Optimal Control of the Fokker-Planck equation

Roberto Guglielmi

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences (ÖAW)
Linz, Austria

INdAM Meeting OCERTO 2016
Optimal Control for Evolutionary PDEs and Related Topics

Cortona, Italy

June 20-24, 2016

Motivation

Consider an Optimal Control Problem (OCP)

$$\min_{u} \tilde{J}(X, u)$$

constrained to a Itô Stochastic Differential Equation (SDE)

$$dX_t = b(X_t, t; u)dt + \sigma(X_t, t)dW_t, \qquad X(0) = x_0$$

Motivation

Consider an Optimal Control Problem (OCP)

$$\min_u \tilde{J}(X, u)$$

constrained to a Itô Stochastic Differential Equation (SDE)

$$dX_t = b(X_t, t; u)dt + \sigma(X_t, t)dW_t, \qquad X(0) = x_0$$

where

- $t \in [0, T_E]$ for a fixed terminal time $T_E > 0$ and
- X_t is a *random variable* representing the state of the SDE
- The control u acts through the drift term b

Motivation

Consider an Optimal Control Problem (OCP)

$$\min_u \tilde{J}(X, u)$$

constrained to a Itô Stochastic Differential Equation (SDE)

$$dX_t = b(X_t, t; u)dt + \sigma(X_t, t)dW_t, \qquad X(0) = x_0$$

where

- $t \in [0, T_E]$ for a fixed terminal time $T_E > 0$ and
- X_t is a *random variable* representing the state of the SDE
- The control u acts through the drift term b

X_t random $\Rightarrow$ The cost functional $\tilde{J}(X, u)$ is a random variable

Choice of the cost functional

Standard approach: consider the averaged objective

$$\min_u \mathbb{E}[\tilde{J}(X,u)] = \min_u \mathbb{E}\left[\int_0^{T_E} L\left(t, X_t, u(t)\right)\,\mathrm{d}t + \psi(X_{T_E})\right]$$

Choice of the cost functional

Standard approach: consider the averaged objective

$$\min_u \mathbb{E}[\tilde{J}(X,u)] = \min_u \mathbb{E}\left[\int_0^{T_E} L\left(t, X_t, u(t)\right)\, \mathrm{d}t + \psi(X_{T_E})\right]$$

Alternative approach: express the objective in terms of the Probability Density Function (PDF) associated with the state X_t, which characterize the shape of its statistical distribution.

Choice of the cost functional

Standard approach: consider the averaged objective

$$\min_u \mathbb{E}[\tilde{J}(X,u)] = \min_u \mathbb{E}\left[\int_0^{T_E} L\left(t, X_t, u(t)\right)\, \mathrm{d}t + \psi(X_{T_E})\right]$$

Alternative approach: express the objective in terms of the Probability Density Function (PDF) associated with the state X_t, which characterize the shape of its statistical distribution.

Related works: deterministic objectives defined by

- the Kullback-Leibler distance (G. Jumarie 1992, M. Kárný 1996) or
- the square distance (M.G. Forbes, M. Guay, J.F. Forbes 2004, Wang 1999)

between the state PDF and a desired one.

However, stochastic models are needed to obtain the PDF by averaging or by interpolation

The Fokker-Planck Equation /1

New approach pursued by Annunziato and Borzì (2010, 2013): Reformulate the objective using the underlying PDF

$$y(x,t) := \int_\Omega \tilde{y}(x,t;z,0)\rho(z,0)\mathrm{d}z$$

$t > 0$, $\rho(z,0)$ given initial density probability,
$\tilde{y}$ transition density probability distribution function

$$\tilde{y}(x,t;z,s) := \mathbb{P}\{X(t) \in (x, x+\mathrm{d}x) : X(s) = z\}\,, \quad t > s$$

and control the PDF directly.

The next essential step:

the PDF evolves according to the Fokker-Planck Equation (or forward Kolmogorov Equation)

The Fokker-Planck Equation /1

New approach pursued by Annunziato and Borzì (2010, 2013): Reformulate the objective using the underlying PDF

$$y(x,t) := \int_\Omega \tilde{y}(x,t;z,0)\rho(z,0)\mathrm{d}z$$

$t > 0$, $\rho(z,0)$ given initial density probability,
$\tilde{y}$ transition density probability distribution function

$$\tilde{y}(x,t;z,s) := \mathbb{P}\{X(t) \in (x, x+\mathrm{d}x) : X(s) = z\}, \quad t > s$$

and control the PDF directly.

The next essential step:

the PDF evolves according to the Fokker-Planck Equation (or forward Kolmogorov Equation)

The Fokker-Planck Equation /2

Fokker-Planck Equation

$$\begin{cases} \partial_t y(x,t) - \frac{1}{2}\partial_{xx}^2\Big(\sigma(x,t)^2 y(x,t)\Big) + \partial_x\Big(b(x,t;u)y(x,t)\Big) = 0 \\ y(\cdot,0) = y_0 \end{cases}$$

The Fokker-Planck Equation /2

Fokker-Planck Equation

$$\begin{cases} \partial_t y(x,t) - \frac{1}{2}\partial_{xx}^2\Big(\sigma(x,t)^2 y(x,t)\Big) + \partial_x\Big(b(x,t;u)y(x,t)\Big) = 0 \\ y(\cdot,0) = y_0 \end{cases}$$

where $y \colon \mathbb{R} \times [0,\infty[\to \mathbb{R}_{\geq 0}$ is the PDF
constrained to $\int_{\mathbb{R}} y(x,t)\mathrm{d}x = 1 \quad \forall t > 0$,

$y_0 \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ is the initial PDF $\left(\int_{\mathbb{R}} y_0(x)\,\mathrm{d}x = 1\right)$,

$\sigma \colon \mathbb{R} \times [0,\infty[\to \mathbb{R}$ and $b \colon \mathbb{R} \times [0,\infty[\times \mathbb{R} \to \mathbb{R}$
are given by the SDE

