Optimal error estimates of parabolic optimal control problems with a moving point source

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Joint work with Boris Vexler, TU München

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Outline

Model Problem

$$\min_{q,u} J(q,u) := \frac{1}{2}\int_0^T \|u(t)-\widehat{u}(t)\|^2_{L^2(\Omega)}dt + \frac{\alpha}{2}\int_0^T |q(t)|^2 dt$$

subject to

$$\begin{aligned} u_t(t,x) - \Delta u(t,x) &= q(t)\delta_{\gamma(t)}, & (t,x) &\in I\times\Omega,\\ u(t,x) &= 0, & (t,x) &\in I\times\partial\Omega,\\ u(0,x) &= 0, & x &\in \Omega. \end{aligned}$$

subject to pointwise control constraints

$$q_a \le q(t) \le q_b \quad \text{a. e. in } I.$$

Here,

- $I=[0,T]$, $\alpha>0$
- $\widehat{u}\in L^2(I;L^\infty(\Omega))$ is given
- $\delta_{\gamma(t)}$ is the Dirac delta function at point $x_t=\gamma(t)$ at each t, i.e.

$$\int_\Omega \delta_{\gamma(t)}(x)v(x)dx = v(\gamma(t)).$$

Assumptions:

Assumption 1

$\Omega \subset \mathbb{R}^2$, is a convex polygonal domain

Assumption 2

$$\gamma(t) \in C^1(\bar{I}) \quad \max_{t\in\bar{I}} |\gamma'(t)| \leq C_\gamma.$$

Assumption 3

$$\gamma(t) \subset \Omega_0 \subset\subset \Omega, \quad \text{for any} \quad t \in I.$$

Literature

- Fixed Pointwise Control :
  - J.-L. Lions, 1972
  - J. Droniou and J.-P. Raymond, 2000
  - W. Gong, M. Hinze, and Z. Zhou, 2014
  - D. Leykekhman and B. Vexler, 2013
  - D. Leykekhman and B. Vexler, 2016

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- Moving Pointwise Control:
  - P.A. Nguyen and J.-P. Raymond, 2001
  - S. Anita and J.-P. Raymond, 2003
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  - W. Gong, and N. Yan, 2016 mistake in the proof?

Optimality system

State equation

$$
\begin{aligned}
\partial_t \bar{u} - \Delta \bar{u} &= \bar{q}(t)\, \delta_{\gamma(t)} && \text{in } I \times \Omega, \\
\bar{u} &= 0 && \text{on } I \times \partial\Omega, \\
\bar{u}(0) &= 0 && \text{in } \Omega.
\end{aligned}
$$

Adjoint equation

$$
\begin{aligned}
-\partial_t \bar{z} - \Delta \bar{z} &= \bar{u} - u_d && \text{in } I \times \Omega, \\
\bar{z} &= 0 && \text{on } I \times \partial\Omega, \\
\bar{z}(T) &= 0 && \text{in } \Omega.
\end{aligned}
$$

Optimality condition

$$
\bar{q}(t) = P_{[q_a, q_b]} \left(-\frac{1}{\alpha} \bar{z}(t, \gamma(t)) \right) \quad \text{for almost all } \; t \in (0, T)
$$

Regularity

State

$\bar{u} \in L^2(I; W_0^{1,s}(\Omega)) \cap H^1(I; W^{-1,s}(\Omega)), \quad s < 2$

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- $\bar{z} \in L^2(I; H^2(\Omega)) \cap H^1(I; L^2(\Omega))$
  $\bar{z} \in L^r(I; W^{2,p}(\Omega_0)) \cap W^{1,r}(I; L^p(\Omega_0))$, $p < \infty$,$r < \infty$, $\Omega_0 \subset\subset \Omega$

