OPTIMAL CONTROL PROBLEM FOR THE SOBOLEV TYPE SEMI-LINEAR MODEL

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A lot of initial-boundary value problems for the equations and the systems of equations not resolved with respect to time derivative are considered in the framework of abstract Sobolev type equations[1]

$$L\,\dot{x} = Mx + Bu, \quad \ker L \neq \{0\}, \tag{1}$$

$$L\,\dot{x} + Mx + N(x) = u, \quad \ker L \neq \{0\}. \tag{2}$$

that make up the vast field of non-classical equations of mathematical physics. The Cauchy problem

$$x(0) - x_0 = 0 \tag{3}$$

for degenerate equations (1) or (2) is unsolvable for arbitrary initial values. We consider the Showalter – Sidorov problem[2]

$$L(x(0) - x_0) = 0 \tag{4}$$

for degenerate equation (2), which is a natural generalization of the Cauchy problem. The optimal control problem

$$J(x, u) \to \inf \tag{5}$$

for a Sobolev type equation was first considered by Georgy A. Sviridyuk and Alexander A. Efremov. This research provided the basis for a branch of optimal control studies referring to linear and semi-linear Sobolev type equations.

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[1] Showalter R.E. The Sobolev Equation. *Applicable Analysis*, 1975, vol. 5, no. 1, pp. 15–22; vol. 5, no. 2, pp. 81–89.

[2] Sviridyuk G.A., Zagrebina S.A. The Showalter – Sidorov Problem as a Phenomena of the Sobolev Type Equations. *The Bulletin of Irkutsk State University. Series: Mathematics*, 2010, vol. 3, no. 1, pp. 104–125.

Structure

- The Showalter – Sidorov problem for the abstract Sobolev type semi-linear equation
- Optimal control problem for the abstract Sobolev type semi-linear equation
- The Showalter – Sidorov problem for the Fitz Hugh – Nagumo model
- Optimal control problem for the Fitz Hugh – Nagumo model
- The problem of a start control and final observation for the Fitz Hugh – Nagumo model

Let $\mathcal{H} = (\mathcal{H}; \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space identified with its dual. Let $(\mathfrak{H}, \mathfrak{H}^*)$ and $(\mathfrak{B}, \mathfrak{B}^*)$ be dual (with respect to the duality $\langle \cdot, \cdot \rangle$) pairs of reflexive Banach spaces, moreover, let the embeddings

$$\mathfrak{H} \hookrightarrow \mathfrak{B} \hookrightarrow \mathcal{H} \hookrightarrow \mathfrak{B}^* \hookrightarrow \mathfrak{H}^* \tag{6}$$

be dense and continuous.

Let
(i) $L \in \mathcal{L}(\mathfrak{H}; \mathfrak{H}^*)$ be a linear self-adjoint positive semidefinite Fredholm operator with orthonormal (in the sense of $\mathcal{H}$) set $\{\varphi_k\}$ of eigenvectors is a basis in the space $\mathfrak{H}$;

(ii) $M \in \mathcal{L}(\mathfrak{B}; \mathfrak{B}^*)$ be a linear self-adjoint positive operator;

(iii) $N \in C^r(\mathfrak{B}; \mathfrak{B}^*)$ be an s-monotone operator (i.e. $\left\langle N_y^{\prime} x, x \right\rangle > 0,\ \forall x, y \in \mathfrak{B} \backslash \{0\}$) and a p-coercive operator (i.e. $\langle N(x), x \rangle \geq C_N \|x\|^p$ and $\|N(x)\|_* \leq C^N \|x\|^{p-1}$ for some constants C_N, $C^N \in \mathbb{R}_+$ and $p \in [2, +\infty)$ and for any $x \in \mathfrak{B}$, where $\|\cdot\|$, $\|\cdot\|_*$ are the norms in the spaces $\mathfrak{B}$ and $\mathfrak{B}^*$ respectively).

