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Nonlinear System of PDEs for Biofilm Development

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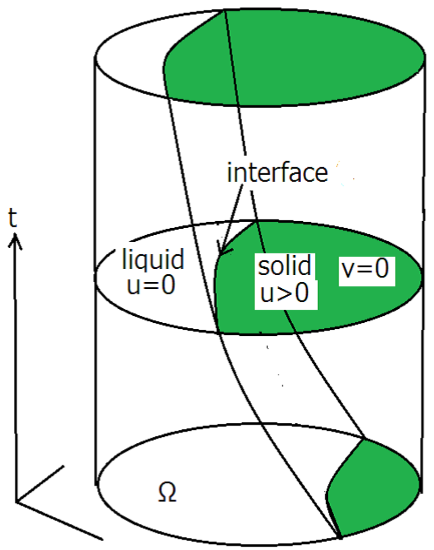
Contents

1. Biofilm development

Eberl-Parker-van Loosdrecht [3](2001) (Sharp interface model)

$\Omega \subset \mathbf{R}^3$, $Q = \Omega \times (0, T)$, Q_ℓ (liquid region), Q_s (solid region),

u: biomass density, $\mathbf{v}$: velocity of the fluid, w: nutrient concentration



$$(B) \qquad u_t - \Delta\beta(u) + bu = f(w)u, \ \ 0 \le u < u^*, \text{in } Q,$$

$$(NS) \qquad \mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla P}{\rho}, \ \ \text{div } \mathbf{v} = 0, \ \ \text{in } Q_\ell,$$

$$\mathbf{v} = 0 \text{ in } Q_s,$$

$$(N) \qquad w_t - \text{div}(d(u)\nabla w) + \mathbf{v}\cdot\nabla w = -f(w)u,$$

$$0 \le w \le 1, \text{in } Q,$$

· β is an increasing function on $[0, u^*)$, $\beta(0) = \beta'(0) = 0$, $\beta(u) \uparrow \infty \ (u \uparrow u^*)$,

· $f(w)u$ is called the nutrient consumption:

$$f(w) = \frac{k_1 w}{k_2 + w} \text{ with constants } k_1, \ k_2 > 0, \ \text{(growth rate)}$$

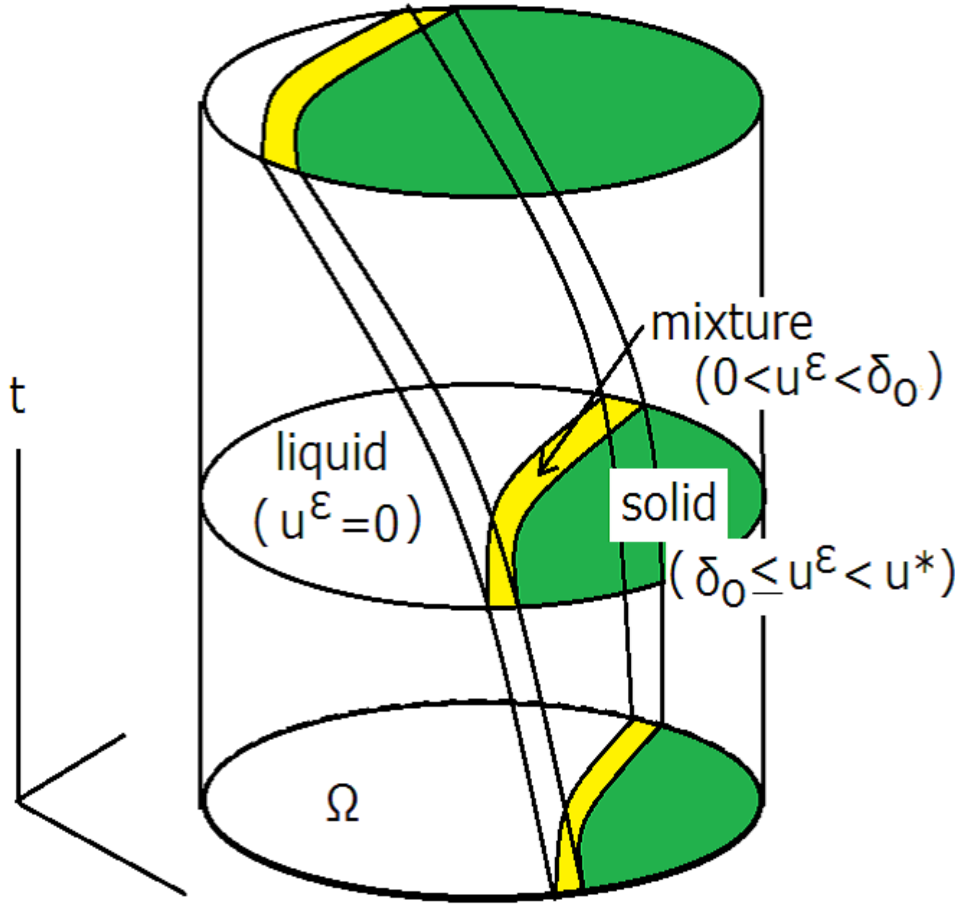
· $d(u)$ is bounded, strictly positive and smooth in $u \in [0, u^*]$,

