Control Strategies for the Fokker-Planck Equation

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Motivation

- dragged Brownian particle following Langevin equation

  $dX_s = Y(s)ds,\ dY_s = -\beta Y_s ds + F(X,s)ds + \sqrt{2\beta kT/m}dB(s)$

  particle confined by potential V: force $F(X,s) = -\nabla V(X,s)$
  $\beta >>,\ t = \frac{s}{\beta},\ \nu = \frac{kT}{m},$

- Smoluchowski equation

$$dX_t = -\nabla V(X,t) + \sqrt{2\nu}dB_t$$

- atomic force microscopy, single molecule pulling, optical trapping

- collinate light from laser into aperature of microscope objetive

- control by optical tweezer $V(X_t,t) = W(X_t) + u(X_t,t).$

The Fokker-Planck equation

Consider probability distribution function

$$\rho(x,t)\mathrm{d}x = \mathbb{P}[X_t \in [x, x+\mathrm{d}x)]$$

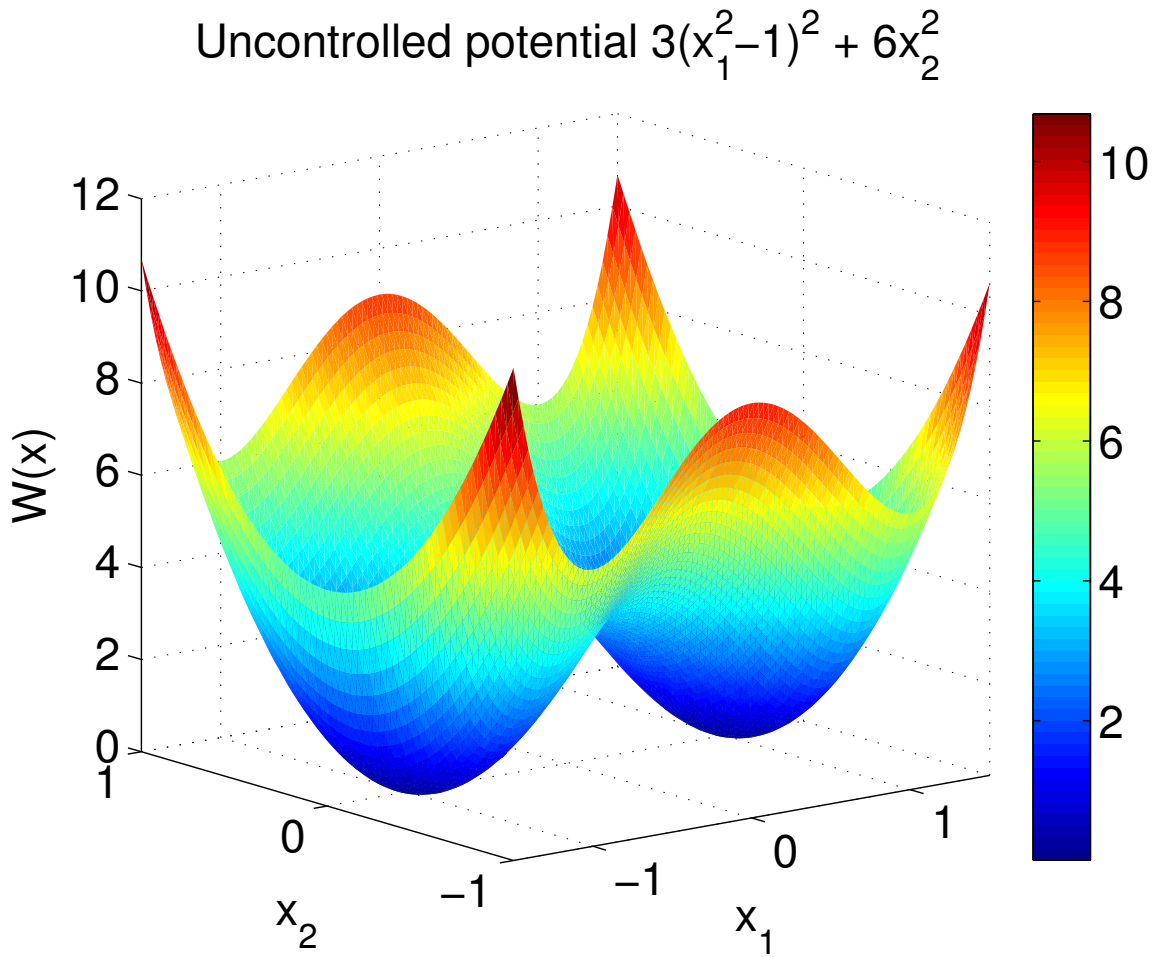
Fokker-Planck equation

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla V) && \text{in } \Omega \times (0,\infty), \\
0 &= (\nu \nabla \rho + \rho \nabla V) \cdot \vec{n} && \text{on } \Gamma \times (0,\infty), \\
\rho(x,0) &= \rho_0(x) && \text{in } \Omega,
\end{aligned}$$

- $\Omega \subset \mathbb{R}^n$ bounded open set with boundary $\Gamma = \partial\Omega$,
- ρ_0 initial probability distribution with $\int_\Omega \rho_0(x)\mathrm{d}x = 1$.

[Hartmann et al.'13]

The Fokker-Planck equation



The Fokker-Planck equation

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Control of the Fokker-Planck equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla V) && \text{in } \Omega \times (0, \infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla V) \cdot \vec{n} && \text{on } \Gamma \times (0, \infty), \\ \rho(x, 0) &= \rho_0(x) && \text{in } \Omega, \end{aligned}$$

- system converges to stationary distribution $\rho_\infty(x)$, (Boltzmann distribution)
- particles have to cross energy barrier between potential wells, $\rightsquigarrow$ may be inadequately slow, $\sim \exp(\Delta W / \nu)$
- can we control the potential $V(x, t) = W(x) + \alpha(x) u(t)$,
- if we can, what are "good" choices for α?

