Optimal control of surface PDEs

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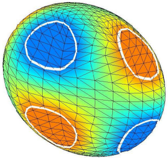
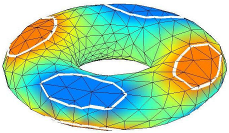
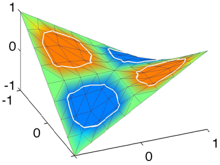
Outline

- **Optimal control of elliptic and parabolic PDEs with constraints on the control and/or the state on a stationary surface;**
- **Optimal control of parabolic PDEs on evolving surfaces;**
- **Next steps**







Approach

We follow the planar case to treat optimal control problems on surfaces.
Concerning discretization, the approximation of the surface has to be considered;

$\rightarrow$ Elliptic case

- Use Dziuk's finite element concept for the Laplace Beltrami operator of 1988 for the discretization of the state equation;
- Use variational discretization (H. COAP 2005) for the discretization of the control problem;
- Use a semismooth Newton method of Hintermüller, Ito and Kunisch; Michael Ulbrich (SIOPT 2003) to solve the resulting nonlinear systems.

$\rightarrow$ Parabolic case

- Combine dg in time discretization with the spatial FE treatment of the Laplace Beltrami operator in the case of stationary surfaces;
- Use the ESFEM of Dziuk and Elliott (Acta Numerica 2013) for the discretization of the parabolic PDE on the evolving surface;
- Alternatively, use the pullback approach of H. Kröner (arXiv:1501.07900 2016) for the treatment of the evolving surfaces;
- Use variational discretization (H. COAP 2005) for the discretization of the control problem.
- Use a semismooth Newton method of Hintermüller et al. (SIOPT 2003) and M. Ulbrich (SIOPT 2003) to solve the resulting nonlinear systems.

Problem statement, elliptic case

Let $\Gamma \subset \mathbb{R}^{n+1}$, $n = 1, 2$ denote an n-dimensional, sufficiently smooth, closed hypersurface.

Consider

$$(P) \quad \begin{cases} \min\limits_{u \in L^2(\Gamma),\, y \in H^1(\Gamma)} J(u, y) = \dfrac{1}{2}\|y - z\|^2_{L^2(\Gamma)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{subject to} \quad u \in U_{ad} \text{ and} \\ \displaystyle\int_\Gamma \nabla_\Gamma y \nabla_\Gamma \varphi + \mathrm{c} y \varphi \, d\Gamma = \int_\Gamma u \varphi \, d\Gamma \,, \forall \varphi \in H^1(\Gamma), \end{cases}$$

where $U_{ad} = \{v \in L^2(\Gamma) \mid a \le v \le b\}$, $a < b \in \mathbb{R}$.

For simplicity we assume Γ to be compact and $\mathrm{c} = 1$.

It follows by standard arguments that problem (P) admits a unique solution $u \in U_{ad}$ with unique associated state $y =: Su \in H^2(\Gamma)$, and

$$u = \mathrm{P}_{U_{ad}}\left(-\frac{1}{\alpha}p(u)\right), \quad p(u) := S^*(Su - z).$$

Technicalities

We assume that Γ is of class C^2. As an embedded, compact hypersurface in $\mathbb{R}^{n+1}$ it is orientable with an exterior unit normal field ν and hence the zero level set of a signed distance function d such that $|d(x)| = \text{dist}(x, \Gamma)$ and

$$\nu(x) = \frac{\nabla d(x)}{\|\nabla d(x)\|} \text{ for } x \in \Gamma.$$

Further, there exists a neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of Γ, such that d is also of class C^2 on $\mathcal{N}$ and the projection

$$a : \mathcal{N} \to \Gamma\,, \quad a(x) = x - d(x)\nabla d(x)$$

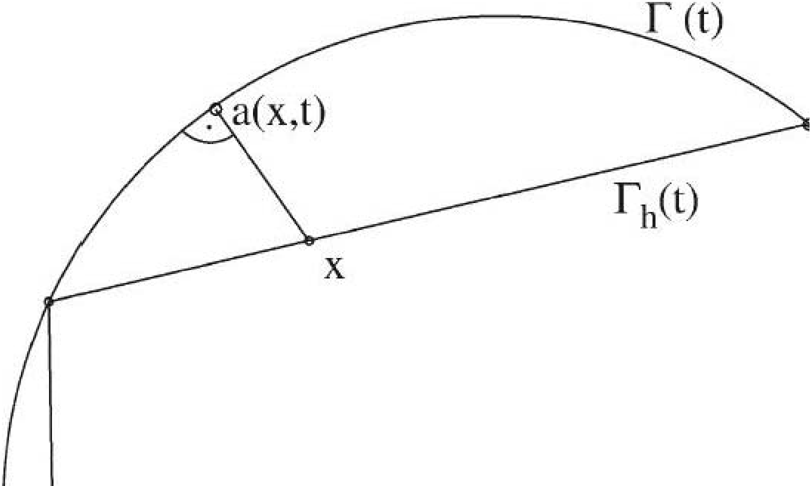
is unique with $\nabla d(x) = \nu(a(x))$, see e.g. Gilbarg-Trudinger 1998.
For $\phi : \Gamma \to \mathbb{R}$ we with $\bar{\phi}(x) = \phi(a(x))$ have

