Feedback stabilization of the Cahn-Hilliard type system for phase separation

Gabriela Marinoschi
Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest

Based on a joint work with

Viorel Barbu, Pierluigi Colli, Gianni Gilardi

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Problem presentation

The Cahn-Hilliard system (CH)

$$\varphi_t - \Delta\left(-\nu\Delta\varphi + F'(\varphi) - \gamma_0\theta\right) = 0, \text{ in } (0,\infty)\times\Omega,$$

$$(\theta + l_0\varphi)_t - \Delta\theta = 0, \text{ in } (0,\infty)\times\Omega,$$

$$\frac{\partial\theta}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = \frac{\partial\Delta\varphi}{\partial\nu} = 0, \text{ on } (0,\infty)\times\partial\Omega,$$
$$\theta(0) = \theta_0, \ \varphi(0) = \varphi_0, \text{ in } \Omega.$$

$$\varphi = \mathit{order\ parameter}, \ \theta = \mathit{temperature}, \ F(\varphi) = \frac{(\varphi^2-1)^2}{4}$$

$$\nu, \ l_0, \ \gamma_0 > 0 \ \mathit{problem\ parameters}$$

Problem presentation

A simple analysis of the linearized system around a stationary state $(\varphi_\infty, \theta_\infty)$ reveals that, in general, not all of the solutions to the stationary system are aymptotically stable and so, their stabilization via a feedback controller with support in an arbitrary subset $\omega \subset \Omega$ is of crucial importance.

The Cahn-Hilliard system (CH)

$$\varphi_t - \Delta\left(-\nu\Delta\varphi + F'(\varphi) - \gamma_0\theta\right) = 1^*_\omega v, \text{ in } (0,\infty)\times\Omega,$$

$$(\theta + l_0\varphi)_t - \Delta\theta = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega,$$

$$\frac{\partial\theta}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = \frac{\partial\Delta\varphi}{\partial\nu} = 0, \text{ on } (0,\infty)\times\partial\Omega,$$
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Problem presentation

$$\omega \text{ open bounded subset of } \Omega \subset \mathbb{R}^d, \; d = 1, 2, 3$$

$$\omega_0 \text{ is an open subset of } \omega$$

$$1^*_\omega \in C_0^\infty(\Omega), \quad \text{supp } 1^*_\omega \subset \omega, \quad 1^*_\omega > 0 \text{ on } \omega_0$$

Problem presentation

Problem

The aim is to stabilize exponentially the solution to (CH) *around a stationary solution* $(\varphi_\infty, \theta_\infty)$ *by the means of a feedback control*

$$(v, u) = \mathcal{F}(\varphi, \theta)$$

Namely

$$\lim_{t\to\infty}(\varphi(t), \theta(t)) = (\varphi_\infty, \theta_\infty),$$

with exponential decay, as the initial datum (φ_0, θ_0) *is in a neighborhood of* $(\varphi_\infty, \theta_\infty)$.

Problem presentation

Lemma

The stationary system

$$\begin{aligned} \nu\Delta^2\varphi_\infty - \Delta(F'(\varphi_\infty) - \gamma_0\theta) &= 0, \ \text{in } \Omega, \\ -\Delta\theta_\infty &= 0, \ \text{in } \Omega, \\ \frac{\partial\varphi_\infty}{\partial\nu} &= \frac{\partial\Delta\varphi_\infty}{\partial\nu} = \frac{\partial\theta_\infty}{\partial\nu} = 0, \ \text{on } \Gamma \end{aligned}$$

has at least a solution

$$\theta_\infty = constant, \ \varphi_\infty \in H^4(\Omega).$$

Problem presentation

- The stabilization technique already used for Navier-Stokes equations and nonlinear parabolic systems is based on the design of the feedback controller as linear combination of the unstable modes of the corresponding linearized system.

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- V. Barbu, Stabilization of Navier-Stokes Flows, Springer-Verlag, London, 2011

Presentation outline

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1. Problem presentation. Preliminaries
2. Proof of the stabilization of the linear system by a finite dimensional control (contructed as a linear combination of the unstable eigenvalues of the linear operator)
3. Construction of the feedback control $\mathcal{F}$ and proof of its properties
4. Proof of the existence of a unique solution to the *nonlinear closed loop system* (with $(v, u) = \mathcal{F}(\theta, \varphi)$) and stabilization of this solution.

