(v^\varepsilon) - I_\varepsilon(u^\varepsilon \vee v^\varepsilon) \le I_\varepsilon(u^\varepsilon).\\
&I_\varepsilon(u^\varepsilon \vee v^\varepsilon) \le I_\varepsilon(u^\varepsilon) + I_\varepsilon(v^\varepsilon) - I_\varepsilon(u^\varepsilon \wedge v^\varepsilon) \le I_\varepsilon(v^\varepsilon).
\end{aligned}$$

Thus,

$$\begin{aligned}
\tilde{u}^\varepsilon := u^\varepsilon \wedge v^\varepsilon \in \arg \min_{w(0)=u_0} I_\varepsilon(w),\\
\tilde{v}^\varepsilon := u^\varepsilon \vee v^\varepsilon \in \arg \min_{w(0)=v_0} I_\varepsilon(w),\\
\tilde{u}^\varepsilon \le \tilde{v}^\varepsilon.
\end{aligned}$$

Passing to the limit $\varepsilon \to 0$, $\exists u, v$ solutions to $(\mathscr{P})$ such that $u(0) = u_0$, $v(0) = v_0$, and $u \le v$.

Comparison principle, examples

Example 1.
Comparison principle for doubly nonlinear problem:

$$\begin{array}{ll} \alpha & (\dot{u}) - \nabla\cdot(B|\nabla u|^{m-2}\nabla u) + C|u|^{m-2}u - D|u|^{q-2}u = 0. \\ + & \text{B.C.} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{No uniqueness!} \end{array}$$

Example 2.
Comparison principle for the fractional (nonlocal) heat equation:

$$\begin{aligned} \dot{u} + (-\Delta)^s u + \gamma u = f. \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^d \setminus \Omega \end{aligned}$$