$$\nabla_\Gamma \phi = (I - \nu\nu^T)\nabla\bar{\phi},$$

with the euclidean projection $(I - \nu\nu^T)$ onto the tangential space of Γ.
The Laplace-Beltrami operator $\Delta_\Gamma : H^1(\Gamma) \to H^1(\Gamma)^*$ in its weak form reads

$$y \mapsto -\int_\Gamma \nabla_\Gamma y \nabla_\Gamma(\,\cdot\,)\,\mathrm{d}\Gamma \in H^1(\Gamma)^*\,.$$

Approximation of ***Gamma***



Discretization of the surface Γ

Following Dziuk 1988, we consider polyhedral

$$\Gamma^h = \bigcup_{i \in I_h} T_h^i$$

consisting of triangles T_h^i with corners on Γ, whose maximum diameter is denoted by h, yielding a regular and, if needed, quasiuniform partition.
We assume

$$a(\Gamma^h) = \Gamma$$

in a one-to-one manner. In order to compare functions defined on Γ^h with functions on Γ we use a to lift a function $y \in L^2(\Gamma^h)$ to Γ

$$y^l(a(x)) = y(x) \quad \forall x \in \Gamma^h,$$

and for $y \in L^2(\Gamma)$ and sufficiently small $h > 0$ we define the inverse lift

$$y_l(x) = y(a(x)) \quad \forall x \in \Gamma^h.$$

For small enough mesh parameters h the lift operation $(\cdot)_l : L^2(\Gamma) \to L^2(\Gamma^h)$ defines a linear homeomorphism with inverse $(\cdot)^l$. Moreover, there exists $C > 0$ such that

$$1 - Ch^2 \le \|(\cdot)_l\|^2_{\mathcal{L}(L^2(\Gamma), L^2(\Gamma^h))}, \|(\cdot)^l\|^2_{\mathcal{L}(L^2(\Gamma^h), L^2(\Gamma))} \le 1 + Ch^2 \,,$$

Discrete control problem

Problem (P) is approximated by the following sequence of optimal control problems

$$(P_h) \quad \begin{cases} \min\limits_{u \in L^2(\Gamma^h),\, y \in V_h} J(u,y) = \frac{1}{2}\|y - z_l\|^2_{L^2(\Gamma^h)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma^h)} \\ \text{subject to} \quad u \in U^h_{ad} \text{ and} \\ y_h = S_h u, \end{cases}$$

with $U^h_{ad} = \{v \in L^2(\Gamma^h) \mid a \leq v \leq b\}$. For some $v \in L^2(\Gamma^h)$ let $S_h v := y_h \in V_h$ solve

$$\int_{\Gamma^h} \nabla_{\Gamma^h} y_h \nabla_{\Gamma^h} \varphi + c y_h \varphi \, \mathrm{d}\Gamma^h = \int_{\Gamma^h} v \varphi \, \mathrm{d}\Gamma^h \quad \forall \varphi \in V_h,$$

where

$$V_h = \left\{\varphi \in C^0\left(\Gamma^h\right) \,\middle|\, \forall i \in I_h : \varphi|_{T_h^i} \in \mathcal{P}^1(T_h^i)\right\} \subset H^1(\Gamma^h).$$

Discrete Laplace-Beltrami operator (Dziuk 88)

Dziuk 88 explained, how to implement $S_h : L^2(\Gamma^h) \to L^2(\Gamma^h)$. The key ingredient is the discrete Laplace-Beltrami operator

$$-\Delta_{\Gamma_h} y = v \text{ on } \Gamma_h,$$

which for $v \in H^1(\Gamma_h)^*$ is defined by

$$\int_{\Gamma_h} \nabla_{\Gamma_h} y \nabla_{\Gamma_h} \phi d\Gamma_h = \langle v, \phi \rangle_{H^1(\Gamma_h), H^1(\Gamma_h)^*}.$$