· $b, \ \nu, \ \rho$ are positive constants, P is the pressure of the fluid.

A relaxation model

We postulate that there is a thin layer between solid and liquid.

It is determined by the spatial average of u: $u^\varepsilon := \rho_\varepsilon * u$, $\delta_0 \in (0, u^*)$

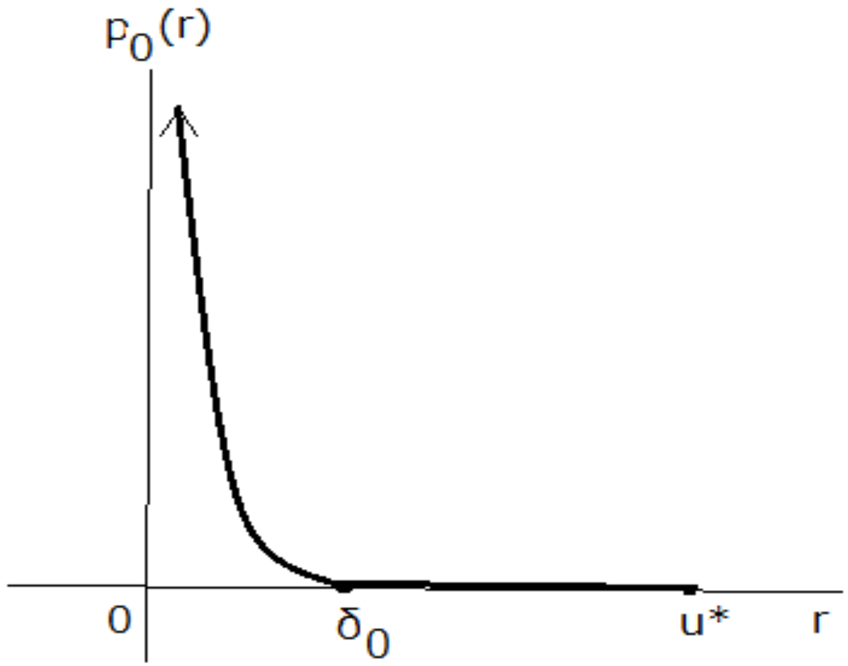


(solid) $\delta_0 \le u^\varepsilon(x,t) < u^*$, $\mathbf{v} = 0$

(mixture) $0 < u^\varepsilon < \delta_0$,

$0 \le |\mathbf{v}| \le p_0(u^\varepsilon)$

(liquid) $u^\varepsilon = 0$, no constraint on $\mathbf{v}$



2. Approximate problems and variational formulation

(1) Weak formulation of (B): $u_t - \Delta\beta(u) + \mathbf{v}\cdot\nabla u^\varepsilon + bu \ni f(w)u$ in Q

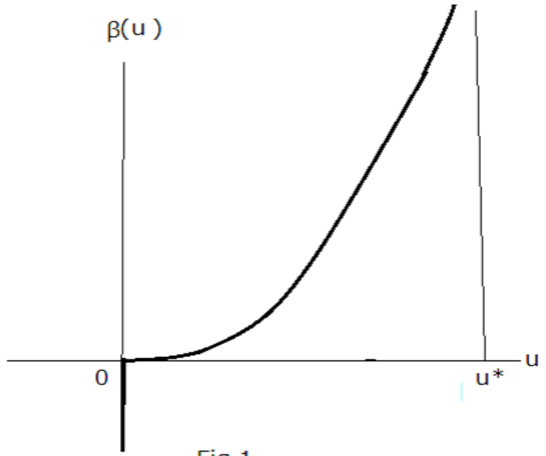


Fig.1

$$H := L^2(\Omega),\ V := H^1(\Omega),\ V \subset H \subset V^*$$

$$|z|_V = \{|\nabla z|_H^2 + n_0 \int_\Gamma |z|^2 d\Gamma\}^{\frac{1}{2}},\ n_0 > 0;$$