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- $\bar{z} \in L^2(I; H^2(\Omega)) \cap H^1(I; L^2(\Omega))$
  $\bar{z} \in L^r(I; W^{2,p}(\Omega_0)) \cap W^{1,r}(I; L^p(\Omega_0))$, $p < \infty$,$r < \infty$, $\Omega_0 \subset\subset \Omega$
  $\bar{z} \in C^\gamma(\bar{I}; C(\overline{\Omega_0})), \quad \gamma < 1$

Finite element discretization in space and time

Discretization concept

- state/adjoint variable: dG(0) in time (implicit Euler), P_1 in space
- control variable: dG(0) in time
- maximal time step k, maximal spacial mesh size h

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Discretized optimal control problem

$$\text{Minimize } \; J(q_{kh}, u_{kh}) = \tfrac{1}{2}\|u_{kh} - u_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha}{2}\|q_{kh}\|^2_{L^2(0,T)}$$

subject to $u_{kh} \in X_{kh}$, $q_{kh} \in Q_{kh,\text{ad}} = Q_{kh} \cap Q_{\text{ad}}$

$$B(u_{kh}, \varphi_{kh}) = \int_0^T q_{kh}(t)\varphi_{kh}(t, \gamma_k(t))dt \quad \text{for all} \quad \varphi_{kh} \in X_{kh},$$

where $\gamma_k \,|_{I_m} = \gamma(t_m) := \gamma_{k,m}$.

$\rightarrow$ $B(v, \varphi) = \sum_{m=1}^{M} \langle v_t, \varphi \rangle_{I_m \times \Omega} + (\nabla v, \nabla \varphi)_{I \times \Omega} + \sum_{m=2}^{M} ([v]_{m-1}, \varphi^+_{m-1})_\Omega + (v^+_0, \varphi^+_0)_\Omega$

Main result

Theorem

Let $\bar{q}$ be optimal control for the problem and $\bar{q}_{kh}$ be the optimal discrete solution. Then there exists a constant C independent of h and k such that

$$\|\bar{q} - \bar{q}_{kh}\|_{L^2(I)} \leq C|\ln h|^2 \ln \frac{T}{k} \left(k + h^2\right) + C_\gamma \ln \frac{T}{k} k.$$

Main difficulty of the proof

The reduced cost functional $j\colon Q \to \mathbb{R}$ on the control space $Q = L^2(I)$

$$j(q) = J(q, u(q)),$$

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$$j'(q)(\delta q) = \int_I (\alpha q(t) + z(t, \gamma(t)))\, \delta q(t)\, dt,$$

where $z = z(q)$ is the solution of the adjoint equation.

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where $z = z(q)$ is the solution of the adjoint equation.

Similarly, we define the discrete reduced cost functional $j_{kh}\colon Q \to \mathbb{R}$ by

$$j_{kh}(q) = J(q, u_{kh}(q)),$$

Note, the control variable q is not explicitly discretized.

The directional derivative of j_{kh}

$$j'_{kh}(q)(\delta q) = \int_I (\alpha q(t) + z_{kh}(t, \gamma_k(t)))\, \delta q(t)\, dt,$$

where $z_{kh} = z_{kh}(q)$ is the solution of the discrete adjoint equation.

Main difficulty of the proof

Due to the quadratic structure of discrete reduced functional j_{kh} the second derivative $j''_{kh}(q)(p,p)$ is independent of q and there holds

Coercivity

$$j''_{kh}(q)(p,p) \geq \alpha \|p\|_I^2 \quad \text{for all} \;\; p \in Q.$$

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Using optimality conditions and the fact that $\bar{q}, \bar{q}_{kh} \in Q_{\text{ad}}$ we obtain

$$-j'_{kh}(\bar{q}_{kh})(\bar{q} - \bar{q}_{kh}) \leq 0 \leq -j'(\bar{q})(\bar{q} - \bar{q}_{kh}).$$

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Using the coercivity, we get