Preliminaries: the transformed nonlinear system

$$\sigma = \alpha_0(\theta + l_0\varphi), \quad \alpha_0 = \sqrt{\frac{\gamma_0}{l_0}}, \quad l := \gamma_0 l_0$$

$$\varphi_t + \nu\Delta^2\varphi - \Delta F'(\varphi) - l\Delta\varphi + \gamma\Delta\sigma = 1^*_\omega v, \text{ in } (0,\infty)\times\Omega,$$

$$\sigma_t - \Delta\sigma + \gamma\Delta\varphi = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega,$$

$$\varphi(0) = \varphi_0, \ \sigma(0) = \sigma_0, \text{ in } \Omega,$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\Delta\varphi}{\partial\nu} = \frac{\partial\sigma}{\partial\nu} = 0, \text{ in } (0,\infty)\times\Gamma.$$

Preliminaries: the new nonlinear system (NNS)

$$y = \varphi - \varphi_\infty, \; z = \sigma - \sigma_\infty$$

$$y_t + \nu\Delta^2 y - \Delta(F''(\varphi_\infty)y) - l\Delta y + \gamma\Delta z = 1^*_\omega v + \Delta F_r(y), \text{ in } (0,\infty)\times\Omega,$$

$$z_t - \Delta z + \gamma\Delta y = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega,$$

$$\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \text{ in } (0,\infty)\times\Gamma,$$

$$y(0) = y_0 = \varphi_0 - \varphi_\infty, \; z(0) = z_0 = \sigma_0 - \sigma_\infty, \text{ in } \Omega.$$

Preliminaries: the transformed nonlinear system

$$y_t + \nu \Delta^2 y - \Delta((\overline{F''_\infty} + g(x))y) - l\Delta y + \gamma \Delta z = 1^*_\omega v + \Delta F_r(y),$$

$$\overline{F''_\infty} := \frac{1}{m_\Omega} \int_\Omega F''(\varphi_\infty(\xi)) d\xi$$

$$g(x) = \frac{3}{m_\Omega} \int_\Omega (\varphi^2_\infty(x) - \varphi^2_\infty(\xi)) d\xi.$$

Preliminaries: the transformed nonlinear system (NS)

$$y_t + \nu\Delta^2 y - F_l \Delta y + \gamma\Delta z = 1^*_\omega v + \Delta(F_r(y) + g(x)y), \text{ in } (0,\infty)\times\Omega,$$

$$z_t - \Delta z + \gamma\Delta y = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega,$$

$$\frac{\partial y}{\partial\nu} = \frac{\partial \Delta y}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, \text{ in } (0,\infty)\times\Gamma,$$

$$y(0) = y_0 = \varphi_0 - \varphi_\infty, \; z(0) = z_0 = \sigma_0 - \sigma_\infty, \text{ in } \Omega.$$

Preliminaries: the linear system (LS)

$$
\begin{aligned}
&y_t + \nu\Delta^2 y - F_l\Delta y + \gamma\Delta z = 1^*_\omega v, \text{ in } (0,\infty)\times\Omega,\\
&z_t - \Delta z + \gamma\Delta y = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega,\\
&\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \text{ in } (0,\infty)\times\Gamma.\\
&y(0) = y_0,\ z(0) = z_0, \text{ in } \Omega.
\end{aligned}
$$

Functional framework

$$H = L^2(\Omega),\ \ V = H^1(\Omega),\ \ V' = (H^1(\Omega))'$$

Introduce $\mathcal{A} : D(\mathcal{A}) \subset H \times H \to H \times H$,

$$\begin{aligned} D(\mathcal{A}) \quad = \quad & \Big\{ w = (y,z) \in H^2(\Omega) \times H^1(\Omega);\ \mathcal{A}w \in H \times H, \\ & \frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma \Big\} \end{aligned}$$

$$\mathcal{A} = \begin{bmatrix} \nu\Delta^2 - F_l\Delta & \gamma\Delta \\ \gamma\Delta & -\Delta \end{bmatrix}$$

The linear system

Lemma

The operator $\mathcal{A}$ is quasi m-accretive on $H \times H$ and its resolvent is compact.