Set $\{\varphi_k\}$ of eigenvectors of operator L is total in $\mathfrak{B}$, therefore, we seek Galerkin approximation to the solution of problem (2), (4) in the form

$$x^m(t) = \sum_{k=1}^{m} a_k(t)\varphi_k, \quad m > \dim \ker L.$$

Here the coefficients $a_k = a_k(t)$, $k = 1, ..., m$ are defined by the next problem

$$\langle L\dot{x}^m, \varphi_k\rangle + \langle M(x^m), \varphi_k\rangle = \langle u, \varphi_k\rangle, \tag{7}$$

$$\langle L(x^m(0) - x_0), \varphi_k\rangle = 0, \ k = 1, ..., m. \tag{8}$$

Suppose $T_m \in \mathbb{R}_+$, $T_m = T_m(x_0)$, $\mathfrak{H}_i^m = \text{span}\{\varphi_1, \varphi_2, ..., \varphi_m\}$, $\mathfrak{H}^m = \mathfrak{H}_1^m \times \mathfrak{H}_2^m$.

Lemma 1.

For any $x_0 \in \mathfrak{H}$ there exists a unique local solution $x^m \in C^r(0, T_m; \mathfrak{H}^m)$ of the problem (7), (8).

We denote the projection P along $\mathrm{coker} L$ onto $\mathrm{im} L \bigcap \mathfrak{B}^*$. Let us introduce the set

$$\mathrm{coim} L = \{x \in \mathfrak{H} : \langle x, \varphi \rangle = 0, \ \forall \varphi \in \ker L\}.$$

Consider the space

$$\mathfrak{X} = \{x | \ x \in L_\infty(0,T; \mathrm{coim} L) \cap L_p(0,T;\mathfrak{B}), \quad P\dot{x} \in L_2(0,T; \mathrm{coim} L)\}.$$

Consider

$$L\ \dot{x} + Mx + N(x) = u, \ \ker L \neq \{0\}, \tag{2}$$

$$L(x(0) - x_0) = 0. \tag{4}$$

Definition 1.

Vector function $x \in \mathfrak{X}$ is called a solution of equation (2) if it satisfies the following equality:

$$\int_0^T \varphi(t) \left[\left\langle L \frac{d}{dt} x, \psi \right\rangle + \langle Mx + N(x), \psi \rangle \right] dt = \int_0^T \varphi(t) \langle u, \psi \rangle \, dt \forall w \in \mathfrak{H}, \forall \varphi \in L_2(0,T). \tag{9}$$

Theorem 1.

For arbitrary $x_0 \in \mathfrak{B}$, $T \in \mathbb{R}_+$, $u \in L_q(0,T;\mathfrak{B}^*)$ there exists a unique solution $x \in \mathfrak{X}$ of problem (2), (4).

Let $T \in \mathbb{R}_+$. Consider the space $\mathfrak{U} = L_q(0, T; \mathfrak{B}^*), p^{-1} + q^{-1} = 1$, and define a closed convex set $\mathfrak{U}_{ad} \subset \mathfrak{U}$. Consider the optimal control problem (2), (4), (5), where the penalty function is given by the relation

$$J(x, u) = \frac{1}{p} \int_0^T \|x(t) - z_d(t)\|_{\mathfrak{B}}^p \, dt + \frac{N}{q} \int_0^T \|u(t)\|_{\mathfrak{B}^*}^q \, dt,$$

$z_d = z_d(t)$ is the desired state.

Definition 2.

A pair $(\hat{x}, \hat{u}) \in \mathfrak{X} \times \mathfrak{U}_{ad}$ is called optimal control for problem (2), (4), (5) if

$$J(\hat{x}, \hat{u}) = \inf_{(x,u)} J(x, u),$$

where all pairs $(x, u) \in \mathfrak{X} \times \mathfrak{U}_{ad}$ satisfy problem (2), (4) in the sense of definition 1.

Theorem 2.

For arbitrary $x_0 \in \mathfrak{B}, T \in \mathbb{R}_+$ there exists solution of problem (2), (4), (5).