$< \cdot, \cdot >$: $V^* \times V \to \mathbf{R}$, duality;

$F : V \to V^*$, duality map,

$$< F\tilde{u}, z > := \int_\Omega \nabla\tilde{u}\cdot\nabla z dx + n_0 \int_\Gamma \tilde{u} z d\Gamma$$

$(F\tilde{u} \approx -\Delta\tilde{u}$ in $\Omega, -\frac{\partial\tilde{u}}{\partial n} = n_0\tilde{u}$ on $\Gamma)$

Given $w \in L^2(0,T;V)$, $\mathbf{v} \in L^2(0,T;V^3)$ and $u_0 \in H$, we consider:

$$(B; w, \mathbf{v}, u_0)\quad \begin{cases} u \in W^{1,2}(0,T;V^*),\ \tilde{u} \in L^2(0,T;V),\ \tilde{u} \in \beta(u) \text{ a.e. on } Q, \\ u'(t) + F\tilde{u}(t) + \mathbf{v}(t)\cdot\nabla u^\varepsilon(t) + bu(t) = f(w(t))u(t) \text{ in } V^*, \\ u(\cdot,0) = u_0 \text{ in } V^* \end{cases}$$

where β: maximal monotone, $\beta(0) \in 0$, $\beta'_+(0) = 0$, $\beta(r) \uparrow \infty$ $(r \uparrow u^*)$,

See Brézis [1] (1973) and Damlamian-K. [2] (1999) for $-\Delta\beta$.

(2) Approximate Navier-Stokes variational inequality

(NS) $\quad \mathbf{v}_t - \nu\Delta\mathbf{v} + \partial I_{K(u^\varepsilon;t)}(\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v} \ni -\frac{\nabla P}{\rho}$, div$\mathbf{v} = 0$, in Q,

$\mathcal{D}_\sigma(\Omega) := \{\mathbf{z} \in \mathcal{D}(\Omega)^3 \mid \text{div } \mathbf{z} = 0 \text{ in } \Omega\}$,

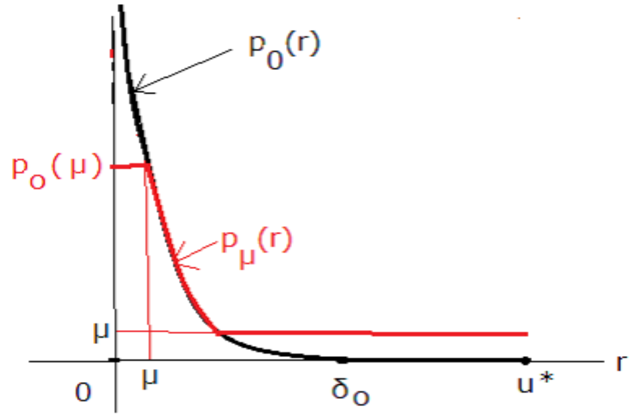
$\mathbf{H}_\sigma := \overline{\mathcal{D}_\sigma(\Omega)}$ in $L^2(\Omega)^3$,

$\mathbf{V}_\sigma := \overline{\mathcal{D}_\sigma(\Omega)}$ in $H_0^1(\Omega)^3$,

$\mathbf{V}_\sigma^*$: dual space of $\mathbf{V}_\sigma$, $\mathbf{V}_\sigma \subset \mathbf{H}_\sigma \subset \mathbf{V}_\sigma^*$.

$K(u^\varepsilon;t) := \{\mathbf{z} \in \mathbf{V}_\sigma \,|\, |\mathbf{z}| \le p_0(u^\varepsilon(\cdot,t)) \text{ in } \Omega\}$.