A Fokker-Planck control framework

Deterministic PDE-constrained Optimal Control Problem

$$\min_u \mathbb{E}[\tilde{J}(X,u)] \quad \rightsquigarrow \quad \min_u J(y,u)$$

s.t. Itô SDE $\rightsquigarrow$ s.t. Fokker-Planck PDE

$$dX_t = b(u)dt + \sigma dW_t\,, \qquad y_t - \frac{1}{2}(\sigma^2 y)_{xx} + (b(u)y)_x = 0$$

A Fokker-Planck control framework

Deterministic PDE-constrained Optimal Control Problem

$\min_u \mathbb{E}[\tilde{J}(X,u)]$	$\rightsquigarrow$	$\min_u J(y,u)$
s.t. Itô SDE	$\rightsquigarrow$	s.t. Fokker-Planck PDE
$dX_t = b(u)dt + \sigma dW_t\,,$		$y_t - \frac{1}{2}(\sigma^2 y)_{xx} + (b(u)y)_x = 0$

1) the class of objectives described by $\min_u J(y,u)$ is larger than that expressed by $\min_u \mathbb{E}[\tilde{J}(X,u)]$, indeed

$$\mathbb{E}\left[\int_0^{T_E} L(t,X_t,u(t)) + \psi(X_{T_E})\right] = \int_{\mathbb{R}^d}\int_0^{T_E} L(t,x,u)y(t,x) + \int_{\mathbb{R}^d} \psi(x)y(T_E,x)\,.$$

A Fokker-Planck control framework

Deterministic PDE-constrained Optimal Control Problem

$\min_u \mathbb{E}[\tilde{J}(X,u)]$	$\rightsquigarrow$	$\min_u J(y,u)$
s.t. Itô SDE	$\rightsquigarrow$	s.t. Fokker-Planck PDE
$dX_t = b(u)dt + \sigma dW_t\,,$		$y_t - \frac{1}{2}(\sigma^2 y)_{xx} + (b(u)y)_x = 0$

1) the class of objectives described by $\min_u J(y,u)$ is larger than that expressed by $\min_u \mathbb{E}[\tilde{J}(X,u)]$, indeed

$$\mathbb{E}\left[\int_0^{T_E} L(t,X_t,u(t)) + \psi(X_{T_E})\right] = \int_{\mathbb{R}^d}\int_0^{T_E} L(t,x,u)y(t,x) + \int_{\mathbb{R}^d}\psi(x)y(T_E,x)\,.$$

2) Bilinear control through the coefficient of the divergence term

MPC for the Fokker-Planck equation

In order to build a real-time sub-optimal control feedback law we apply a Model Predictive Control scheme, also known as Receding Horizon scheme, to the Fokker-Planck equation.

Basic idea of MPC:

OCP on a long (possibly infinite) time horizon	$\rightsquigarrow$	Several iterative OCPs on (shorter) finite time horizons

Related References

- Stochastic optimal control with cost functional expressed by the PDF: Jumarie 1992, Kárný 1996, Wang 1999, Forbes, Forbes, Guay 2004
- FP-MPC approach: Annunziato and Borzì 2010, 2013, ...
- Existence of optimal controls for bilinear controls: Addou and Benbrik 2002, for a control $u = u(t)$
- Analysis of the FPE with general coefficients: Aronson 1968, Risken 1989, Figalli 2008, Le Bris and Lions 2008, Porretta 2015
- Controllability of the FPE: Blaquière 1992, Porretta 2014
- Connection with mean field type control: Bensoussan, Frehse, Yam 2013

Existing Work by Annunziato and Borzì, 2010, 2013

Track a desired PDF over a given time interval.

Optimal Control Problem

$\Omega \subset \mathbb{R}$ open, $u_a, u_b \in \mathbb{R}$ with $u_a < u_b$, $y_d \in L^2(\Omega)$ and $\lambda > 0$. Consider the following OCP on $[0, T]$:

$$\min_u J(y, u) := \frac{1}{2} \|y(\cdot, T) - y_d(\cdot, T)\|^2_{L^2(\Omega)} + \frac{\lambda}{2} |u|^2$$

s.t.

$$\begin{cases} \partial_t y - \frac{1}{2}\partial^2_{xx}\left(\sigma^2 y\right) + \partial_x \left(b(u) y\right) = 0 & \text{in } \Omega \times (0, T) \\ y(\cdot, 0) = y_n & \text{in } \Omega \\ y = 0 & \text{in } \partial\Omega \times (0, T) \end{cases} \tag{1}$$

$$u \in U_{ad} := \{u \in \mathbb{R} \,|u_a \leq u \leq u_b\}$$
$$b(x; u) := \gamma(x) + u \text{ with } \gamma \in C^1(\Omega)$$

Existing Work by Annunziato and Borzì, 2010, 2013

Track a desired PDF over a given time interval.

Optimal Control Problem

$\Omega \subset \mathbb{R}$ open, $u_a, u_b \in \mathbb{R}$ with $u_a < u_b$, $y_d \in L^2(\Omega)$ and $\lambda > 0$.
Consider the following OCP on $[0, T]$:

$$\min_u J(y,u) := \frac{1}{2} \|y(\cdot,T) - y_d(\cdot,T)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}|u|^2$$

s.t.

$$\begin{cases} \partial_t y - \frac{1}{2}\partial^2_{xx}\left(\sigma^2 y\right) + \partial_x \left(b(u) y\right) = 0 & \text{in } \Omega \times (0,T) \\ y(\cdot,0) = y_n & \text{in } \Omega \\ y = 0 & \text{in } \partial\Omega \times (0,T) \end{cases} \tag{1}$$

$$u \in U_{ad} := \{u \in \mathbb{R} \,|u_a \le u \le u_b\}$$
$$b(x;u) := \gamma(x) + u \text{ with } \gamma \in C^1(\Omega)$$

Space-Time Dependent Controls

Collaboration with Arthur Fleig and Lars Grüne, Univ. Bayreuth
Consider a control function $u(x, t)$

Proposition (Aronson)

Let the following assumptions hold:

- $\sigma \in C^1(\Omega)$ such that
$$\sigma(x,t) \geq \theta \quad \forall (x,t) \in Q\,, \text{ for some constant } \theta > 0$$
- $b \in L^q(0, T_E; L^p(\Omega))$ with $2 < p, q \leq \infty$ and $\frac{d}{2p} + \frac{1}{q} < \frac{1}{2}$.
- $y_0 \in L^2(\Omega)$ is nonnegative and bounded.