Let $\phi_1, \ldots, \phi_N$ denote the piecewise linear, continuous a nodal basis of V_h, and let $y_h = \sum_{i=1}^{N} y_i \phi_i$. Then the solution $y_h \in V_h$ of $-\Delta_{\Gamma_h} y_h = v$ is defined through the solution of the linear system

$$\sum_{i=1}^{N} y_i \int_{\Gamma_h} \nabla_{\Gamma_h} \phi_i \nabla_{\Gamma_h} \phi_k d\Gamma_h = \langle v, \phi_k \rangle_{H^1(\Gamma_h) H^1(\Gamma_h)^*} \text{ for } k = 1, \ldots, N.$$

It remains to compute integrals $\int_{T_h^j} \nabla_{\Gamma_h} \phi_i \nabla_{\Gamma_h} \phi_k dT_h^j$ which can be done by parametrising functions over the unit triangle $\hat{T} \subset \mathbb{R}^2$.

Error estimate for the control problem (Vierling, H. JCM 2012)

It follows by standard arguments that problem (P_h) admits a unique solution $u_h \in U_{ad}^h$ with unique associated state $y_h \in V_h$, and

$$u_h = \mathrm{P}_{U_{ad}^h}\left(-\frac{1}{\alpha} p_h(u_h)\right), \quad p_h(u_h) := S_h^*(S_h u_h - z_l).$$

With the help of

$$\left(\mathrm{P}_{U_{ad}^h}(v_l)\right)^l = \mathrm{P}_{U_{ad}}(v) \text{ for all } v \in L^2(\Gamma)$$

and the error estimate of Dziuk 1988

$$\|(\cdot)^l S_h(\cdot)_l - S\|_{\mathcal{L}(L^2(\Gamma), L^2(\Gamma))} \le Ch^2.$$

we now can prove

$$\alpha \|u_h^l - u\|_{L^2(\Gamma)}^2 + \|y_h^l - y\|_{L^2(\Gamma)}^2 \le C\left(\frac{1}{\alpha}\left\|\left((\cdot)^l S_h^*(\cdot)_l - S^*\right)(y-z)\right\|_{L^2(\Gamma)}^2 \right.$$
$$\left. + \left\|\left((\cdot)^l S_h(\cdot)_l - S\right) u\right\|_{L^2(\Gamma)}^2\right) \le Ch^4.$$

Sketch of proof

u_l is the $L^2(\Gamma^h)$-orthogonal projection of $-\left(\frac{1}{\alpha}p(u)\right)_l$, thus a feasible test control,

$$\langle \alpha u_h + p_h(u_h), u_l - u_h\rangle_{L^2(\Gamma^h)} \geq 0.$$

On the other hand u_l is the $L^2(\Gamma^h)$-orthogonal projection of $-\frac{1}{\alpha}p(u)_l$, thus

$$\langle -\frac{1}{\alpha}p(u)_l - u_l, u_h - u_l\rangle_{L^2(\Gamma^h)} \leq 0.$$

Adding these inequalities yields

$$\alpha\|u_l - u_h\|^2_{L^2(\Gamma^h)} \leq \langle (p_h(u_h) - p(u)_l), u_l - u_h\rangle_{L^2(\Gamma^h)}$$
$$= \langle p_h(u_h) - S_h^*(y-z)_l, u_l - u_h\rangle_{L^2(\Gamma^h)} + \langle S_h^*(y-z)_l - p(u)_l, u_l - u_h\rangle_{L^2(\Gamma^h)}.$$

The first addend is estimated via

$$\langle p_h(u_h) - S_h^*(y-z)_l, u_l - u_h\rangle_{L^2(\Gamma^h)} \leq -\frac{1}{2}\|y_h - y_l\|^2_{L^2(\Gamma^h)} + \frac{1}{2}\|S_h u_l - y_l\|^2_{L^2(\Gamma^h)}.$$

The second addend satisfies

$$\langle S_h^*(y-z)_l - p(u)_l, u_l - u_h\rangle_{L^2(\Gamma^h)} \leq \frac{\alpha}{2}\|u_l - u_h\|^2_{L^2(\Gamma^h)} + \frac{1}{2\alpha}\|S_h^*(y-z)_l - p(u)_l\|^2_{L^2(\Gamma^h)}.$$

Together this yields

$$\alpha\|u_l - u_h\|^2_{L^2(\Gamma^h)} + \|y_h - y_l\|^2_{L^2(\Gamma^h)} \leq \frac{1}{\alpha}\|S_h^*(y-z)_l - p(u)_l\|^2_{L^2(\Gamma^h)} + \|S_h u_l - y_l\|^2_{L^2(\Gamma^h)}$$