Assumptions:

- $W \in W^{2,\max(2,n+\varepsilon)}(\Omega), \varepsilon > 0$
- $\alpha \in W^{2,\max(2,n+\varepsilon)}(\Omega), \varepsilon > 0$ with $\nabla \alpha \cdot \vec{n} = 0$ on Γ

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Outline

Motivation

The Fokker Planck equation

A Riccati-based local feedback law

A Lyapunov-based global feedback law

A numerical study

Solutions to the Fokker-Planck equation

For $T > 0$ we call ρ a (variational) solution on $(0, T)$ if

$$\rho \in W(0,T) = L^2(0,T; H^1(\Omega)) \cap H^1(0,T; (H^1(\Omega))^*)$$

and for a.e. $t \in (0,T)$

$$\langle \rho_t(t), v \rangle + \langle \nu \nabla \rho(t) + \rho(t) \nabla W, \nabla v \rangle + u(t) \langle \rho(t) \nabla \alpha, \nabla v \rangle = 0 \ \forall v \in H^1(\Omega)$$

$$\rho(0) = \rho_0.$$

Proposition

(i) $u \in L^2(0,T), \rho_0 \in L^2(\Omega) \Rightarrow \exists!$ solution $\rho \in W(0,T)$

(ii) If moreover $\Delta\alpha \in L^\infty(\Omega)$

$\Rightarrow \rho_t \in L^2(0,T; L^2(\Omega)),\ \rho \in C([0,T]; H^1(\Omega))$

Proposition

Let $u \in L^2(0,T)$ and $\rho_0 \in L^2(\Omega)$.

(i) For every $t \in [0,T]$ we have $\int_\Omega \rho(t)\,\mathrm{d}x = \int_\Omega \rho_0\,\mathrm{d}x$.

(ii) If $\rho_0 \geq 0$ a.e. on Ω, then $\rho(x,t) \geq 0\ \forall t > 0$ and a.e. on Ω.

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A bilinear control system

Consider the bilinear control system

$$\begin{aligned}\dot{\rho}(t) &= \mathcal{A}\rho(t) + u(t)\mathcal{N}\rho(t),\\ \rho(0) &= \rho_0,\end{aligned}$$

where the operators $\mathcal{A}$ and $\mathcal{N}$ are defined as follows

$$\begin{aligned}\mathcal{A}\colon \mathcal{D}(\mathcal{A}) &\subset L^2(\Omega) \to L^2(\Omega),\\ \mathcal{D}(\mathcal{A}) &= \left\{\rho \in H^2(\Omega)\,|\,(\nu\nabla\rho + \rho\nabla W)\cdot\vec{n} = 0 \text{ on } \Gamma\right\},\\ \mathcal{A}\rho &= \nu\Delta\rho + \nabla\cdot(\rho\nabla W),\end{aligned}$$

$$\mathcal{N}\colon H^1(\Omega) \to L^2(\Omega), \quad \mathcal{N}\rho = \nabla\cdot(\rho\nabla\alpha).$$

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$$\begin{aligned}\dot{\rho}(t) &= \mathcal{A}\rho(t) + u(t)\mathcal{N}\rho(t),\\ \rho(0) &= \rho_0,\end{aligned}$$

its $L^2(\Omega)$-adjoints are given by

$$\begin{aligned}\mathcal{A}^*\colon \mathcal{D}(\mathcal{A}^*) \subset L^2(\Omega) &\to L^2(\Omega),\\ \mathcal{D}(\mathcal{A}^*) &= \left\{\varphi \in H^2(\Omega)\,|(\nu\nabla\varphi)\cdot\vec{n} = 0 \text{ on } \Gamma\right\},\\ \mathcal{A}^*\phi &= \nu\Delta\varphi - \nabla W\cdot\nabla\varphi,\end{aligned}$$

$$\mathcal{N}^*\colon H^1(\Omega) \to L^2(\Omega), \quad \mathcal{N}^*\varphi = -\nabla\varphi\cdot\nabla\alpha.$$

The symmetrized operator

cp. [Risken.'96]

We introduce $\Phi(x) = \log(\nu) + \frac{W(x)}{\nu}$ and the operator

$$\mathcal{A}_s : \mathcal{D}(\mathcal{A}_s) \subset L^2(\Omega) \to L^2(\Omega),$$

$$\mathcal{D}(\mathcal{A}_s) = \left\{ \varrho \in H^2(\Omega) \,\middle|\, (\nu \nabla \varrho + \frac{1}{2} \varrho \nabla W) \cdot \vec{n} = 0 \text{ on } \Gamma \right\},$$

$$\mathcal{A}_s = e^{\frac{\Phi}{2}} \mathcal{A} e^{-\frac{\Phi}{2}}.$$

Lemma

(i) $\mathcal{A}_s = \mathcal{A}_s^*$, $\sigma(\mathcal{A}_s) = \sigma_p(\mathcal{A}_s) = \sigma(\mathcal{A}) \subset \overline{\mathbb{R}}_-$, eigenfunctions $\{\varrho_i\}_{i=0}^{\infty}$ of $\mathcal{A}_s$ form a complete orthogonal set

(ii) ψ_i eigenfunction of $\mathcal{A} \Leftrightarrow e^{\frac{\Phi}{2}} \psi_i$ eigenfunction of $\mathcal{A}_s$
ψ_i eigenfunction of $\mathcal{A} \Leftrightarrow e^{\Phi} \psi_i$ is an eigenfunction of $\mathcal{A}^*$

(iii) $\rho_\infty = e^{-\Phi}$ is an eigenfunction of $\mathcal{A}$ with $\mathcal{A}\rho_\infty = 0$

A shifted problem

Instead of ρ, consider the shifted state $y := \rho - \rho_\infty$

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y(t) + u(t)\mathcal{N}y(t) + \mathcal{B}u(t), \\ y(0) &= \rho_0 - \rho_\infty, \end{aligned}$$

with $\mathcal{B} = \mathcal{N}\rho_\infty$.