Then there exists a unique *nonnegative* weak solution y to the Fokker-Planck IBVP (1)

Space-Time Dependent Controls

Collaboration with Arthur Fleig and Lars Grüne, Univ. Bayreuth
Consider a control function $u(x, t)$

Proposition (Aronson)

Let the following assumptions hold:

- $\sigma \in C^1(\Omega)$ such that

$$\sigma(x,t) \geq \theta \quad \forall (x,t) \in Q\,, \text{ for some constant } \theta > 0$$

- $b \in L^q(0, T_E; L^p(\Omega))$ with $2 < p, q \leq \infty$ and $\frac{d}{2p} + \frac{1}{q} < \frac{1}{2}$.
- $y_0 \in L^2(\Omega)$ is nonnegative and bounded.

Then there exists a unique *nonnegative* weak solution y to the Fokker-Planck IBVP (1)

Towards existence of Optimal Solutions

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $H := L^2(\Omega), V := H_0^1(\Omega)$ and V' dual of V.
The Fokker-Planck equation $\mathcal{E}(y_0, u, f)$ can be rewritten as

$$\begin{cases} \dot{y}(t) + Ay(t) + div(b(t, u(t))y(t)) = f(t) & \text{in } V'\,, \; t \in (0, T) \\ y(0) = y_0\,, & \end{cases}$$

where $y_0 \in H$, $A : V \to V'$ linear and continuous, $f \in L^2(0, T; V')$,
$b\colon \mathbb{R}^{d+1} \times \mathcal{U} \to \mathbb{R}^d, (x, t; u) \mapsto b(x, t; u(x, t))$ satisfies

$$\sum_{i=1}^{d} |b_i(x, t; u)|^2 \leq M(1 + |u(x, t)|^2) \quad \forall x \in \Omega\,, \; \forall t \in [0, T]\,,$$

and $\forall u \in \mathcal{U} := L^q(0, T; L^\infty(\Omega; \mathbb{R}^d))$, for some $q > 2$,

Rmk: optimization problem in a Banach space

Towards existence of Optimal Solutions

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $H := L^2(\Omega), V := H_0^1(\Omega)$ and V' dual of V.
The Fokker-Planck equation $\mathcal{E}(y_0, u, f)$ can be rewritten as

$$\begin{cases} \dot{y}(t) + Ay(t) + div(b(t, u(t))y(t)) = f(t) & \text{in } V', \; t \in (0, T) \\ y(0) = y_0 \, , \end{cases}$$

where $y_0 \in H$, $A : V \to V'$ linear and continuous, $f \in L^2(0, T; V')$, $b \colon \mathbb{R}^{d+1} \times \mathcal{U} \to \mathbb{R}^d, (x, t; u) \mapsto b(x, t; u(x, t))$ satisfies

$$\sum_{i=1}^{d} |b_i(x, t; u)|^2 \leq M(1 + |u(x, t)|^2) \quad \forall x \in \Omega \, , \; \forall t \in [0, T] \, ,$$

and $\forall u \in \mathcal{U} := L^q(0, T; L^\infty(\Omega; \mathbb{R}^d))$, for some $q > 2$,

Rmk: optimization problem in a Banach space

Let $y_0 \in H$, $f \in L^2(0, T; V')$ and $u \in \mathcal{U}$. Then a solution y of the Fokker-Planck equation satisfies the estimates

$$\begin{aligned}
|y|^2_{L^\infty(0,T;H)} &\le Ce^{c|u|^2_{\mathcal{U}}} \left(|y(0)|^2_H + |f|^2_{L^2(0,T;V')}\right) , \\
|y|^2_{L^2(0,T;V)} &\le C(1 + |u|^2_{\mathcal{U}} e^{c|u|^2_{\mathcal{U}}}) \left(|y(0)|^2_H + |f|^2_{L^2(0,T;V')}\right) , \\
|\dot{y}|^2_{L^2(0,T;V')} &\le C(1 + |u|^2_{\mathcal{U}} e^{c|u|^2_{\mathcal{U}}}) \left(|y(0)|^2_H + |f|^2_{L^2(0,T;V')}\right) ,
\end{aligned}$$

for some positive constants c, C.

Existence of Optimal Controls

Theorem (Fleig - G.)

Let $y_0 \in V$, $y_d \in H$, $u_a, u_b \in L^\infty(0,T; L^\infty(\Omega;\mathbb{R}^d))$,

consider $\min_{u\in\mathcal{U}_{ad}} J(y,u)$, where y is the unique solution to

$$\begin{cases} \dot{y}(t) + Ay(t) + div(b(t,u(t)), y(t)) = 0 \quad \text{in } V',\ t \in (0,T) \\ y(0) = y_0, \end{cases}$$

$\mathcal{U}_{ad} := \{u \in \mathcal{U} : u_a(x,t) \le u(x,t) \le u_b(x,t) \text{ for a.e. } (x,t) \in Q\}$.

Then there exists a pair

$$(\bar{y}, \bar{u}) \in C([0,T], H) \times \mathcal{U}_{ad}$$

such that $\bar{y}$ is a solution of $\mathcal{E}(y_0, \bar{u}, 0)$ and $\bar{u}$ minimizes J in $\mathcal{U}_{ad}$.