Sketch of proof

u_I is the $L^2(\Gamma^h)$-orthogonal projection of $-\left(\frac{1}{\alpha}p(u)\right)_I$, thus a feasible test control,

$$\langle \alpha u_h + p_h(u_h), u_I - u_h \rangle_{L^2(\Gamma^h)} \geq 0.$$

On the other hand u_I is the $L^2(\Gamma^h)$-orthogonal projection of $-\frac{1}{\alpha}p(u)_I$, thus

$$\langle -\frac{1}{\alpha}p(u)_I - u_I, u_h - u_I \rangle_{L^2(\Gamma^h)} \leq 0.$$

Adding these inequalities yields

$$\begin{aligned}\alpha \|u_I - u_h\|^2_{L^2(\Gamma^h)} &\leq \langle (p_h(u_h) - p(u)_I)\,, u_I - u_h \rangle_{L^2(\Gamma^h)} \\ &= \langle p_h(u_h) - S_h^*(y-z)_I, u_I - u_h \rangle_{L^2(\Gamma^h)} + \langle S_h^*(y-z)_I - p(u)_I, u_I - u_h \rangle_{L^2(\Gamma^h)}.\end{aligned}$$

The first addend is estimated via

$$\langle p_h(u_h) - S_h^*(y-z)_I, u_I - u_h \rangle_{L^2(\Gamma^h)} \leq -\frac{1}{2}\|y_h - y_I\|^2_{L^2(\Gamma^h)} + \frac{1}{2}\|S_h u_I - y_I\|^2_{L^2(\Gamma^h)}.$$

The second addend satisfies

$$\langle S_h^*(y-z)_I - p(u)_I, u_I - u_h \rangle_{L^2(\Gamma^h)} \leq \frac{\alpha}{2}\|u_I - u_h\|^2_{L^2(\Gamma^h)} + \frac{1}{2\alpha}\|S_h^*(y-z)_I - p(u)_I\|^2_{L^2(\Gamma^h)}.$$

Together this yields

$$\alpha\|u_I - u_h\|^2_{L^2(\Gamma^h)} + \|y_h - y_I\|^2_{L^2(\Gamma^h)} \leq \frac{1}{\alpha}\|S_h^*(y-z)_I - p(u)_I\|^2_{L^2(\Gamma^h)} + \|S_h u_I - y_I\|^2_{L^2(\Gamma^h)}$$

Sketch of proof

u_l is the $L^2(\Gamma^h)$-orthogonal projection of $-\left(\frac{1}{\alpha}p(u)\right)_l$, thus a feasible test control,

$$\langle \alpha u_h + p_h(u_h), u_l - u_h \rangle_{L^2(\Gamma^h)} \geq 0.$$

On the other hand u_l is the $L^2(\Gamma^h)$-orthogonal projection of $-\frac{1}{\alpha}p(u)_l$, thus

$$\langle -\frac{1}{\alpha}p(u)_l - u_l, u_h - u_l \rangle_{L^2(\Gamma^h)} \leq 0.$$

Adding these inequalities yields

$$\begin{aligned} \alpha \| u_l - u_h \|^2_{L^2(\Gamma^h)} &\leq \langle (p_h(u_h) - p(u)_l), u_l - u_h \rangle_{L^2(\Gamma^h)} \\ &= \langle p_h(u_h) - S_h^*(y - z)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} + \langle S_h^*(y - z)_l - p(u)_l, u_l - u_h \rangle_{L^2(\Gamma^h)}. \end{aligned}$$

The first addend is estimated via

$$\langle p_h(u_h) - S_h^*(y - z)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} \leq -\frac{1}{2}\| y_h - y_l \|^2_{L^2(\Gamma^h)} + \frac{1}{2}\| S_h u_l - y_l \|^2_{L^2(\Gamma^h)}.$$

The second addend satisfies

$$\langle S_h^*(y - z)_l - p(u)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} \leq \frac{\alpha}{2}\| u_l - u_h \|^2_{L^2(\Gamma^h)} + \frac{1}{2\alpha}\| S_h^*(y - z)_l - p(u)_l \|^2_{L^2(\Gamma^h)}.$$

Together this yields

$$\alpha \| u_l - u_h \|^2_{L^2(\Gamma^h)} + \| y_h - y_l \|^2_{L^2(\Gamma^h)} \leq \frac{1}{\alpha}\| S_h^*(y - z)_l - p(u)_l \|^2_{L^2(\Gamma^h)} + \| S_h u_l - y_l \|^2_{L^2(\Gamma^h)}$$