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$$\begin{aligned} \alpha \|\bar{q} - \bar{q}_{kh}\|_I^2 &\leq j''_{kh}(\bar{q})(\bar{q} - \bar{q}_{kh}, \bar{q} - \bar{q}_{kh}) \\ &= j'_{kh}(\bar{q})(\bar{q} - \bar{q}_{kh}) - j'_{kh}(\bar{q}_{kh})(\bar{q} - \bar{q}_{kh}) \end{aligned}$$

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&= (z(\bar{q})(t, \gamma(t)) - z_{kh}(\bar{q})(t, \gamma_k(t)), \bar{q} - \bar{q}_{kh})_I
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&= (z(\bar{q})(t,\gamma(t)) - z_{kh}(\bar{q})(t,\gamma_k(t)), \bar{q} - \bar{q}_{kh})_I \\
&\leq \left(\int_0^T |z(\bar{q})(t,\gamma(t)) - z_{kh}(\bar{q})(t,\gamma_k(t))|^2\, dt\right)^{\frac{1}{2}} \|\bar{q} - \bar{q}_{kh}\|_I.
\end{aligned}$$

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Thus the main task is to estimate

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However, in the case of moving point sources

$$\int_0^T |z(\bar{q})(t,\gamma_k(t)) - z_{kh}(\bar{q})(t,\gamma_k(t))|^2\,dt \nleq \sup_{x\in\Omega_0}\int_0^T |z(\bar{q})(t,x) - z_{kh}(\bar{q})(t,x)|^2\,dt.$$

So the error estimates from the previous talk can not be applied.

Best approximation results

Instead the analysis requires:

Theorem (Global best approximation result)

Let v be the solution of the heat equation and $v_{kh} \in X_{kh}$ it's dG(0)cG(1) approximation, i.e. $B(v - v_{kh}, \varphi_{kh}) = 0$ for all $\varphi_{kh} \in X_{kh}$. Then

$$\int_I |(v - v_{kh})(t, \gamma_k)|^2 dt \le c\ell_h \inf_{\chi \in X_{kh}} \left(\|v - \chi\|^2_{L^2(I;L^\infty(\Omega))} + \|v - \pi_k v\|^2_{L^2(I;L^\infty(\Omega))} \right),$$

where $\gamma_k \mid_{I_m} = \gamma(t_m) := \gamma_{k,m}$.

and

Interior Best approximation results

Theorem (Interior (local) best approximation result)

Let $B_{d,m} := B_d(\gamma(t_m))$ denote a ball of radius d centered at $\gamma(t_m)$. Let v be the solution of the heat equation and $v_{kh} \in X_{kh}$ it's dG(0)cG(1) approximation. Let $d > 4h$. Then there exists a constant C independent of h, k and d such that

$$\int_0^T |(v - v_{kh})(t, \gamma_k(t))|^2 dt$$
$$\leq C\ell_{kh} \inf_{\chi \in X_{k,h}^{q,r}} \left\{ \sum_{m=1}^{M} \left(\|v-\chi\|^2_{L^2(I_m;L^\infty(B_{d,m}))} + \|\pi_k v - \chi\|^2_{L^2(I_m;L^\infty(B_{d,m}))} \right) \right.$$
$$\left. +d^{-2} \left(\|v-\chi\|^2_{L^2(I;L^2(\Omega))} + \|\pi_k v - \chi\|^2_{L^2(I;L^2(\Omega))} + h^2 \|\nabla(v-\chi)\|^2_{L^2(I;L^2(\Omega))} \right) \right\}. \tag{1}$$

Conclusions

- Optimal error estimates
    1. Global and interior (almost) best approximations for

$$\int_I |(v - v_{kh})(t, \gamma_k)|^2 dt$$

- No coupling between k and h
- Rather weak assumptions on $\gamma(t)$

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- 3D case

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- 3D case much more technical
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- 3D case   much more technical
- Optimal error estimates for controls along moving low dimensional manifolds.
  Error estimates in W.Gong, N.Yan 2016 are suboptimal in 3D

THANK YOU