Let $(y_0, z_0) \in H \times H$ and $U = (v, u) \in L^2(0, T; H \times H)$. Then, problem

$$\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) = 1^*_\omega U(t), \text{ a.e. } t \in (0, \infty), \qquad (LS)$$

$$(y(0), z(0)) = (y_0, z_0)$$

has a unique solution

$$\begin{aligned} (y, z) \quad \in \quad & C([0, T]; H \times H) \cap L^2(0, T; H^2(\Omega) \times H^1(\Omega)) \\ & \cap C((0, T]; H^2(\Omega) \times H^1(\Omega)) \end{aligned}$$

for all $T > 0$.

The linear system

$\mathcal{A}$ is self-adjoint $\Longrightarrow$ its eigenvalues are real and semi-simple.
The eigenvectors corresponding to distinct eigenvalues are orthogonal.
The resolvent of $\mathcal{A}$ is compact $\Longrightarrow$ there exists a finite number N of nonpositive eigenvalues

$$\lambda_i \leq 0, \ \text{for } i = 1, ..., N.$$

$$\lambda_1 < \lambda_2 < ... < \lambda_N < \lambda_{N+1} < ...$$

Let $(\varphi_i, \psi_i)_{i \geq 1}$ be the eigenvectors.

2. Stabilization of the linear system by a finite dimensional controller

$$U(t,x) = (v(t,x), u(t,x)) = \sum_{j=1}^{N} w_j(t)(\varphi_j(x), \psi_j(x))$$

The open loop linear system (OLS)

$$\begin{aligned}
\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= \sum_{j=1}^{N} w_j(t) 1_\omega^*(\varphi_j, \psi_j), \text{ a.e. } t > 0, \\
(y(0), z(0)) &= (y^0, z^0).
\end{aligned}$$

2. Stabilization of the linear system by a finite dimensional controller

Lemma

There exist $w_j \in L^2(\mathbb{R}^+)$, $j = 1, ..., N$, *such that the controller*

$$U(t,x) = (v(t,x), u(t,x)) = \sum_{j=1}^{N} w_j(t)(\varphi_j(x), \psi_j(x)), \ \ t \geq 0, \ \ x \in \Omega,$$

stabilizes exponentially (OLS), *that is its solution* (y, z) *satisfies*

$$\|y(t)\|_H + \|z(t)\|_H \leq C_P e^{-kt} \left(\|y^0\|_H + \|z^0\|_H \right), \ \textit{for all } t > 0$$

$$\left(\sum_{j=1}^{N} \int_0^\infty |w_j(t)|^2 \, dt \right)^{1/2} \leq C_P \left(\|y^0\|_H + \|z^0\|_H \right).$$

C_P *is a positive constant depending on the problem parameters* (ν, γ, l) *and* $\|\varphi_\infty\|_{L^2(\Omega)}$.

2. Stabilization of the linear system by a finite dimensional controller

Let $T_0 > 0$ be arbitrary fixed.
The solution is represented by a sum of two pairs of functions such that the functions in the first pair vanishes at $t = T_0$ and the functions in the second pair decrease exponential to 0, as $t \to \infty$.

$$(y(t,x), z(t,x)) = \sum_{j=1}^{\infty} \xi_j(t)(\varphi_j(x), \psi_j(x)), \ (t,x) \in (0,\infty) \times \Omega,$$

$$\xi_i' + \lambda_i \xi_i = \sum_{j=1}^{N} w_j d_{ij}, \quad \xi_i(0) = \xi_{i0}, \text{ for } i \geq 1,$$

$$d_{ij} = \int_{\Omega} 1^*_{\omega}(\varphi_i \varphi_j + \psi_i \psi_j) dx, \ j = 1, ..., N, \ i = 1, ...$$

2. Stabilization of the linear system by a finite dimensional controller

Proof.