Necessary Optimality Conditions

Let $y_d \in L^2(0,T;H)$, $y_\Omega \in H$, $\alpha, \beta, \lambda \geq 0$ with $\max\{\alpha, \beta\} > 0$.

$$J(u) := \frac{\alpha}{2} \|y - y_d\|^2_{L^2(0,T;H)} + \frac{\beta}{2} \|y(T) - y_\Omega\|^2_H + \frac{\lambda}{2} \|u\|^2_{L^2(0,T;H)} \,.$$

We derive the first-order necessary optimality system for $\bar{u}(x,t)$

$$\begin{aligned}
&\partial_t \bar{y} - \sum_{i,j=1}^{d} \partial^2_{ij}(a_{ij}\bar{y}) + \sum_{i=1}^{d} \partial_i\big(\bar{u}_i \bar{y}\big) = 0 \,, && \text{in } Q \,,\\
&-\partial_t \bar{p} - \sum_{i,j=1}^{d} a_{ij}\partial^2_{ij}\bar{p} - \sum_{i=1}^{d} \bar{u}_i \partial_i \bar{p} = \alpha[\bar{y} - y_d] \,, && \text{in } Q \,,\\
&\bar{y} = \bar{p} = 0 && \text{on } \Sigma \,,\\
&\bar{y}(0) = y_0 \,, \quad \bar{p}(T) = \beta[\bar{y}(T) - y_\Omega] \,, && \text{in } \Omega \,,\\
&\iint_Q [\bar{y}\partial_i \bar{p} + \lambda \bar{u}_i]\,(u_i - \bar{u}_i)\,\mathrm{d}x\mathrm{d}t \geq 0 && \forall u \in \mathcal{U}_{ad} \,, i = 1, \ldots, d \,.
\end{aligned}$$

Numerical Example

Consider the Ornstein-Uhlenbeck process with

$$\sigma(x,t) \equiv \bar{\sigma} = 0.8, \qquad b(x,t,u) := u - x$$

on $\Omega :=]-5,5[$ with $u_a = -10, u_b = 10, \lambda = 0.001,$ and $T_E = 5$.

Numerical Example

Consider the Ornstein-Uhlenbeck process with

$$\sigma(x,t) \equiv \bar{\sigma} = 0.8, \qquad b(x,t,u) := u - x$$

on $\Omega :=]-5,5[$ with $u_a = -10, u_b = 10, \lambda = 0.001,$ and $T_E = 5$.

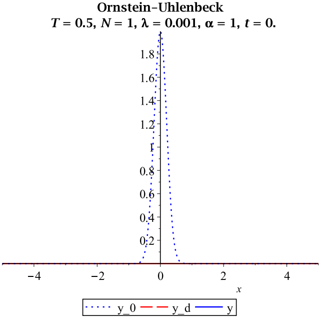
The target and initial PDF are given by

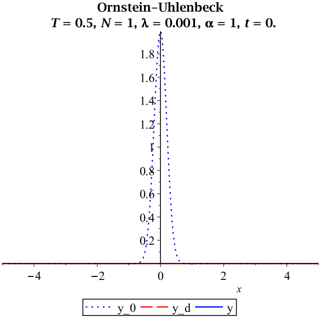
$$y_d(x,t) := \frac{\exp\left(-\frac{[x-2\sin(\pi t/5)]^2}{2\cdot 0.2^2}\right)}{\sqrt{2\pi\cdot 0.2^2}}$$

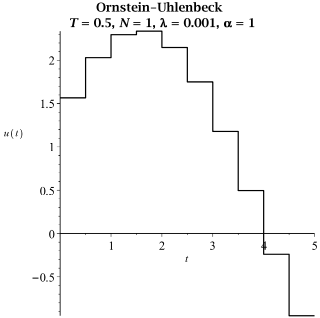
and

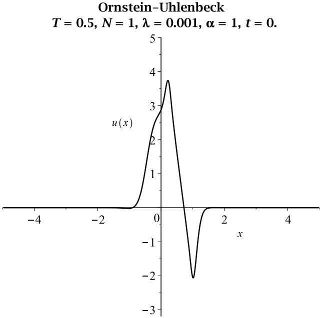
$$y_0(x) := y_d(x,0) = \frac{\exp\left(-\frac{x^2}{2\cdot 0.2^2}\right)}{\sqrt{2\pi\cdot 0.2^2}},$$

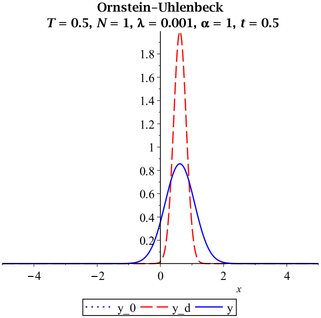
respectively.

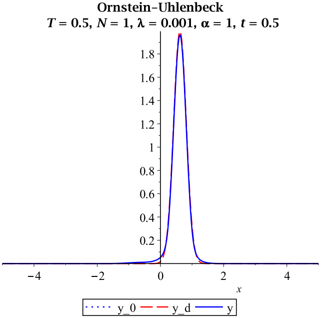

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

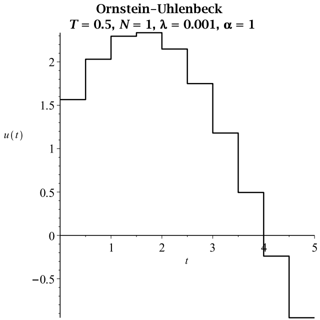

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

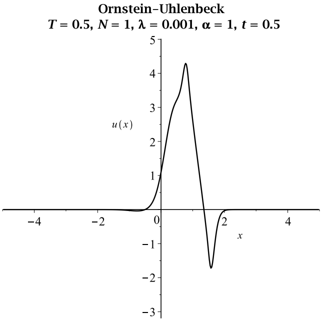

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
−0.5
1
2
3
4
5
t

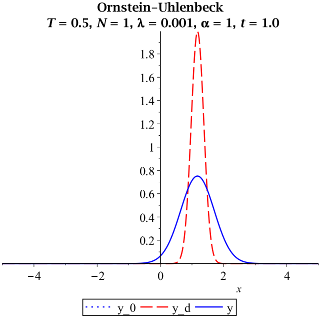

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
5
4
3
u(x)
2
1
−4
−2
0
2
4
x
−1
−2
−3