$K_\mu(u^\varepsilon;t) := \{\mathbf{z} \in \mathbf{V}_\sigma \,|\, |\mathbf{z}| \le p_\mu(u^\varepsilon(\cdot,t)) \text{ in } \Omega\}$.



$(NS)_\mu$ $\quad \mathbf{v}_t - \nu\Delta\mathbf{v} + \partial I_{K_\mu(u^\varepsilon;t)}(\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v} \ni -\frac{\nabla P}{\rho}$, div$\mathbf{v} = 0$, in Q,

$$(NS;u,\mathbf{v}_0)_\mu \quad \begin{cases} \mathbf{v} \in W^{1,2}(0,T;\mathbf{H}_\sigma), |\mathbf{v}| \le p_\mu(u^\varepsilon) \text{ a.e. on } Q; \\ \int_\Omega \mathbf{v}'(t)\cdot(\mathbf{v}(t)-\eta)dx + \nu\int_\Omega \nabla\mathbf{v}(t)\cdot\nabla(\mathbf{v}(t)-\eta)dx \\ \qquad + \int_\Omega (\mathbf{v}(t)\cdot\nabla)\mathbf{v}(t)\cdot(\mathbf{v}(t)-\eta)dx \le 0, \\ \qquad\qquad \forall \eta \in \mathbf{V}_\sigma,\ |\eta| \le p_\mu(u^\varepsilon(\cdot,t)) \text{ a.e. on } \Omega, \\ \mathbf{v}(\cdot,0) = \mathbf{v}_0 \text{ in } \mathbf{H}_\sigma, \end{cases}$$

See Fukao-K.[3](2015) and Gokieli-K-Niezgódka [6](2016).

(3) Variational formulation of nutrient transport/consumption

(N) $w_t - \mathrm{div}(d(u^\varepsilon)\nabla w) + \mathbf{v}\cdot\nabla w + \partial I_{K(N)}(w) \ni -f(w)u$ in Q

where $K(N) := \{z \in V \mid 0 \le z \le 1, |\nabla z| \le N \text{ in } \Omega\}$ for a const. $N > 0$.

$$(N; u, \mathbf{v}, w_0) \begin{cases} w \in W^{1,2}(0,T;H) \cap C([0,T];V),\ 0 \le w \le 1,\ |\nabla w| \le N; \\ \int_\Omega w'(t)(w(t)-z)dx + \int_\Omega d(u^\varepsilon(t))\nabla w(t)\cdot\nabla(w(t)-z)dx \\ + \int_\Omega \mathbf{v}(t)\cdot\nabla w(t)(w(t)-z)dx \le -\int_\Omega f(w(t))u(t)(w(t)-z)dx, \\ \qquad \forall z \in V,\ 0 \le z \le 1,\ |\nabla z| \le N, \\ w(0) = w_0. \end{cases}$$

Approximate full problem P_μ: For each parameter $\mu > 0$ a triplet $\{u_\mu, \mathbf{v}_\mu, w_\mu\}$ is called a solution of P_μ, if $u_\mu, \mathbf{v}_\mu$ and w_μ satisfy the system

$(B; w_\mu, \mathbf{v}_\mu, u_0)$, $(NS; u_\mu, \mathbf{v}_0)_\mu$, $(N; u_\mu, \mathbf{v}_\mu, w_0)$

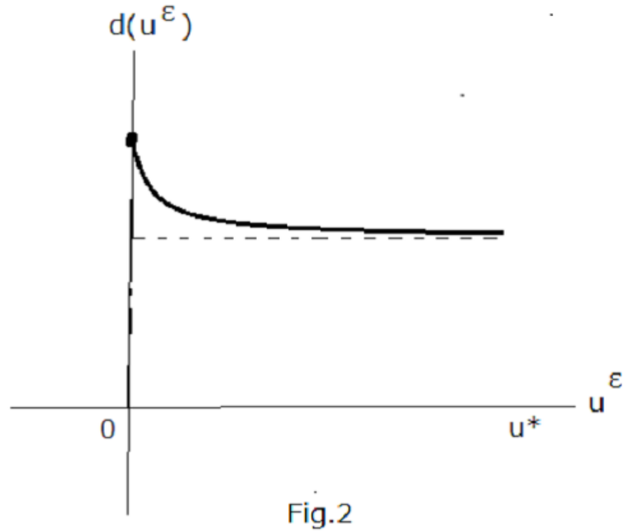


Fig.2

3. Convergence and an existence result

Assume that the parameter $\varepsilon > 0$ is fixed,

$$u_0 \in H,\ \widehat{\beta}(u_0) \in L^1(\Omega) \text{ (hence } 0 \le u_0 \le u^*),\ \widehat{\beta} : \text{primitive of } \beta,$$