(i) System from $i = 1, ..., N$ is null controllable in T_0

$$\xi_i(T_0) = 0 \text{ for all } i = 1, ..., N.$$

Ingredients: system $\{\sqrt{1^*_\omega}\varphi_j, \sqrt{1^*_\omega}\psi_j\}_{j=1}^N$ is linearly independent on ω, Kalman Lemma

(ii) System from $i = N + 1, ...$is stabilized exponentially in origin

$$\begin{aligned} |\xi_i(t)| &\leq C_2 e^{-\lambda_{N+1} t} \left(|\xi_{i0}| + \left(\int_0^{T_0} |w_j(s)|^2 \, ds \right)^{1/2} \right) \\ \text{for } t &> T_0, \ i \geq N+1 \end{aligned}$$

□

3. Construction of the feedback control and properties

Let

$$A : D(A) \subset H \to H, \qquad A = -\Delta + I,$$

$$D(A) = \left\{ w \in H^2(\Omega);\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma \right\}.$$

Define A^α, $\alpha \geq 0$.
A^α is a linear continuous positive and self-adjoint operator on H

$$D(A^\alpha) = \{ w \in H; \|A^\alpha w\|_H < \infty \}$$

$$\|w\|_{D(A^\alpha)} = \|A^\alpha w\|_H .$$

$$D(A^\alpha) \subset H^{2\alpha}(\Omega) \text{ with equality if and only if } 2\alpha < 3/2$$

$$\begin{aligned} & \Phi(y^0, z^0) & (P) \\ = \quad & \text{Min}\left\{ \frac{1}{2}\int_0^\infty \left(\left\| A^{3/2} y(t) \right\|_H^2 + \left\| A^{3/4} z(t) \right\|_H^2 + \| W(t) \|_{\mathbb{R}^N}^2 \right) dt \right\} \end{aligned}$$

subject to (OLS), for all $W = (w_1, ..., w_N) \in L^2(0, \infty; \mathbb{R}^N)$

3. Construction of the feedback control and properties

Lemma

For each pair $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$, *problem* (P) *has a unique optimal solution* $(\{w_j^*\}_{j=1}^N, y^*, z^*)$.

$$
\begin{aligned}
c_1 \left(\left\| A^{1/2} y^0 \right\|_H^2 + \left\| A^{1/4} z^0 \right\|_H^2 \right) &\leq \Phi(y^0, z^0) \\
&\leq c_2 \left(\left\| A^{1/2} y^0 \right\|_H^2 + \left\| A^{1/4} z^0 \right\|_H^2 \right) \\
\forall (y^0, z^0) &\in D(A) \times D(A^{1/2})
\end{aligned}
$$

There exists a linear positive operator

$$R \in \mathcal{L}(D(A^{1/2}) \times D(A^{1/4}); (D(A^{1/2}) \times (D(A^{1/4}))'$$

such that

$$\Phi(y^0, z^0) = \frac{1}{2}\left(R(y^0, z^0), (y^0, z^0)\right) \text{ for all } (y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4}).$$

Moreover, $R(y^0, z^0)$ *is the Gâteaux derivative of the function* Φ *at* (y^0, z^0)

$$\Phi'(y^0, z^0) = R(y^0, z^0), \text{ for all } (y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$$

and R *is self-adjoint.*

3. Construction of the feedback control and properties

$$
\begin{aligned}
\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= \sum_{j=1}^{N} w_j(t) 1^*_\omega(\varphi_j, \psi_j), \text{ a.e. } t > 0, \\
(y(0), z(0)) &= (y^0, z^0).
\end{aligned}
$$

3. Construction of the feedback control and properties

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= BW(t), \text{ a.e. } t > 0, \\ (y(0), z(0)) &= (y^0, z^0). \end{aligned}$$

$$B:\mathbb{R}^N \to H\times H,\quad B^*:H\times H\to \mathbb{R}^N,$$

$$BW=\begin{bmatrix} \sum_{i=1}^{N} 1_\omega^* \varphi_i w_i \\ \sum_{i=1}^{N} 1_\omega^* \psi_i w_i \end{bmatrix} \text{ for all } W=\begin{bmatrix} w_1 \\ \dots \\ w_N \end{bmatrix}\in \mathbb{R}^N$$

and

$$B^*q=\begin{bmatrix} \int_\Omega 1_\omega^*(\varphi_1 q_1+\psi_1 q_2)dx \\ \dots \\ \int_\Omega 1_\omega^*(\varphi_N q_1+\psi_N q_2)dx \end{bmatrix} \text{ for all } q=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\in H\times H.$$