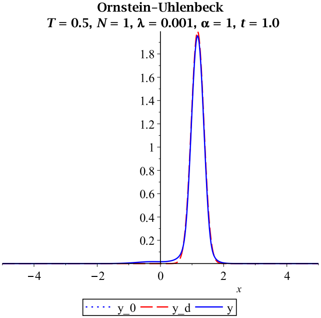

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.5
x
y_0
y_d
y

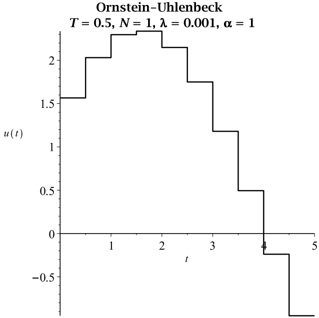

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.5
x
y_0
y_d
y

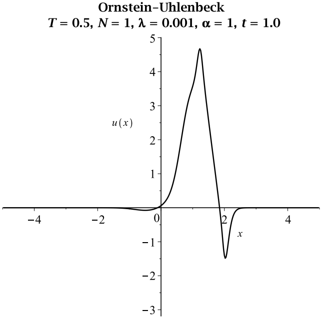

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

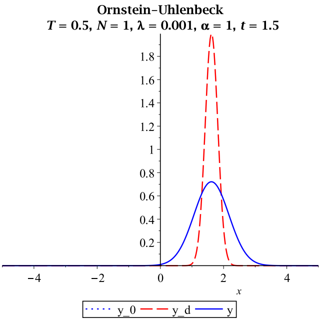

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.5
u(x)
x


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

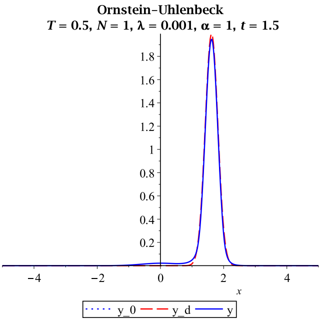

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
−0.5
1
2
3
4
5
t

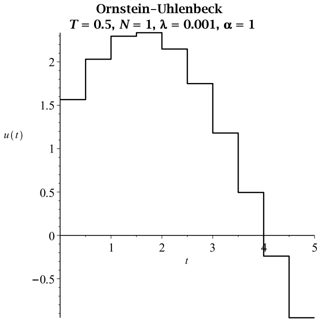

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.0
5
4
3
u(x)
2
1
0
−1
−2
−3
−4
−2
2
4
x



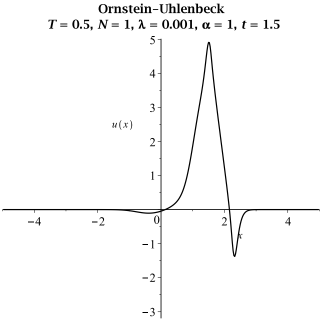
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y



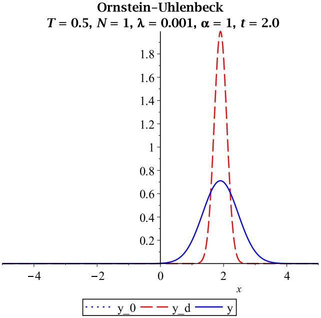
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

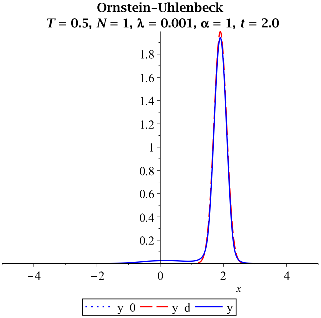


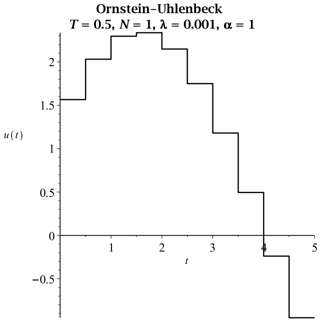
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
−0.5
1
2
3
4
5
t

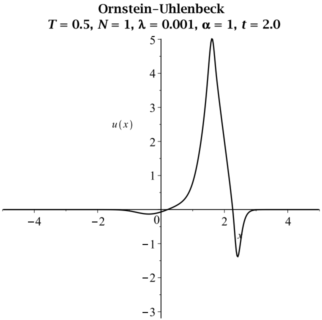


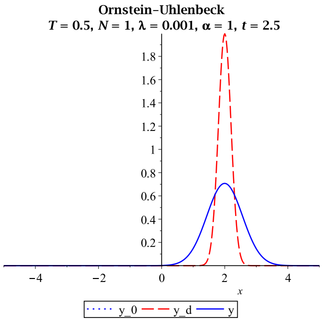
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
5
4
3
u(x)
2
1
−4
−2
0
2
4
x
−1
−2
−3


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

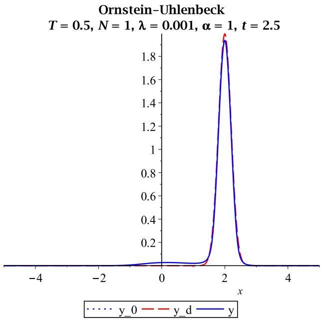

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
−0.5
1
2
3
4
5
t

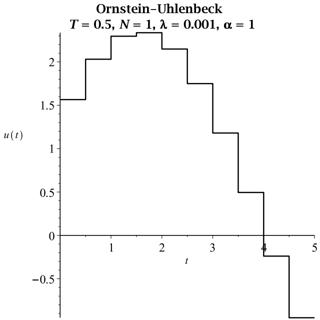

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
5
4
3
u(x)
2
1
−4
−2
0
2
4
x
−1
−2
−3