Sketch of proof

u_l is the $L^2(\Gamma^h)$-orthogonal projection of $-\left(\frac{1}{\alpha}p(u)\right)_l$, thus a feasible test control,

$$\langle \alpha u_h + p_h(u_h), u_l - u_h \rangle_{L^2(\Gamma^h)} \geq 0.$$

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The second addend satisfies

$$\langle S_h^*(y-z)_l - p(u)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} \leq \frac{\alpha}{2}\|u_l - u_h\|^2_{L^2(\Gamma^h)} + \frac{1}{2\alpha}\|S_h^*(y-z)_l - p(u)_l\|^2_{L^2(\Gamma^h)}.$$

Together this yields

$$\alpha\|u_l - u_h\|^2_{L^2(\Gamma^h)} + \|y_h - y_l\|^2_{L^2(\Gamma^h)} \leq \frac{1}{\alpha}\|S_h^*(y-z)_l - p(u)_l\|^2_{L^2(\Gamma^h)} + \|S_h u_l - y_l\|^2_{L^2(\Gamma^h)}$$

Sketch of proof

u_l is the $L^2(\Gamma^h)$-orthogonal projection of $-\left(\frac{1}{\alpha}p(u)\right)_l$, thus a feasible test control,

$$\langle \alpha u_h + p_h(u_h), u_l - u_h \rangle_{L^2(\Gamma^h)} \geq 0.$$

On the other hand u_l is the $L^2(\Gamma^h)$-orthogonal projection of $-\frac{1}{\alpha}p(u)_l$, thus

$$\langle -\frac{1}{\alpha}p(u)_l - u_l, u_h - u_l \rangle_{L^2(\Gamma^h)} \leq 0.$$

Adding these inequalities yields

$$\begin{aligned} \alpha \|u_l - u_h\|^2_{L^2(\Gamma^h)} &\leq \langle (p_h(u_h) - p(u)_l), u_l - u_h \rangle_{L^2(\Gamma^h)} \\ &= \langle p_h(u_h) - S_h^*(y-z)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} + \langle S_h^*(y-z)_l - p(u)_l, u_l - u_h \rangle_{L^2(\Gamma^h)}. \end{aligned}$$

The first addend is estimated via

$$\langle p_h(u_h) - S_h^*(y-z)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} \leq -\frac{1}{2}\|y_h - y_l\|^2_{L^2(\Gamma^h)} + \frac{1}{2}\|S_h u_l - y_l\|^2_{L^2(\Gamma^h)}.$$

The second addend satisfies

$$\langle S_h^*(y-z)_l - p(u)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} \leq \frac{\alpha}{2}\|u_l - u_h\|^2_{L^2(\Gamma^h)} + \frac{1}{2\alpha}\|S_h^*(y-z)_l - p(u)_l\|^2_{L^2(\Gamma^h)}.$$

Together this yields

$$\alpha\|u_l - u_h\|^2_{L^2(\Gamma^h)} + \|y_h - y_l\|^2_{L^2(\Gamma^h)} \leq \frac{1}{\alpha}\|S_h^*(y-z)_l - p(u)_l\|^2_{L^2(\Gamma^h)} + \|S_h u_l - y_l\|^2_{L^2(\Gamma^h)}$$

State constraints (H., H. Kröner 2016)

In the case of additional state constraints, the planar case directly extends to surfaces.

Consider

$$
(P) \quad \begin{cases} \min\limits_{u \in L^2(\Gamma),\, y \in H^1(\Gamma)} J(u,y) = \frac{1}{2}\|y-z\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{subject to } u \in U_{ad} := \{u \in L^2(\Gamma); a \le u \le b\}, \\ \int_\Gamma \nabla_\Gamma y \nabla_\Gamma \varphi + cy\varphi \, d\Gamma = \int_\Gamma u\varphi \, d\Gamma, \forall \varphi \in H^1(\Gamma) \\ \text{and } y \in Y_{ad} := \{y \in L^\infty(S); y \le c\}. \end{cases}
$$

Slater condition: assume that $u \in U_{ad}$ exists with $y(u) < c$.

It then follows by standard arguments that problem (P) admits a unique solution $u \in U_{ad}$ with unique associated state $y(u) =: Su \in H^2(\Gamma)$, and $y(u) \le c$,

$$
u = \mathrm{P}_{U_{ad}}\left(-\frac{1}{\alpha}p(u)\right), \quad p(u) := S^*\left((Su - z) + \mu\right), \mu \ge 0, \int_\Gamma (c-y)d\mu = 0,
$$

where μ is a Radon measure on Γ.