3. Construction of the feedback control and properties

$$(1_\omega^* v, 1_\omega^* u) = 1_\omega^* U = -BB^* R(y^*, z^*)$$

3. Construction of the feedback control and properties

Lemma

Let $W^* = \{w_i^*\}_{i=1}^N$ *and* (y^*, z^*) *be optimal for problem* (P) *corresponding to* $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$*. Then,*

$$W^*(t) = -B^* R(y^*(t), z^*(t)), \text{ for all } t > 0.$$

R has the following properties

$$\begin{aligned} 2c_1 \left\|(y^0, z^0)\right\|^2_{D(A^{1/2})\times D(A^{1/4})} &\leq (R(y^0, z^0), (y^0, z^0))_{H\times H} \\ &\leq 2c_2 \left\|(y^0, z^0)\right\|^2_{D(A^{1/2})\times D(A^{1/4})}, \end{aligned}$$

$$\forall\ (y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$$

Lemma

$$\left\| R(y^0, z^0) \right\|_{H\times H} \leq C_R \left\| (y^0, z^0) \right\|_{D(A)\times D(A^{1/2})},$$

$$\forall\ (y^0, z^0) \in D(A) \times D(A^{1/2})$$

R satisfies the Riccati algebraic equation

$$2\left(R(y^0, z^0), \mathcal{A}(y^0, z^0)\right)_{H\times H} + \left\| B^* R(y^0, z^0) \right\|^2_{\mathbb{R}^N}$$
$$= \left\| A^{3/2} y^0 \right\|^2_H + \left\| A^{3/4} z^0 \right\|^2_H,$$

$\forall\ (y^0, z^0) \in D(A^{3/2}) \times D(A^{3/4})$.

4. Feedback stabilization of the closed loop nonlinear system

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= \mathcal{G}(y(t)) - 1_\omega^* U(t), \text{ a.e. } t > 0, \\ (y(0), z(0)) &= (y_0, z_0), \end{aligned}$$

where (y_0, z_0) is fixed, $\mathcal{G}(y(t)) = (G(y(t)), 0)$ and

$$G(y) = \Delta F_r(y) + \Delta(g(x)y).$$

4. Feedback stabilization of the closed loop nonlinear system

$$
\begin{aligned}
\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= \mathcal{G}(y(t)) - BB^* R(y(t), z(t)), \text{ a.e. } t, \\
(y(0), z(0)) &= (y_0, z_0),
\end{aligned}
$$

where (y_0, z_0) is fixed, $\mathcal{G}(y(t)) = (G(y(t)), 0)$ and

$$G(y) = \Delta F_r(y) + \Delta(g(x)y).$$

4. Feedback stabilization of the closed loop nonlinear system

Theorem

Let $\chi_\infty := \|\nabla\varphi_\infty\|_\infty + \|\Delta\varphi_\infty\|_\infty$. *There exists* $\chi_0 > 0$ *such that the following hold true. If* $\chi_\infty \leq \chi_0$ *there exists* ρ *such that for all pairs*

$$\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})} \leq \rho,$$

the closed loop system has a unique solution

$$\begin{aligned}(y,z) \quad \in \quad & C(0,\infty; H\times H)\cap L^2(0,\infty; D(A^{3/2})\times D(A^{3/4})) \\ & \cap W^{1,2}(0,\infty; (D(A^{1/2})\times D(A^{1/4}))'),\end{aligned}$$

which is exponentially stable, that is

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/4})} \leq C_P e^{-kt}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})}).$$

4. Feedback stabilization of the closed loop nonlinear system

Proof.

Proof is organized in 3 steps: existence, uniqueness and stabilization. □

- Step 1. Existence is proved on every interval $[0, T]$ by the Schauder fixed point theorem.

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- Existence and uniqueness on $[0, \infty)$ follow by those above.