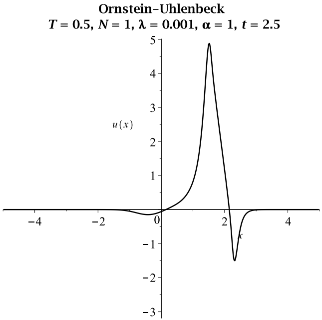
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.5
x
y_0
y_d
y



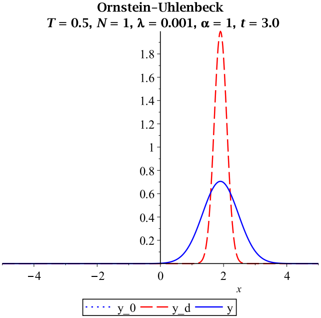
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.5
x
y_0
y_d
y

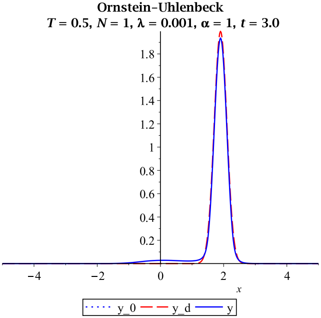


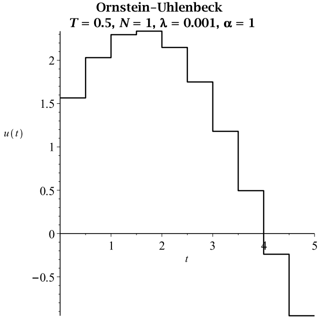
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

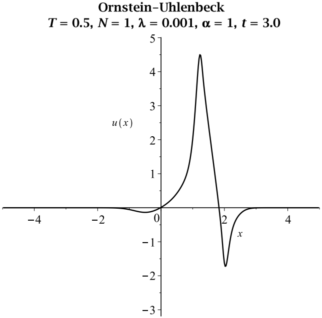


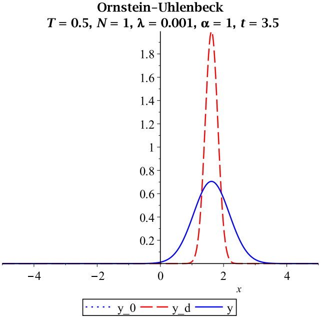
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.5
u(x)
x

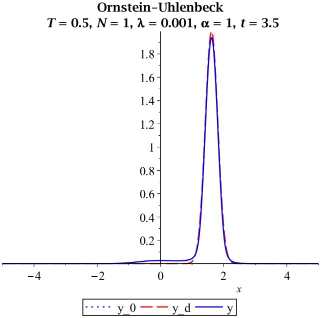

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

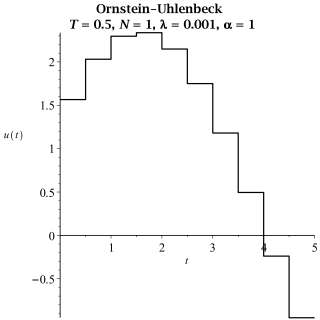

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

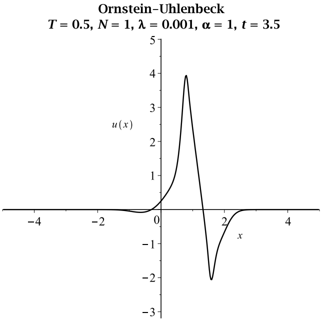

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
−0.5
1
2
3
4
5
t

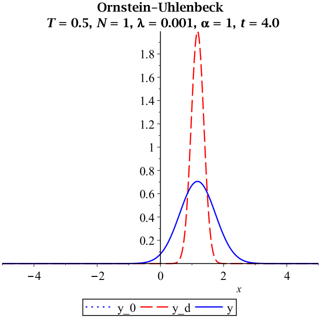

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
5
4
3
u(x)
2
1
−4
−2
0
2
4
x
−1
−2
−3