State constraints discrete

Problem (P) is approximated by the following sequence of optimal control problems

$$(P_h) \quad \begin{cases} \min\limits_{u \in L^2(\Gamma^h),\, y \in V_h} J(u,y) = \frac{1}{2}\|y - z_l\|^2_{L^2(\Gamma^h)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma^h)} \\ \text{subject to } u \in U^h_{ad}, y_h = S_h u, \text{ and } y_h(x_j) \le c \ (j = 1, \dots, nv) \end{cases}$$

with $U^h_{ad} = \{v \in L^2(\Gamma^h) \mid a \le v \le b\}$ and x_j the finite element nodes.
It follows by standard arguments that problem (P_h) admits a unique solution $u_h \in U^h_{ad}$ with unique associated state $y_h \in V_h$, $y_h(x_j) \le c$ $(j = 1, \dots, nv)$, and

$$u_h = \mathrm{P}_{U^h_{ad}}\left(-\frac{1}{\alpha} p_h(u_h)\right), p_h(u_h) := S_h^*\left((S_h u_h - z_l) + \mu_h\right), \mu_h \ge 0, \int_{\Gamma_h} c - y_h \, d\mu_h = 0.$$

Here $\mu_h = \sum_{j=1}^{nv} \mu_j \delta_{x_j}$ with $\mu_j \in \mathbb{R}$.

Error analysis for state constraints (H., H. Kröner 2016, arXiv:1606.03025)

We assume that a Slater condition is satisfied. Then

- No control constraints, then $u \in W^{1,s}(\Gamma)(s \in [1,2))$:

$$\|u - u_h^l\|_{L^2(\Gamma)},\ \|y - y_h^l\|_{H^1(\Gamma)} \leq Ch^{\frac{3}{2}-\frac{d}{2s}}\sqrt{|\log h|}.$$

- Uniform control constraints and state constraints; $u \in L^\infty(\Gamma), u_h \in L^\infty(\Gamma_h)$ uniformly in h:

$$\|u - u_h^l\|_{L^2(\Gamma)},\ \|y - y_h^l\|_{H^1(\Gamma)} \leq Ch|\log h|.$$

Key proof ingredient: uniform error estimates for finite element approximations of elliptic surface PDEs by Demlow SIAM J. Numer. Anal. 47 (2009).

State constraints, parabolic (H., H. Kröner 2016, arXiv:1604.07658)

Discretization of the PDE with cg(0) in space and dg(0) in time with time gridpoints $0 = t_0 < \cdots < t_N = T$ and $\tau := T/N$. Control action $(Bu)(x,t) := \sum_{k=1}^{m} u_i(t) f_i(x)$, control space is $U := L^2(0,T)^m$, control actions $f_i \in H^1(\Gamma)$. There holds

- $\max_{1 \le n \le N} \|y_l(\cdot, t_n) - y_{(h,\tau)}(\cdot, t_n)\|_{L^\infty(\Gamma_h)} \le C\sqrt{|\log h|}(h+\tau)$.

With this we are able to show

- $\alpha\|u - u_{(h,\tau)}\|_U^2 + \tau \sum_{n=1}^{N} \|y_l(\cdot, t_n) - y_{(h,\tau)(\cdot,t_n)}\|_{L^2(\Gamma_h)}^2 \le C\sqrt{|\log h|}(h + \sqrt{\tau})$.

Key: Uniform estimates for the state y under natural regularity conditions combined with variational discretization. Avoid estimates for the adjoint state.

Implementation with semismooth Newton method (H., Vierling OMS 2012)

Combine variational discretization of H. COAP 2005 with finite element techniques for PDEs on surfaces developed by Dziuk 1988 and semi-smooth Newton techniques from Hintermüller et al SIOPT 2003 to solve the $\infty-$dimensional problem

$$G_h(u_h) = u_h - \mathrm{P}_{[a,b]}\left(-\frac{1}{\alpha}p_h(u_h)\right) = 0.$$

The generalized derivative of G_h is given by

$$DG_h(u) = \left(I + \frac{\chi}{\alpha}S_h^* S_h\right),$$

where $\chi = \chi_{\{\gamma\in\Gamma^h; a < -\frac{1}{\alpha}p_h(u)[\gamma] < b\}}$
The semi-smooth Newton method for (P_h) then reads
Given u the next iterate u^+ is computed by performing three steps

1. **Set $((1-\chi)\,u^+)[\gamma] = \left((1-\chi)\mathrm{P}_{[a,b]}\left(-\frac{1}{\alpha}p_h(u)\right)\right)[\gamma] \in \{a,b\}$.**
2. **Solve**
$$\left(I + \frac{\chi}{\alpha}S_h^* S_h\right)\chi u^+ = \frac{\chi}{\alpha}\left(S_h^* z_I - S_h^* S_h\,(1-\chi)\,u^+\right)$$
for χu^+ by CG iteration over $L^2(\mathcal{I}(-\frac{1}{\alpha}p_h(u))$.
3. **Set $u = \chi u^+ + (1-\chi)u^+$.**