$$\mathbf{v}_0 \in \mathbf{V}_\sigma \cap \mathbf{C}(\overline{\Omega}),$$

$$\mathbf{v}_0 = 0 \text{ on a neighborhood of } \overline{\{x \in \Omega \mid u_0^\varepsilon(x) > \delta_0\}}$$

$$w_0 \in V,\ 0 \le w_0 \le 1,\ |\nabla w_0| \le N \text{ on } \Omega.$$

Then we have:

Proposition. For any small $\mu > 0$ problem P_μ has at least one solution $\{u_\mu, \mathbf{v}_\mu, w_\mu\}$.

Proof. Find a fixed point of $\mathcal{S}$ by the Schauder's fixed-point theorem:

Iteration Scheme

$$\begin{array}{lcl}
u \longrightarrow \mathbf{v} \text{ by solving } (NS; u, \mathbf{v}_0)_\mu & \longrightarrow & w \text{ by solving } (N; u, \mathbf{v}, w_0)_\mu \\
\uparrow & & \downarrow \\
\uparrow \quad \bar{u} = \mathcal{S}u & & \downarrow \\
\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow & & \bar{u} \text{ by solving } (B; w, \mathbf{v}, u_0)
\end{array}$$

Theorem (Gokieli-K-Niezgódka [5](2016)). There exists a sequence $\{\mu_n\}$ with $\mu_n \to 0$ such that

$$\begin{aligned}
&u_{\mu_n} \to u \text{ weakly in } W^{1,2}(0,T;V^*),\ u_{\mu_n}(t) \to u(t) \text{ weakly in } H,\ \forall t \in [0,T],\\
&w_{\mu_n} \to w \text{ in } C([0,T];V),\ \text{weakly in } W^{1,2}(0,T;H),\\
&\mathbf{v}_{\mu_n} \to \mathbf{v} \text{ in } L^2(0,T;\mathbf{H}_\sigma) \text{ and weakly in } L^2(0,T;\mathbf{V}_\sigma),
\end{aligned}$$

and that $\{u, \mathbf{v}, w\}$ satisfies $(B; w, \mathbf{v}, u_0), (N; u, \mathbf{v}, w_0)$ and

$$|\mathbf{v}| \le p_0(u^\varepsilon) \text{ a.e. in } Q \text{ (hence } \mathbf{v} = 0 \text{ a.e. in } \{u^\varepsilon(x,t) \ge \delta_0\});$$

$$\int_0^t \int_\Omega \eta' \cdot (\mathbf{v} - \eta) dx d\tau + \nu \int_0^t \int_\Omega \nabla \mathbf{v} \cdot \nabla(\mathbf{v} - \eta) dx d\tau$$

$$+ \int_0^t \int_\Omega (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot (\mathbf{v} - \eta) dx d\tau + \frac{1}{2}|\mathbf{v}(t) - \eta(t)|^2_{\mathbf{H}_\sigma} \le \frac{1}{2}|\mathbf{v}_0 - \eta(0)|^2_{\mathbf{H}_\sigma},$$

$$\int_\Omega \mathbf{v}(t) \cdot \eta(t) dx \text{ is of BV in } t \in [0,T], \quad \mathbf{v}(0) = \mathbf{v}_0 \text{ in } \mathbf{H}_\sigma,$$

$$\forall t \in [0,T],\ \forall \eta \in \mathcal{K}(u^\varepsilon),$$

where $\mathcal{K}(u^\varepsilon) := \left\{ \eta \in C^1([0,T]; \mathbf{W}^{1,4}_{0,\sigma}(\Omega)) \;\middle|\; \begin{array}{l} \text{supp}(\eta) \subset \{u^\varepsilon(x,t) < \delta_0\}, \\ |\eta| \le p_0(u^\varepsilon) \text{ on } Q \end{array} \right\}.$

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Thank you for your attention