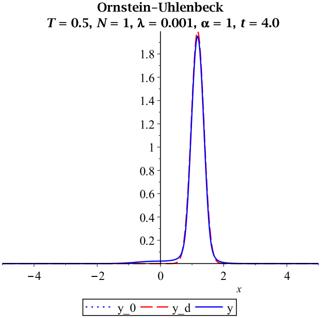

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

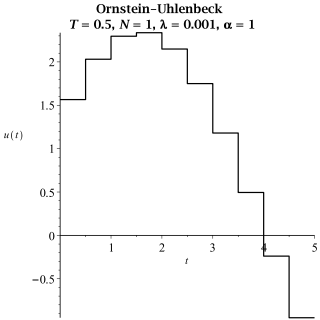

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

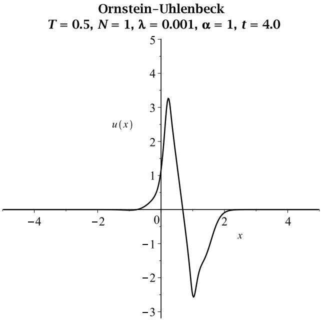

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
−0.5
1
2
3
4
5
t

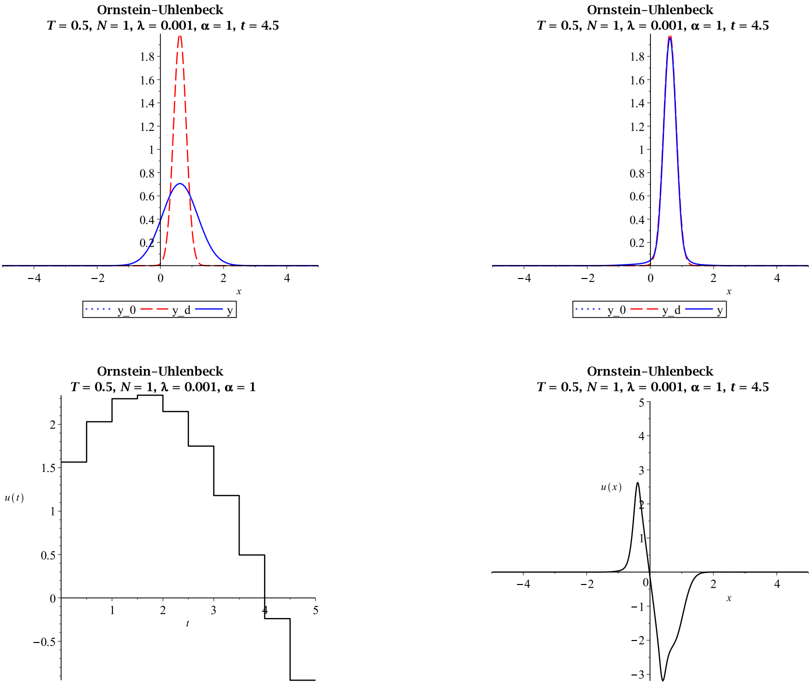

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
5
4
3
u(x)
2
1
0
−1
−2
−3
−4
−2
2
4
x

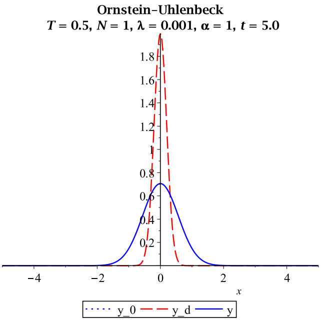

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
x
y_0
y_d
y

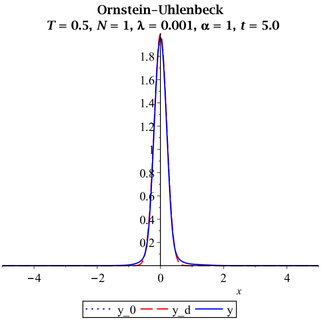

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
x
y_0
y_d
y

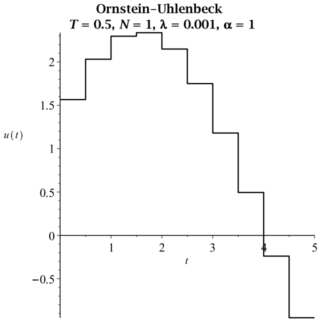

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

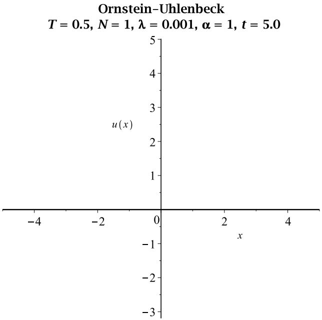

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
u(x)
x


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
x
y_0
y_d
y
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
x
y_0
y_d
y
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
u(x)
x


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
−0.5
1
2
3
4
5
t

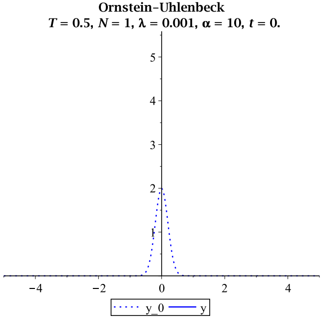

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
5
4
3
u(x)
2
1
0
−1
−2
−3
−4
−2
2
4
x

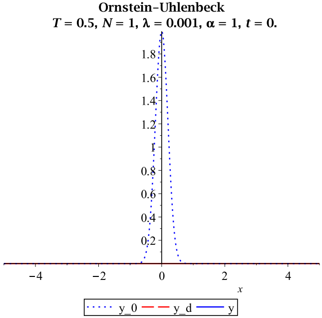
Numerical Examples (2)

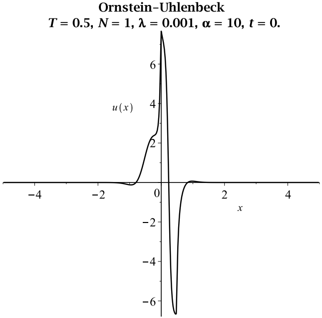
With space-dependent control, larger class of objectives possible:

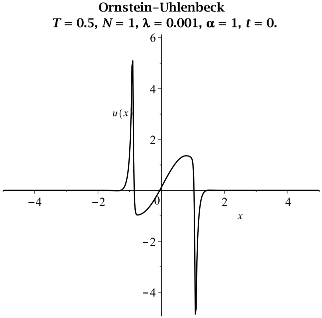
- region avoidance, without prescribing the shape of the PDF, e.g. try to force the state PDF into $[0, 0.5]$.
- Try to track non-smooth targets, e.g.

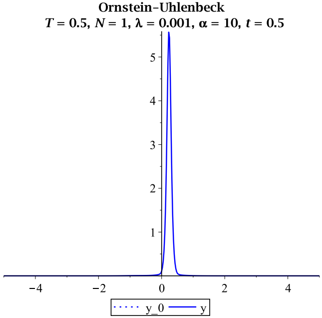
$$y_d(x,t) := \begin{cases} 0.5 & \text{if } x \in [-1 + 0.15t, 1 + 0.15t] \\ 0 & \text{otherwise.} \end{cases}$$