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with the boundary $\partial\Omega$ of class C^∞. Consider the system of Fitz Hugh — Nagumo[3] equations in the cylinder $\Omega \times \mathbb{R}_+$

$$\begin{cases} v_t = \alpha_1 \Delta v + \beta_1 w - \varkappa_1 v + u_1, \\ w_t = \alpha_2 \Delta w + \beta_2 w - \varkappa_2 v - w^3 + u_2. \end{cases} \tag{10}$$

Here,

$w = w(s,t)$ is a function describing the dynamics of membrane potential,

$v = v(s,t)$ is a slow restoring function, associated with the ionic currents,

$\beta_2 \in \mathbb{R}$, $\alpha, \beta_1, \alpha_1, \alpha_2, \varkappa_1, \varkappa_2 \in \mathbb{R}_+$ are fixed parameters:

$\varkappa_1, \beta_1$ are a threshold of excitation and its rate,
α_1 is the electrical conductivity of medium,

α_2 is the repolarization of medium,

$u = (u_1, u_2)$ is an excitation source.

[3] Fitzhugh R. Impulses and Physiological States in Theoretical Models of Nerve Membrane. *Biophysical Journal*, 1961, vol. 1, no. 6, pp. 445–466.

A non-degenerate model, which represents the basic properties of wave excitation, was proposed independently by Fitz Hugh and Nagumo. The system of Fitz Hugh – Nagumo equations consists of two components, often referred to as an activator and an inhibitor, simulating the behavior of chemical elements in membrane. The rate of production of the activator increases as its concentration rises that brings a strong nonlinearity into the system. The inhibitor prevents the growth of the activator concentration. If the rate of change w component substantially exceeds the rate of change of v, then the rate of change of the components of v can be taken equal to zero. For the first time Sviridyuk and Bokareva[4] offered to investigate the degenerate equations system Fitz Hugh – Nagumo in the case when the rate of change of the components of v equal to zero. We consider the degenerate system

$$\begin{cases} -\alpha_1 \Delta v - \beta_1 w + \varkappa_1 v = u_1, \\ w_t - \alpha_2 \Delta w + \beta_2 w + \beta_1 v + w^3 = u_2 \end{cases} \tag{11}$$

with the boundary condition

$$v(s,t) = 0, \ w(s,t) = 0, \ (s,t) \in \partial\Omega \times \mathbb{R} \tag{12}$$

and the Showalter – Sidorov condition

$$w(s,0) - w_0(s) = 0, \ s \in \Omega. \tag{13}$$

[4] Bokareva T.A., Sviridyuk G.A. Whitney Folds in Phase Spaces of Some Semilinear Equations of Sobolev Type. *Mathematical Notes*, 1994, vol. 55, no. 3, pp. 3–10.

Suppose $\mathfrak{H}_i = \overset{\circ}{W}{}_2^1(\Omega)$ and $\mathfrak{B}_i = L_4(\Omega)$, $i = 1, 2$. Consider the Hilbert space $\mathcal{H} = L_2(\Omega) \times L_2(\Omega)$ with the inner product

$$[x, \zeta] = \langle v, \xi \rangle + \langle w, \eta \rangle, \text{where } x = (v, w), \zeta = (\xi, \eta).$$

Define the space as follows $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2$ and $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2$, and by $\mathfrak{H}^*$, $\mathfrak{B}^*$ denote the dual space for the space $\mathfrak{H}$, $\mathfrak{B}$ with respect to the inner product in $\mathcal{H}$, respectively. When $n \leq 4$ there are dense and continuous embeddings

$$\mathfrak{H} \hookrightarrow \mathfrak{B} \hookrightarrow \mathcal{H} \hookrightarrow \mathfrak{B}^* \hookrightarrow \mathfrak{H}^*. \tag{14}$$

Define operators as follows

$$[Lx, \zeta] = \langle w, \eta \rangle, \ \ x, \zeta \in \mathfrak{H},$$

$$[Mx, \zeta] = \langle -\alpha_1 \Delta v - \beta_1 w + \varkappa_1 v, \xi \rangle + \langle -\alpha_2 \Delta w + |\beta_2| w + \beta_1 v, \eta \rangle, \quad x, \zeta \in \mathfrak{H},$$

$$[N(x), \zeta] = \langle w^3, \eta \rangle, \quad x, \zeta \in \mathfrak{B}.$$

Thus, the problem (11) – (13) is reduced to the Showalter – Sidorov problem

$$L(x(0) - x_0) = 0 \tag{4}$$

for an abstract semi-linear equation

$$L \dot{x} + Mx + N(x) = u, \ \ker L \neq \{0\}, \tag{2}$$

where $u = (u_1, u_2)$.