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- Step 2. Uniqueness is proved on $[0, T]$ following by an usual method and using that BB^* is linear continuous from $V' \times V' \to V' \times V'$.
- Existence and uniqueness on $[0, \infty)$ follow by those above.
- Step 3. Estimates using Riccati eq. and the properties of R lead to

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/4})} \leq C_P e^{-kt}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})}).$$

4. Feedback stabilization of the closed loop nonlinear system

Let $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/4})$

$$
\begin{aligned}
S_T \;=\; & \Big\{ (y,z) \in L^2(0,T;H\times H);\ \sup_{t\in(0,T)} \left(\|y(t)\|^2_{D(A^{1/2})} + \|z(t)\|^2_{D(A^{1/4})} \right) \\
& + \int_0^T \left(\left\| A^{3/2} y(t) \right\|^2_H + \left\| A^{3/4} z(t) \right\|^2_H \right) dt \le r^2 \Big\}
\end{aligned}
$$

S_T is a convex closed subset of $L^2(0,T;D(A^{3/2-\varepsilon}) \times H)$, $\varepsilon \in (0, 1/4)$

4. Feedback stabilization of the closed loop nonlinear system

Fix $(\overline{y}, \overline{z}) \in S_T$ and consider the Cauchy problem

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) + BB^* R(y(t), z(t)) &= \mathcal{G}(\overline{y}(t)), \text{ a.e. } t, \\ (y(0), z(0)) &= (y_0, z_0). \end{aligned}$$

Define

$$\Psi_T : S_T \to L^2(0, T; D(A^{3/2-\varepsilon}) \times H), \quad \Psi_T(\overline{y}, \overline{z}) = (y, z)$$

i) $\Psi_T(S_T) \subset S_T$ provided that r is well chosen
ii) $\Psi_T(S_T)$ is relatively compact in $L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$
iii) Ψ_T is continuous in $L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$ norm.

4. Feedback stabilization of the closed loop nonlinear system

- Multiply eq. by $R(y(t), z(t))$ scalarly in $H \times H$ and use Riccati eq.

$$
\begin{aligned}
& \|y(t)\|^2_{D(A^{1/2})} + \|z(t)\|^2_{D(A^{1/4})} + \int_0^t \left(\left\| A^{3/2} y(s) \right\|^2_H + \left\| A^{3/4} z(s) \right| \right. \\
\leq \; & C_P \left(\|y_0\|^2_{D(A^{1/2})} + \|z_0\|^2_{D(A^{1/4})} + \int_0^t \|G(\overline{y}(s))\|^2_H \, ds \right)
\end{aligned}
$$

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$$\leq \quad C_P \left(\|y_0\|^2_{D(A^{1/2})} + \|z_0\|^2_{D(A^{1/4})} + \int_0^t \|G(\overline{y}(s))\|^2_H \, ds \right)$$

- Estimate G norm

$$\int_0^T \|G(\overline{y}(t))\|^2_H \, dt \leq C(r^6 + \|\varphi_\infty\|^2_{2,\infty} r^4 + \|g\|^2_{2,\infty} r^2),$$

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- Fix r

$$C_P \rho^2 + C C_P (r^6 + \|\varphi_\infty\|^2_{2,\infty} r^4 + \|g\|^2_{2,\infty} r^2) \leq r^2$$

4. Feedback stabilization of the closed loop nonlinear system

$$\varphi_\infty \Longrightarrow \chi_\infty := \|\nabla \varphi_\infty\|_\infty + \|\Delta \varphi_\infty\|_\infty$$

$$\chi_0 := \frac{-\|\varphi_\infty\|_\infty + \sqrt{\|\varphi_\infty\|_\infty^2 + CC_P^{-1/2}}}{2}$$

where C is a constant independent of φ_∞ and C_P depends on the problem parameters and $\|\varphi_\infty\|_{L^2(\Omega)}$.
Assume

$$\chi_\infty \leq \chi_0$$

Fix r (depending on $\|\varphi_\infty\|_{2,\infty}$) and $\rho \leq (2C_P)^{-1/2} r$ such that for

$$\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})} \leq \rho,$$

the closed loop system is exponentially stable.

4. Feedback stabilization of the closed loop nonlinear system

$$\chi_\infty := \|\nabla \varphi_\infty\|_\infty + \|\Delta \varphi_\infty\|_\infty \leq \chi_0 \ (\textit{depends on } \ \|\varphi_\infty\|_\infty)$$

Any constant stationary solution can be stabilized.

4. Feedback stabilization of the closed loop nonlinear system

One might work directly with the linearized system assuming the key hypothesis that all unstable eigenvalues $\{\lambda_j\}_{j=1}^{N}$ are semi-simple.
In this case the condition $\|\nabla\varphi_\infty\|_\infty + \|\Delta\varphi_\infty\|_\infty \leq \chi_0$ is no longer necessary.

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Thank you for your attention