Γ the unit sphere in $\mathbb{R}^3$ and $\alpha = 1.5 \cdot 10^{-6}$

We consider

$$\min_{u \in L^2(\Gamma),\, y \in H^1(\Gamma)} J(u,y) \text{ subject to } -\Delta_\Gamma y + y = u - r, \quad -1 \le u \le 1.$$

We choose $z = 52\alpha x_3(x_1^2 - x_2^2)$ **, to obtain the solution**

$$\bar{u} = r = \min\big(1, \max\big(-1, 4x_3(x_1^2 - x_2^2)\big)\big).$$

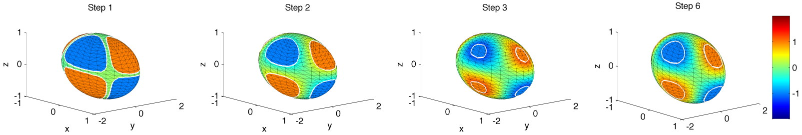


Figure: Selected steps of semismooth Newton method.

reg. refs.	0	1	2	3	4	5
$L2$-error	5.8925e-01	1.4299e-01	3.5120e-02	8.7123e-03	2.2057e-03	5.4855e-04
EOC	-	2.0430	2.0255	2.0112	1.9818	2.0075
# Steps	6	6	6	6	6	6

Table: L^2-error, EOC and number of semismooth Newton iterationsiterations.

$\Gamma = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_3 = x_1 x_2 \wedge x_1, x_2 \in (0,1)\}$ and $\alpha = 10^{-3}$

Consider

$$\min_{u \in L^2(\Gamma),\, y \in H^1(\Gamma)} J(u,y) \text{ subject to } -\Delta_\Gamma y = u - r, \quad y = 0 \text{ on } \partial\Gamma \quad -0.5 \le u \le 0.5$$

with solution $\bar{u} = r = \max\big(-0.5, \min\big(0.5, \sin(\pi x)\sin(\pi y)\big)\big)$.

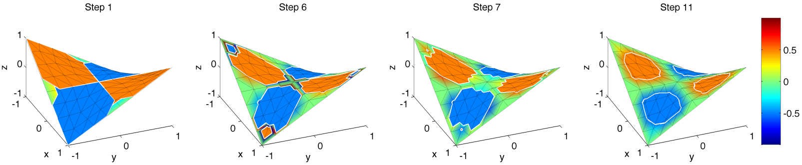


Figure: Selected steps of semismooth Newton method.

reg. refs.	0	1	2	3	4	5
$L2$-error	3.5319e-01	6.6120e-02	1.5904e-02	3.6357e-03	8.8597e-04	2.1769e-04
EOC	-	2.4173	2.0557	2.1291	2.0369	2.0250
# Steps	11	12	12	11	13	12

Table: L^2-error, EOC and number of semismooth Newton iterations.

Γ Torus and $\alpha = 10^{-3}$

Let z such that

$$\min_{u\in L^2(\Gamma),\, y\in H^1(\Gamma)} J(u,y) \text{ subject to } -\Delta_\Gamma y = u - r, \quad -1 \le u \le 1\,, \int_\Gamma y \,d\Gamma = \int_\Gamma u \,d\Gamma = 0$$

is solved by

$$\bar{u} = r = \max\big(-1, \min\big(1, 5xyz\big)\big).$$

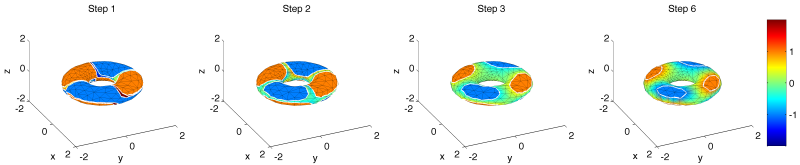


Figure: Selected steps of semismooth Newton method.

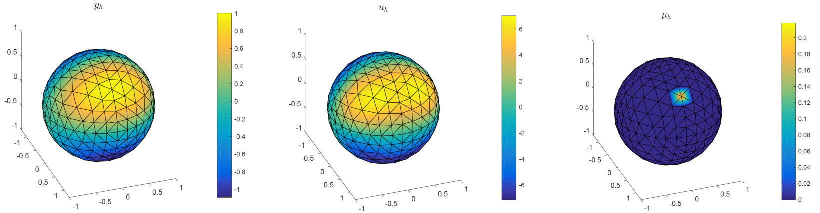
reg. refs.	0	1	2	3	4	5
$L2$-error	3.4603e-01	9.8016e-02	2.6178e-02	6.6283e-03	1.6680e-03	4.1889e-04
EOC	-	1.8198e+00	1.9047e+00	1.9816e+00	1.9905e+00	1.9935e+00
# Steps	9	3	3	3	2	2

Table: L^2-error, EOC and number of semismooth Newton iterations.