Consider the space

$$\mathfrak{X} = \{x|\ v \in L_2(0,T;\mathfrak{H}_1),\ \ w \in L_\infty(0,T;\mathrm{coim}\ L) \cap L_4(0,T;\mathfrak{B}_2)\}$$

where

$$\mathrm{coim}\ L = \{0\} \times \mathfrak{H}_2.$$

Definition 1.

Vector-function $x \in \mathfrak{X}$ at $T \in \mathbb{R}_+$ is called a weak generalized solution of the Showalter – Sidorov problem (11) – (13), if it satisfies the following equality:

$$\int_0^T \varphi(t)\left([L\tfrac{dx}{dt},\zeta] + [Mx + N(x),\zeta]\right)dt = \int_0^T \varphi(t)[u,\zeta]dt, \quad [L(x-x_0),\zeta] = 0$$
$$\forall \zeta \in \mathfrak{H}^*, \varphi \in L_2(0,T).$$

We search an approximate solution of (11) – (13) in the form

$$v_m(s,t) = \sum_{j=1}^{m} a_j(t)\varphi_j(s), \quad w_m(s,t) = \sum_{j=1}^{m} b_j(t)\varphi_j(s), \tag{15}$$

where the coefficients $a_j = a_j(t),\ b_j = b_j(t),\ j = 1, ..., m$ are determined by system of equations

$$\begin{aligned} &\langle -\alpha_1 \Delta v_m - \beta_1 w_m + \varkappa_1 v_m, \varphi_j \rangle = \langle u_1, \varphi_j \rangle, \\ &\langle \frac{\partial w_m}{\partial t} - \alpha_2 \Delta w_m + |\beta_2| w_m + \beta_1 v_m + w_m^3, \varphi_j \rangle = \langle u_2, \varphi_j \rangle,\ j = 1, ..., m, \end{aligned} \tag{16}$$

with condition

$$\langle w(0) - w_0, \varphi_j \rangle = 0. \tag{17}$$

Suppose $T_m \in \mathbb{R}_+$, $T_m = T_m(x_0)$, $\mathfrak{H}_i^m = \operatorname{span}\{\varphi_1,\ \varphi_2, ...,\ \varphi_m\}$, $\mathfrak{H}^m = \mathfrak{H}_1^m \times \mathfrak{H}_2^m$.

Lemma 2.

For any $x_0 \in \mathfrak{H}$ there exists a unique local solution $x^m \in C^r(0, T_m; \mathfrak{H}^m)$ of problem (16), (17).

$$\begin{cases} -\alpha_1\Delta v - \beta_1 w + \varkappa_1 v = u_1, \\ w_t - \alpha_2\Delta w + \beta_2 w + \beta_1 v + w^3 = u_2, \end{cases} \tag{11}$$

$$v(s,t) = 0,\ w(s,t) = 0,\ (s,t) \in \partial\Omega \times \mathbb{R}, \tag{12}$$

$$w(s,0) - w_0(s) = 0,\ s \in \Omega. \tag{13}$$

The following theorem on the uniqueness of the solution of the Showalter – Sidorov problem for the Fitz Hugh – Nagumo model was proved. It establishes the convergence of the approximate solutions to the precise solution.

Theorem 3.

Let $\alpha_1, \alpha_2, \beta_1, \varkappa_1 \in \mathbb{R}_+$, $\beta_2 \in \mathbb{R}_-$ and $n \leq 4$, then for any $x_0 \in \mathfrak{H}$, $T \in \mathbb{R}_+$, $u_1 \in L_2(0,T;\mathfrak{H}_1^*)$, $u_2 \in L_{\frac{4}{3}}(0,T;\mathfrak{B}_2^*)$ there exists a unique solution $x \in \mathfrak{X}$ to problem (11) – (13).