The control operator $\mathcal{B}$ and its adjoint are defined as

$$\begin{aligned} \mathcal{B}\colon \mathbb{R} \to L^2(\Omega), &\quad \mathcal{B}c = c\mathcal{N}\rho_\infty = c\nabla\cdot(\rho_\infty\nabla\alpha), \\ \mathcal{B}^*\colon L^2(\Omega) \to \mathbb{R}, &\quad \mathcal{B}^*v = \langle \mathcal{N}\rho_\infty, v\rangle. \end{aligned}$$

Some useful projections

Projection $\mathcal{P}$ onto $\mathbb{1}^\perp$ along ρ_∞

$$\mathcal{P}\colon L^2(\Omega) \to L^2(\Omega), \quad \mathcal{P}y = y - \int_\Omega y \,\mathrm{d}x\, \rho_\infty,$$

$$\mathcal{Y}_\mathcal{P} = \operatorname{im}(\mathcal{P}) = \left\{ v \in L^2(\Omega)\colon \int_\Omega v \,\mathrm{d}x = 0 \right\}, \quad \ker(\mathcal{P}) = \operatorname{span}\{\rho_\infty\}.$$

Complementary projection $\mathcal{Q}$

$$\mathcal{Q}\colon L^2(\Omega) \to L^2(\Omega), \quad \mathcal{Q}y = (I - \mathcal{P})y = \int_\Omega y \,\mathrm{d}x\, \rho_\infty,$$

$$\operatorname{im}(\mathcal{Q}) = \ker(\mathcal{P}), \quad \ker(\mathcal{Q}) = \operatorname{im}(\mathcal{P}).$$

Orthogonal projection on $\mathcal{Y}_\mathcal{P}$:

$$\Theta\colon L^2(\Omega) \to L^2(\Omega), \quad \Theta y = y - \frac{1}{|\Omega|} \int_\Omega y \,\mathrm{d}x\, \mathbb{1},$$

$$\operatorname{im}(\Theta) = \operatorname{im}(\mathcal{P}) = \mathcal{Y}_\mathcal{P}, \quad \ker(\Theta) = \{\mathbb{1}\}.$$

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Some useful projections cont'd

The $L^2(\Omega)$ adjoint of $\mathcal{P}$ is the projection $\mathcal{P}^*$ onto $\rho_\infty^\perp$ along $\mathbb{1}$

$$\mathcal{P}^* \colon L^2(\Omega) \to L^2(\Omega), \quad \mathcal{P}^* y = y - \int_\Omega \rho_\infty y \, \mathrm{d}x \, \mathbb{1},$$

$$\operatorname{im}(\mathcal{P}^*) = \left\{ v \in L^2(\Omega) \colon \int_\Omega \rho_\infty v \, \mathrm{d}x = 0 \right\}, \quad \ker(\mathcal{P}^*) = \{\mathbb{1}\}.$$

The complementary projection $\mathcal{Q}^*$ is given by

$$\mathcal{Q}^* \colon L^2(\Omega) \to L^2(\Omega), \quad \mathcal{Q}^* y = \int_\Omega \rho_\infty y \, \mathrm{d}x \, \mathbb{1},$$

$$\operatorname{im}(\mathcal{Q}^*) = \ker(\mathcal{P}^*), \quad \ker(\mathcal{Q}^*) = \operatorname{im}(\mathcal{P}^*).$$

Decoupling the problem

By means of $\mathcal{P}$ and $\mathcal{Q}$ we can decompose our state space

$$\mathcal{Y} = L^2(\Omega) = \operatorname{im}(\mathcal{P}) \oplus \operatorname{im}(\mathcal{Q}) =: \mathcal{Y}_\mathcal{P} \oplus \mathcal{Y}_\mathcal{Q},$$
$$y = y_\mathcal{P} + y_\mathcal{Q} = \mathcal{P}y + \mathcal{Q}y,\ y \in L^2(\Omega).$$

$\rightsquigarrow$ applying $\mathcal{P}$ and $\mathcal{Q}$ yields

$$\begin{pmatrix} \dot{y}_\mathcal{P} \\ \dot{y}_\mathcal{Q} \end{pmatrix} = \begin{pmatrix} \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{A} \\ \mathcal{Q}\mathcal{A} & \mathcal{Q}\mathcal{A} \end{pmatrix} \begin{pmatrix} y_\mathcal{P} \\ y_\mathcal{Q} \end{pmatrix} + u \begin{pmatrix} \mathcal{P}\mathcal{N} & \mathcal{P}\mathcal{N} \\ \mathcal{Q}\mathcal{N} & \mathcal{Q}\mathcal{N} \end{pmatrix} \begin{pmatrix} y_\mathcal{P} \\ y_\mathcal{Q} \end{pmatrix} + u \begin{pmatrix} \mathcal{P}\mathcal{B} \\ \mathcal{Q}\mathcal{B} \end{pmatrix}.$$

$\rightsquigarrow$ simplifies to

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$$\dot{y}_\mathcal{P} = \widehat{\mathcal{A}} y_\mathcal{P} + u\widehat{\mathcal{N}} y_\mathcal{P} + \widehat{\mathcal{B}} u, \quad y_\mathcal{P}(0) = \mathcal{P}\rho_0,$$
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$$\dot{y}_\mathcal{P} = \widehat{\mathcal{A}} y_\mathcal{P} + u\widehat{\mathcal{N}} y_\mathcal{P} + \widehat{\mathcal{B}} u, \quad y_\mathcal{P}(0) = \mathcal{P}\rho_0,$$
$$y_\mathcal{Q}(t) = \mathcal{Q}\rho_0 - \rho_\infty = 0,\ t \geq 0.$$

Outline

Motivation

The Fokker Planck equation

A Riccati-based local feedback law

A Lyapunov-based global feedback law

A numerical study

An LQR problem for the linearized system

For $\delta > 0$ let us focus on the linearized system

$$\dot{y}_{\mathcal{P}} = (\widehat{\mathcal{A}} + \delta I) y_{\mathcal{P}}(t) + \widehat{\mathcal{B}} u, \quad y_{\mathcal{P}}(0) = \mathcal{P}\rho_0,$$

together with the quadratic cost functional

$$J(y_{\mathcal{P}}, u) = \frac{1}{2} \int_0^\infty \langle y_{\mathcal{P}}(t), \mathcal{M} y_{\mathcal{P}}(t) \rangle_{L^2(\Omega)} \, \mathrm{d}t + \frac{1}{2} \int_0^\infty |u(t)|^2 \, \mathrm{d}t,$$

where $\mathcal{M} \in \mathcal{L}(\mathcal{Y}_{\mathcal{P}})$ is a self-adjoint nonnegative operator on $\mathcal{Y}_{\mathcal{P}}$.