Γ the unit sphere in $\mathbb{R}^3$, state constraints, order of convergence (EOC)

Problem data: $b = 1$, $z = 8\sin(\pi x)\sin(\pi z)$, $\alpha = 10^{-1}$. Numerical solution on refinement level 8 as reference solution.

Level	$\|u^* - u_h\|_{L^2(S_8)}$	$\|y^* - y_h\|_{H^1(S_8)}$
1-2	2.2543632887749	1.3533941706509
2-3	2.0097197103454	1.1016844875780
3-4	1.9880953229332	1.0304424150678
4-5	1.9707724502506	1.0158452194757
5-6	1.9187206279711	1.0370412238907
6-7	1.8808617316338	1.1614534037133



Control of parabolic PDEs on evolving surfaces

Consider

$$(P)\quad\begin{cases}\min\limits_{u\in L^2(Q)} J(u) = \frac{1}{2}\|S(u)-z\|^2_{L^2(Q)} + \frac{\alpha}{2}\|u\|^2_{L^2(Q)}\\ \text{subject to}\quad a\le u\le b \text{ and } y=S(u),\ \text{where}\\ \underbrace{\int_{\Gamma(t)} \dot{y}\varphi + \nabla_{\Gamma(t)}y\nabla_{\Gamma(t)}\varphi + cy\varphi\, d\Gamma(t) = \int_{\Gamma(t)} u\varphi\, d\Gamma(t)\,, \forall\varphi\in H^1(\Gamma(t)),}_{:\Leftrightarrow y=S(u)}\end{cases}$$

and its variational discrete counterpart

$$(P_d)\quad\begin{cases}\min\limits_{u\in L^2(Q_d)} J(u) = \frac{1}{2}\|S_d(u)-z_l\|^2_{L^2(Q)} + \frac{\alpha}{2}\|u\|^2_{L^2(Q_d)}\\ \text{subject to}\quad a\le u\le b,\ \text{where}\end{cases}$$

$S_d(u)$ **denotes the ESFEM solution associated to** $S(u)$**, where** $d=(k,h)$.

Both, (P) **and** (P_d) **admit unique solutions in** $u\in L^2(Q)$ **and** $u_d\in L^2(Q_d)$.

Parabolic optimal control on moving surfaces - Vierling IFB 2014

With the help of L^2 error estimates of Dziuk and Elliott (Math. Comp. 2013) Vierling among other things for smooth data shows the following error estimate for $k \sim h^2$;

$$\alpha\|u_h^l - u\|^2_{L^2(Q)} + \|y_h^l - y\|^2_{L^2(Q)} \leq Ch^4.$$

Moving torus

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Parabolic optimal control on moving surfaces - pullback-approach

- **Formulate and discretize the evolving parabolic surface PDE on the reference surface $\Gamma(0)$. This leads to the approximation of a parabolic PDE with time-dependent coefficients on a fixed surface, whose numerical analysis can be found in H. Kröner (arXiv:1604.04665) and H. Kröner and Maryia Borukhava (arXiv:1501.07900). The optimal control problem then turns into a parabolic optimal control problem with time-dependent coefficients on the fixed reference surface. Then the earlier analysis presented for a fixed surface applies (H., H. Kröner arXiv:1604.07658).**

- **Control of the evolution of the surface. This is a shape optimization problem and in the pullback-setting turns into a 'control in the coefficients' problem.**

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Summary

We considered the numerical approximation of optimal control problems with elliptic and parabolic PDEs on surfaces.

- In the elliptic case we
  - used Dziuk's finite element concept for the Laplace Beltrami operator of 1988 for the discretization of the state equation;
  - used variational discretization (H. COAP 2005) for the discretization of the control problem;
  - used a semismooth Newton method of Hintermüller, Ito and Kunisch; M. Ulbrich (SIOPT 2003) to solve the resulting nonlinear systems.
- In the parabolic case
  - provide error analysis for stationary surface;
  - Vierling used ESFEM from Dziuk and Elliott (Acta Numerica 2013) for the discretization of the parabolic PDE on the evolving surface combined with variational discretization (H. COAP 2005) and a semismooth Newton method of Hintermüller, Ito and Kunisch; M. Ulbrich (SIOPT 2003) to treat the parabolic optimal control problem on evolving surfaces;
  - we propose and analyze the pullback approach for treating parabolic optimal control problems on evolving surfaces in the presence of control and/or state constraints.

Next steps

- **Improve time integration.**
- **Control of the evolution of the surface.**

Thank you for your attention!

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