The optimal control problem for the Fitz Hugh – Nagumo model

For the system of equations

$$\begin{cases} -\alpha_1 \Delta v - \beta_1 w + \varkappa_1 v = u_1, \\ \frac{dw}{dt} - \alpha_2 \Delta w + \beta_2 w + \beta_1 v + w^3 = u_2 \end{cases} \tag{11}$$

with the boundary condition

$$v(s,t) = 0,\ w(s,t) = 0,\ (s,t) \in \partial\Omega \times \mathbb{R} \tag{12}$$

and the Showalter – Sidorov condition

$$w(s,0) - w_0(s) = 0,\ s \in \Omega \tag{13}$$

consider the optimal control problem

$$J(x,u) \to \inf. \tag{5}$$

Construct a control space

$$\mathfrak{U} = \{u = (u_1, u_2) : u_1 \in L_2(0,T;\mathfrak{H}_1), u_2 \in L_{\frac{4}{3}}(0,T;\mathfrak{B}_2)\},$$

choose a closed and convex set $\mathfrak{U}_{ad} \subset \mathfrak{U}$.

In the cylinder $Q_T = \Omega \times (0,T)$ define the penalty functional

$$J(x,u) = \vartheta \int_0^T \left[\|v - v_d\|^2_{\mathfrak{H}_1} + \|w - w_d\|^4_{\mathfrak{B}_2} \right] dt + \\ +(1-\vartheta) \int_0^T [\|u_1\|^2_{\mathfrak{H}_1^*} + \|u_2\|^{\frac{4}{3}}_{\mathfrak{B}_2^*}] dt, \vartheta \in (0,1), \tag{18}$$

where $x_d = (v_d, w_d)$ is the target state of the system.
The following theorem establishes the existence of an optimal control problem of solutions to Showalter – Sidorov problem for the Fitz Hugh – Nagumo model.

Theorem 4.

Let $\alpha_1, \alpha_2, \beta_1, \varkappa_1 \in \mathbb{R}_+$, $\beta_2 \in \mathbb{R}_-$ and $n \leq 4$, then for any $x_0 \in \mathfrak{H}$, $T \in \mathbb{R}_+$ there exists an optimal control for problem (11) – (13).

The problem of a start control and final observation

For the system of homogeneous equations

$$\left\{ \begin{array}{l} 0 = \alpha_1 \Delta v + \beta_1 w - \varkappa_1 v, \\ \frac{dw}{dt} = \alpha_2 \Delta w + \beta_2 w - \varkappa_2 v - w^3 \end{array} \right. \tag{19}$$

with initial-boundary value conditions

$$v(s,t) = 0, w(s,t) = 0, \ (s,t) \in \partial\Omega \times \mathbb{R}_+, \tag{20}$$

$$w(s,0) = u(s), \ s \in \Omega \tag{21}$$

consider the problem of a start control and final observation

$$J(x(T), u) \to \inf . \tag{22}$$

Construct a control space $\mathfrak{U} = \mathfrak{B}_2$ and select closed and convex set $\mathfrak{U}_{ad} \subset \mathfrak{U}$. Define the functional by formula:

$$J(x,u) = \vartheta \|v(T) - v_f\|^2_{\mathfrak{H}_1} + \vartheta \|w(T) - w_f\|^4_{\mathfrak{B}_2} + (1-\vartheta)\|u\|^4_{\mathfrak{B}_2}, \qquad (23)$$

where $x_f(s) = (v_f(s), w_f(s))$ is the required state of the system, which it achieves with a minimum initial impact of the passage of time $t = T$.

Definition 4.

A pair $(\hat{x}(T), \hat{u}) \in [\mathfrak{H}_1 \times \mathfrak{B}_2] \times \mathfrak{U}_{ad}$ is called a solution of (19) – (22), if

$$J(\hat{x}(T), \hat{u}) = \inf_{(x,u)} J(x(T), u),$$

and all pairs $(\hat{x}, \hat{u})$ satisfy problem (19) – (21) in the sense of definition 1. Vector- function $\hat{u}$ is called the starting control for problem (19) – (22).

The following theorem establishes the existence of a start control and final observation of the Showalter – Sidorov problem for the Fitz Hugh – Nagumo model.

Theorem 5.

Let $\alpha_1, \alpha_2, \beta_1, \varkappa_1 \in \mathbb{R}_+$, $\beta_2 \in \mathbb{R}_-$, $T \in \mathbb{R}_+$ *and* $n \leq 4$, *then there exists a solution* $(\hat{x}(T), \hat{u})$ *problem (19) – (22).*

Thank you for attention!