Riccati-based feedback law: $u = -\widehat{\mathcal{B}}^* \widehat{\Pi} y_{\mathcal{P}}$

The choice of α

Recall that we have $\widehat{\mathcal{B}} = \mathcal{B} = \mathcal{N}\rho_\infty = \nabla \cdot (\rho_\infty \nabla \alpha)$

$\rightsquigarrow$ what is a good choice for α?

Let φ_i be eigenfunctions of $\mathcal{A}^*$ associated to the eigenvalues

$$-\delta \leq \lambda_q \leq \cdots \leq \lambda_2 < 0 = \lambda_1.$$

Determine α as solution to the elliptic equation

$$\begin{aligned} \nabla \cdot (\rho_\infty \nabla \alpha) &= \mathcal{P} \sum_{i=2}^{q} e^{-\Phi} \varphi_i \quad \text{in } \Omega, \\ (\rho_\infty \nabla \alpha) \cdot \vec{n} &= 0 \qquad\qquad\qquad\quad \text{on } \Gamma. \end{aligned}$$

$\Rightarrow \exists!$ solution $\alpha \in W^{2,\max(2,n+\varepsilon)}(\Omega)/\mathbb{R}$

The ∞-dimensional Hautus test

Consider now the operator Riccati equation

$$(\widehat{\mathcal{A}} + \delta I)^* \widehat{\Pi} + \widehat{\Pi}(\widehat{\mathcal{A}} + \delta I) - \widehat{\Pi}\widehat{\mathcal{B}}\widehat{\mathcal{B}}^* \widehat{\Pi} + \mathcal{M} = 0.$$

The pair $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$ is δ-stabilizable if

$$\ker(\lambda I - \widehat{\mathcal{A}}^*) \cap \ker(\widehat{\mathcal{B}}^*) = \{0\} \quad \text{for } \lambda \in \overline{\mathbb{C}}_{-\delta} \cap \sigma(\widehat{\mathcal{A}}^*).$$

Assume that (λ_j, ϕ_j), $j \in \{2, \ldots, q\}$ is an eigenpair of $\widehat{\mathcal{A}}^*$.

$$\widehat{\mathcal{B}}^* \phi_j = \langle \widehat{\mathcal{B}}, \phi_j \rangle = \left\langle \mathcal{P} \sum_{i=2}^{q} e^{-\Phi} \varphi_i, \phi_j \right\rangle = \sum_{i=2}^{q} \langle e^{-\Phi} \varphi_i, \varphi_j \rangle = \| e^{-\frac{\Phi}{2}} \varphi_j \|^2$$

e.g. [Curtain/Zwart'95]

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e.g. [Curtain/Zwart'95]

The Riccati equation on $L^2(\Omega)$

Consider the two Riccati equations

$$(1)\quad (\mathcal{A}+\delta\mathcal{P})^*\Pi+\Pi(\mathcal{A}+\delta\mathcal{P})-\Pi\mathcal{B}\mathcal{B}^*\Pi+\mathcal{P}^*\mathcal{M}\mathcal{P}=0,\quad \Pi\in\mathcal{L}(\mathcal{Y}),$$

$$(2)\quad (\widehat{\mathcal{A}}^*+\delta I)\widehat{\Pi}+\widehat{\Pi}(\widehat{\mathcal{A}}+\delta I)-\widehat{\Pi}\widehat{\mathcal{B}}\widehat{\mathcal{B}}^*\widehat{\Pi}+\mathcal{M}=0,\quad \widehat{\Pi}\in\mathcal{L}(\mathcal{Y}_{\mathcal{P}}).$$

Lemma

(i) If $\widehat{\Pi}\in\mathcal{L}(\mathcal{Y}_{\mathcal{P}})$ is a solution to (2), then for all $\gamma\in\mathbb{R}$, $\Pi=\mathcal{P}^*\widehat{\Pi}\mathcal{P}+\gamma\mathbb{1}\mathbb{1}^*\in\mathcal{L}(\mathcal{Y})$ is a solution to (1).

(ii) If the Hautus criterion is satisfied, then a solution $\Pi\in\mathcal{L}(\mathcal{Y})$ to (1) exists and $\exists\gamma\in\mathbb{R}$ such that $\Pi=P^*\widehat{\Pi}P+\gamma\mathbb{1}\mathbb{1}^*$ is a solution to (2) where $\widehat{\Pi}:=\Theta\Pi I_{\mathcal{P}}\in\mathcal{L}(\mathcal{Y}_{\mathcal{P}})$.

Local stabilization of the nonlinear system

What happens if we apply $u = -\widehat{\mathcal{B}}^*\widehat{\Pi}y_{\mathcal{P}}$ to the nonlinear system

$$\dot{y}_{\mathcal{P}} = \widehat{\mathcal{A}}y_{\mathcal{P}} + u\widehat{\mathcal{N}}y_{\mathcal{P}} + \widehat{\mathcal{B}}u, \quad y_{\mathcal{P}}(0) = \mathcal{P}\rho_0.$$

Theorem

There exist $C, \widetilde{C} > 0$ such that if $\|\rho_0\|_{L^2(\Omega)} \leq \frac{3}{16C^2\widetilde{C}}$, then

$$\dot{y}_{\mathcal{P}} = (\widehat{\mathcal{A}} - \widehat{\mathcal{B}}\widehat{\mathcal{B}}^*\widehat{\Pi})y_{\mathcal{P}} - (\widehat{\mathcal{B}}^*\widehat{\Pi}y_{\mathcal{P}})\widehat{\mathcal{N}}y_{\mathcal{P}}, \quad y_{\mathcal{P}}(0) = \mathcal{P}\rho_0,$$

admits a unique solution

$$y_{\mathcal{P}} \in W_{\mathcal{P}}^{\infty} := L^2(0,\infty; H^1(\Omega) \cap \mathcal{Y}_{\mathcal{P}}) \cap H^1(0,\infty; [H^1(\Omega) \cap \mathcal{Y}_{\mathcal{P}}]')$$

satisfying $\|e^{\delta t}y_{\mathcal{P}}\|_{W_{\mathcal{P}}^{\infty}} \leq \frac{1}{4C\widetilde{C}}$. In particular, $\|y_{\mathcal{P}}\|_{L^2(\Omega)} \leq \widehat{C}e^{-\delta t}\|\rho_0\|_{L^2(\Omega)}$.

Compare e.g.: M. Badra, I. Lasiecka, J.-P. Raymond, R. Triggiani,...

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An alternative choice for α

Let (λ_2, ψ_2) denote the second eigenpair of $\widehat{\mathcal{A}}$.

Determine α as solution to the elliptic equation

$$\begin{aligned} \nabla \cdot (\rho_\infty \nabla \alpha) &= \psi_2 \quad \text{in } \Omega, \\ (\rho_\infty \nabla \alpha) \cdot \vec{n} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Further choose $\mu > 0$ such that

$$\langle (\mu I - \widehat{\mathcal{A}}) y_{\mathcal{P}}, y_{\mathcal{P}} \rangle_{L^2(\Omega)} \geq \langle y_{\mathcal{P}}, y_{\mathcal{P}} \rangle_{H^1(\Omega)}, \quad \text{for all } y_{\mathcal{P}} \in \mathcal{D}(\widehat{\mathcal{A}}).$$

Determine the solution Υ to the Lyapunov equation:

$$\langle \Upsilon y_{\mathcal{P}}, \widehat{\mathcal{A}} z_{\mathcal{P}} \rangle + \langle \widehat{\mathcal{A}} y_{\mathcal{P}}, \Upsilon z_{\mathcal{P}} \rangle = -2\mu \langle y_{\mathcal{P}}, z_{\mathcal{P}} \rangle, \ y_{\mathcal{P}}, z_{\mathcal{P}} \in \mathcal{D}(\widehat{\mathcal{A}}).$$

A global feedback law

$$\langle \Upsilon y_{\mathcal{P}}, \widehat{\mathcal{A}} z_{\mathcal{P}} \rangle + \langle \widehat{\mathcal{A}} y_{\mathcal{P}}, \Upsilon z_{\mathcal{P}} \rangle = -2\mu \langle y_{\mathcal{P}}, z_{\mathcal{P}} \rangle, \;\; y_{\mathcal{P}}, z_{\mathcal{P}} \in \mathcal{D}(\widehat{\mathcal{A}}).$$

Theorem

Let μ and Υ be as before. Consider the system

$$\dot{y}_{\mathcal{P}} = \widehat{\mathcal{A}} y_{\mathcal{P}} + u \widehat{\mathcal{N}} y_{\mathcal{P}} + \widehat{\mathcal{B}} u, \quad y_{\mathcal{P}}(0) = \mathcal{P}\rho_0.$$

where the control u is defined by the feedback law

$$u = -\langle \widehat{\mathcal{B}} + \widehat{\mathcal{N}} y_{\mathcal{P}}, \Upsilon y_{\mathcal{P}} + y_{\mathcal{P}} \rangle.$$

Then $V(y_{\mathcal{P}}) := \langle y_{\mathcal{P}}, \Upsilon y_{\mathcal{P}} + y_{\mathcal{P}} \rangle$ is a global Lyapunov function.

Theorem

Let $\lambda_i, i = 2, 3, \ldots$ denote the eigenvalues of $\widehat{\mathcal{A}}$. Assume that

$$\widetilde{\lambda}_2 := \lambda_2 - \|\psi_2\|^2 + \frac{\mu}{\lambda_2}\|\psi_2\|^2 \neq \lambda_j, \quad j = 3, \ldots$$

Then for the spectrum of the linearized closed loop operator it holds that

$$\sigma(\widehat{\mathcal{A}} - \widehat{\mathcal{B}}\widehat{\mathcal{B}}^* \Upsilon - \widehat{\mathcal{B}}\widehat{\mathcal{B}}^*) = \{\widetilde{\lambda}_2\} \cup \{\lambda_j\}, \quad j \geq 3.$$

Outline

Motivation

The Fokker Planck equation

A Riccati-based local feedback law

A Lyapunov-based global feedback law

A numerical study

Finite-dimensional properties

Assume spatial discretization yields $A, N, \rho^d, e = \frac{1}{d}\left(1, \ldots, 1\right)^T$.

- $\int_\Omega \rho(t)\,\mathrm{d}x = \int_\Omega \rho_0\,\mathrm{d}x \rightsquigarrow e^T\rho^d(t) = e^T\rho_0^d = 1$
- $\rho_0 \geq 0 \Rightarrow \rho(x,t) \geq 0\ \forall t \rightsquigarrow A$ is a Metzler matrix
- $A_s = DAD^{-1}$ with $D = \mathrm{diag}(e^{\frac{\Phi^d}{2}})$ is a symmetric matrix
- $A^T e = N^T e = 0 = A\rho_\infty^d$
- $\mu = \max(\varepsilon, \varepsilon + \frac{1}{2}\lambda_{\max}(\widehat{A} + \widehat{A}^T))$

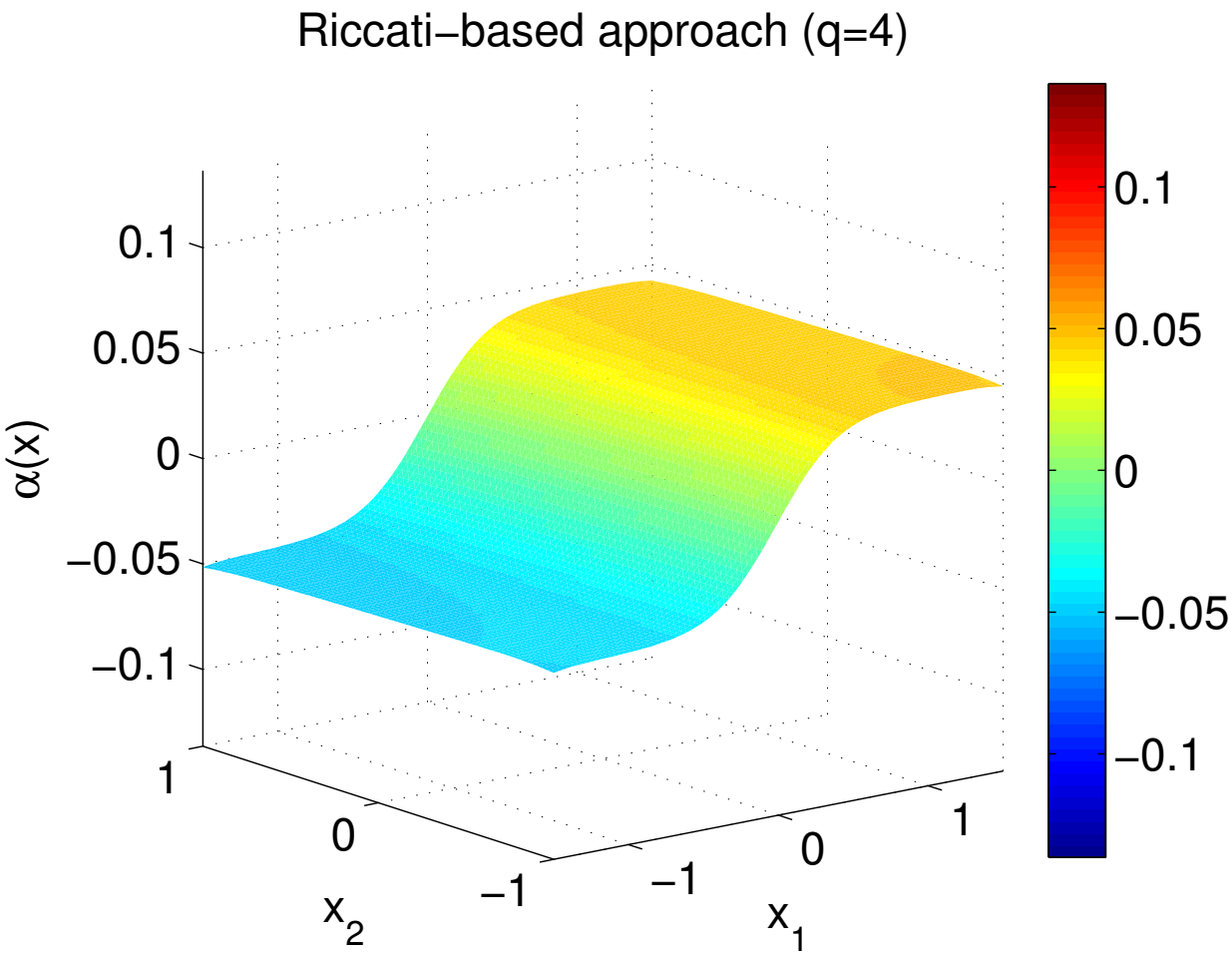
A two dimensional double well potential

As a numerical example, let us consider

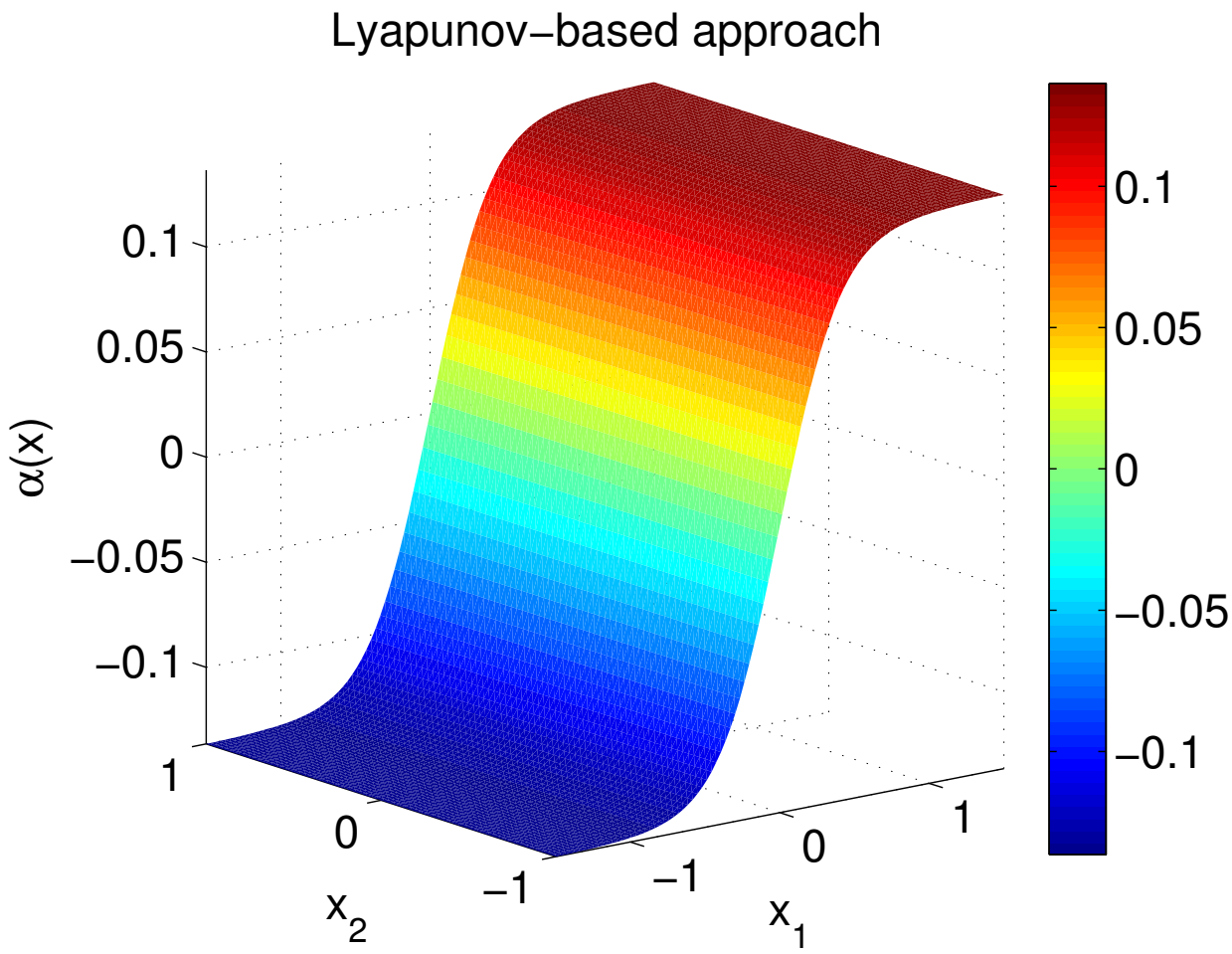
$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla W) + u \nabla \cdot (\rho \nabla \alpha) && \text{in } \Omega \times (0, \infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla W) \cdot \vec{n} && \text{on } \Gamma \times (0, \infty), \\ \rho(x, 0) &= \rho_0(x) && \text{in } \Omega. \end{aligned}$$

- $\Omega = (-1.5, 1.5) \times (-1.1)$
- $W(x) = 3(x_1^2 - 1)^2 + 6x_2^2$
- finite differences with $n_x \cdot n_y = 96 \cdot 64 = 6144$ grid points
- upwind scheme for convective terms

A comparison of different α



A comparison of different α



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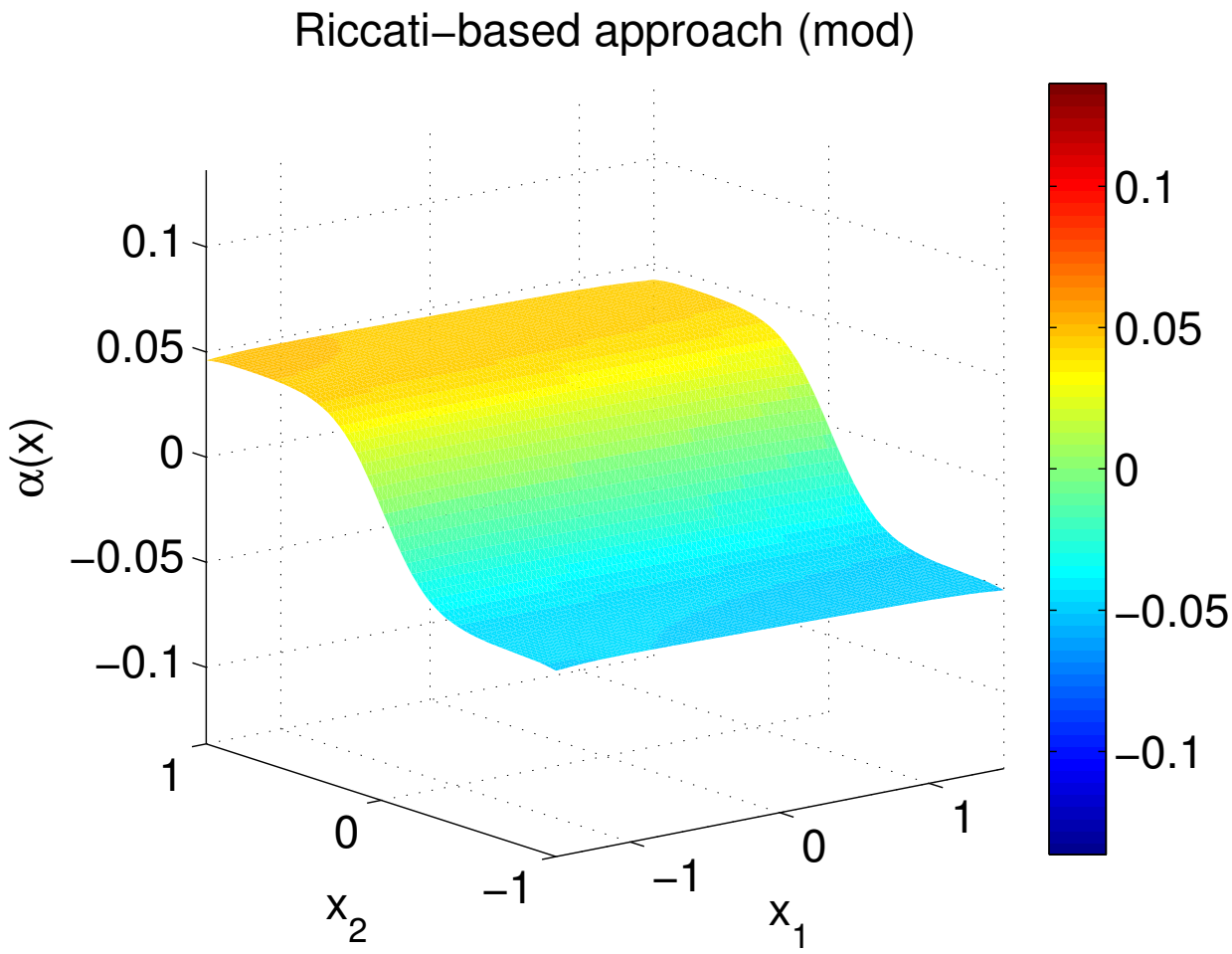

Some numerical results

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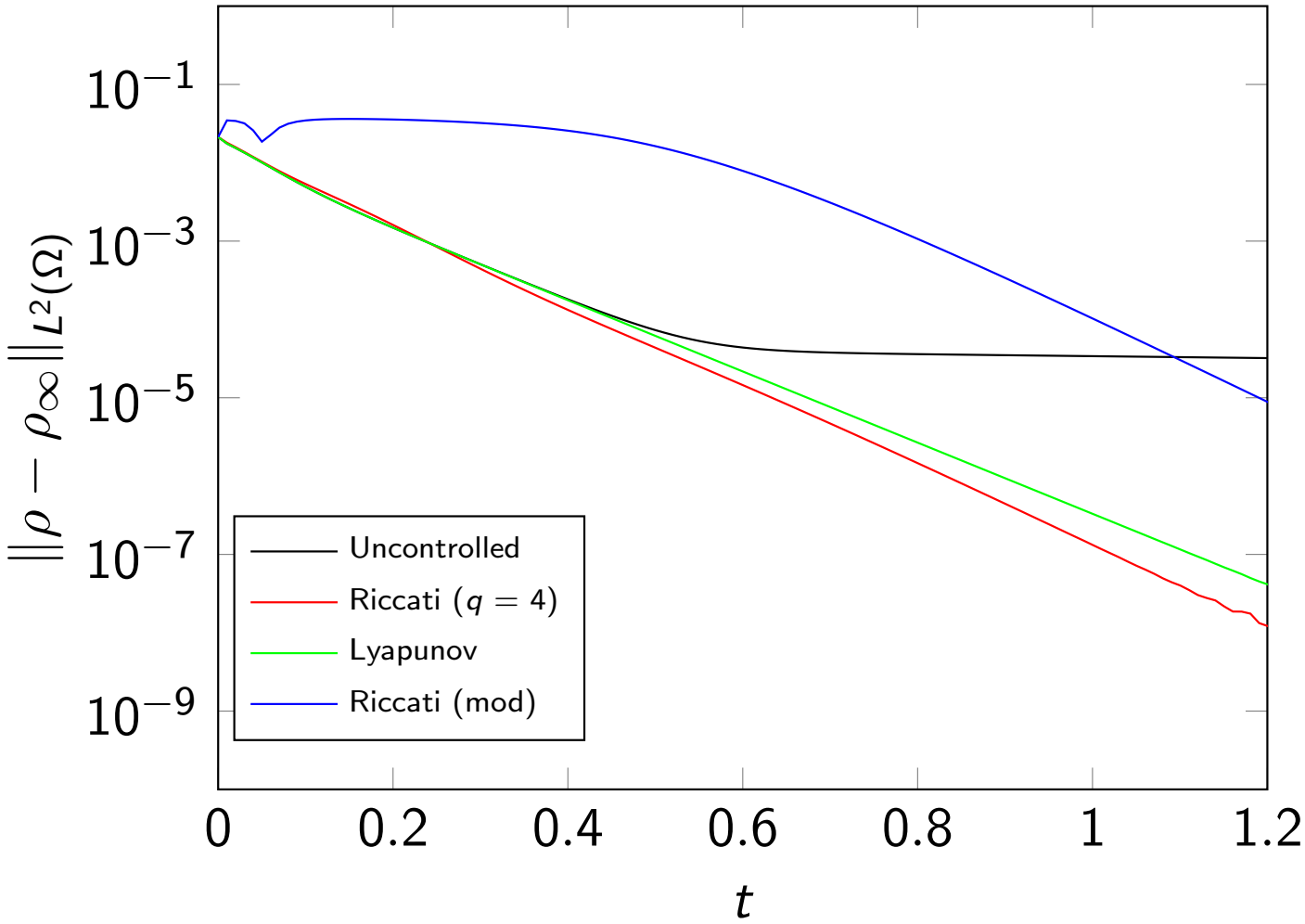

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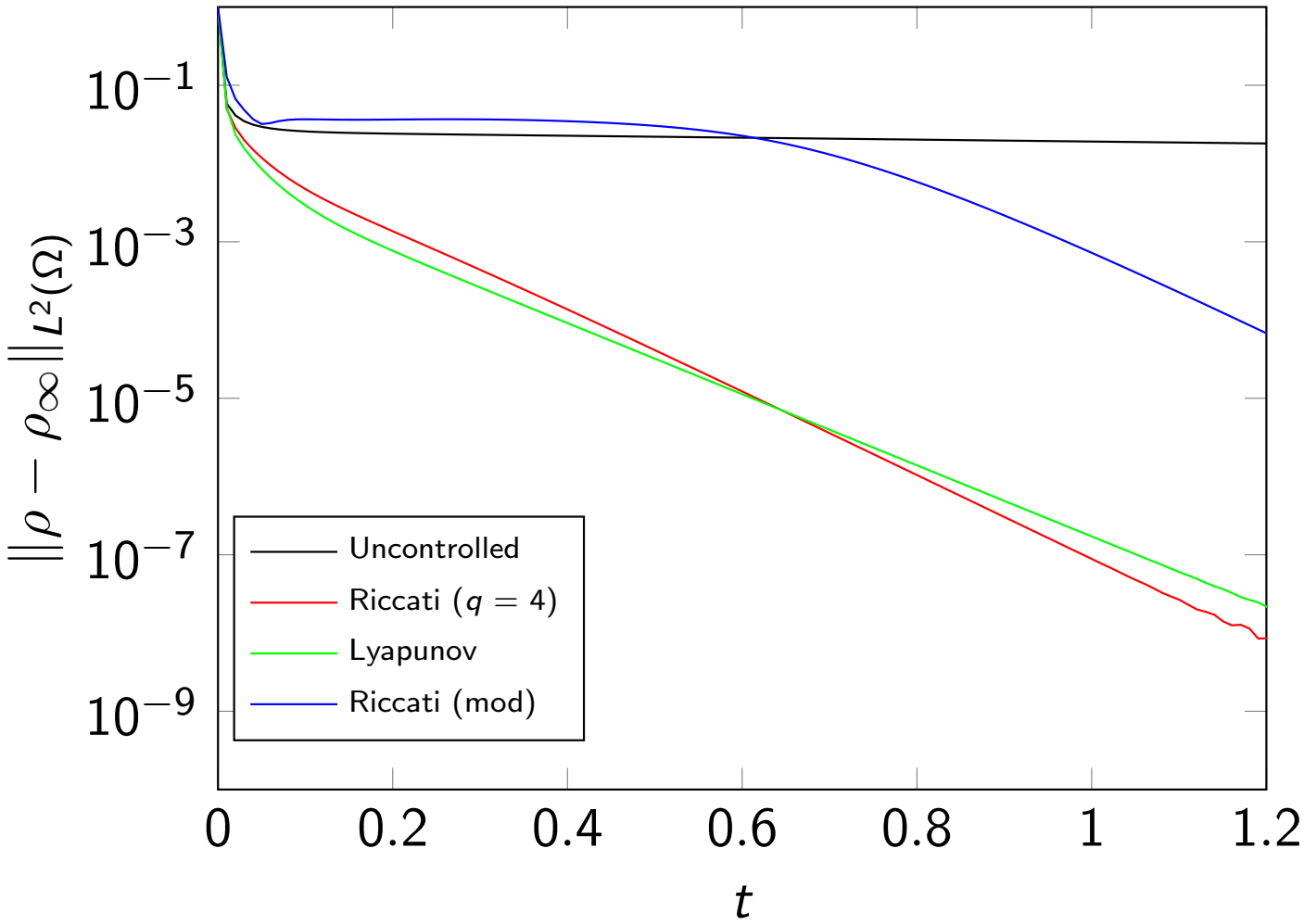

Conclusion

- Fokker-Planck equation yields a bilinear control system.
- Decoupled the system by spectral projections.
- Riccati-based feedback law $\Rightarrow$ local stabilization.
- Lyapunov-based feedback law $\Rightarrow$ global stabilization .
- Numerically efficient approaches?
- Optimal feedback law for bilinear system?

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Thank you for your attention!