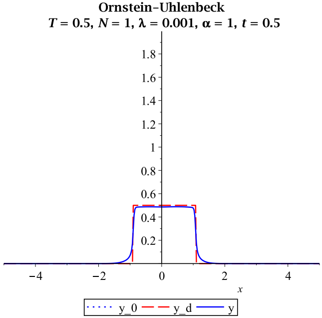

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 0.
y_0
y

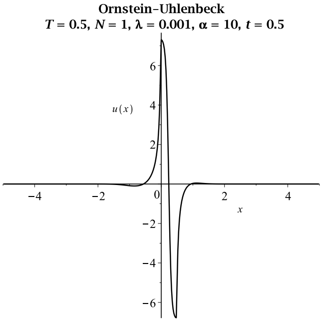

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
x
y_0
y_d
y

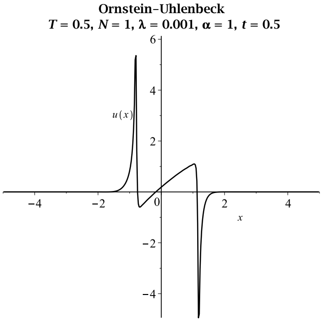

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 0.
u(x)
x

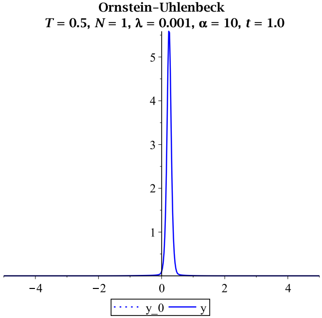

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
u(x)
x

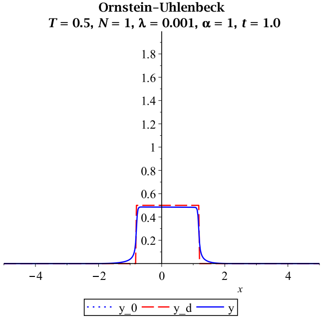

Ornstein–Uhlenbeck
$T = 0.5$, $N = 1$, $\lambda = 0.001$, $\alpha = 10$, $t = 0.5$
y_0
y

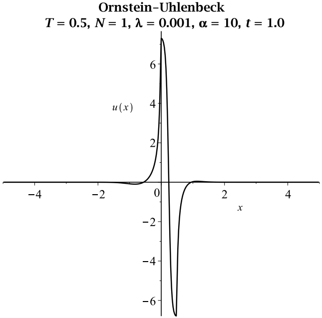

Ornstein–Uhlenbeck
$T = 0.5$, $N = 1$, $\lambda = 0.001$, $\alpha = 1$, $t = 0.5$
x
y_0
y_d
y

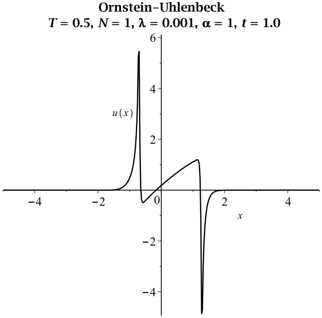

Ornstein–Uhlenbeck
$T = 0.5$, $N = 1$, $\lambda = 0.001$, $\alpha = 10$, $t = 0.5$
u(x)
x

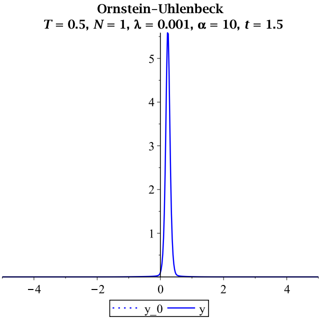

Ornstein–Uhlenbeck
$T = 0.5$, $N = 1$, $\lambda = 0.001$, $\alpha = 1$, $t = 0.5$
u(x)
x

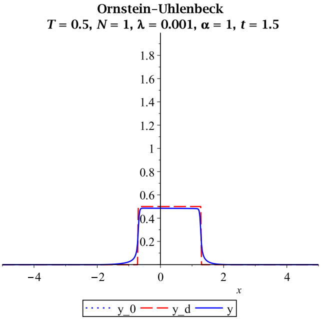

Ornstein–Uhlenbeck
$T = 0.5$, $N = 1$, $\lambda = 0.001$, $\alpha = 10$, $t = 1.0$
y_0
y

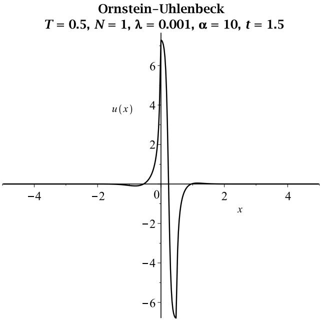

Ornstein–Uhlenbeck
$T = 0.5$, $N = 1$, $\lambda = 0.001$, $\alpha = 1$, $t = 1.0$
x
y_0
y_d
y

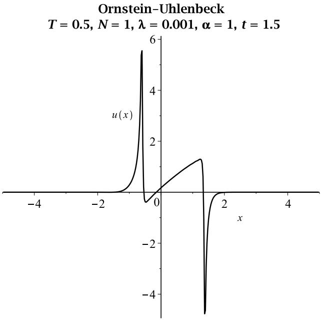

Ornstein–Uhlenbeck
$T = 0.5$, $N = 1$, $\lambda = 0.001$, $\alpha = 10$, $t = 1.0$
u(x)
x

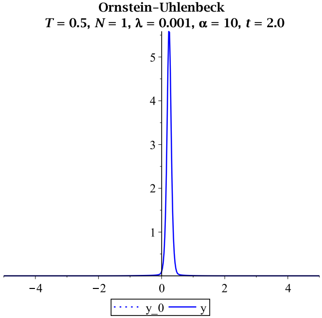

Ornstein–Uhlenbeck
$T = 0.5$, $N = 1$, $\lambda = 0.001$, $\alpha = 1$, $t = 1.0$
u(x)
x

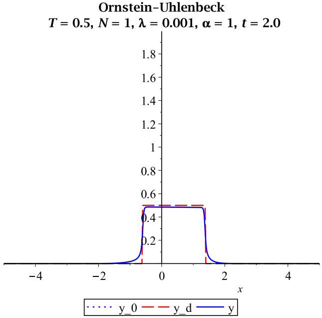

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 1.5
y_0
y

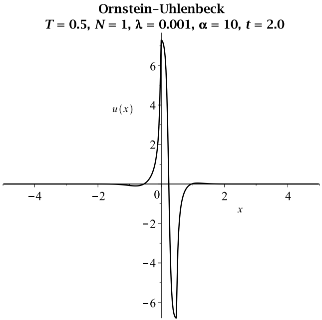

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
x
y_0
y_d
y

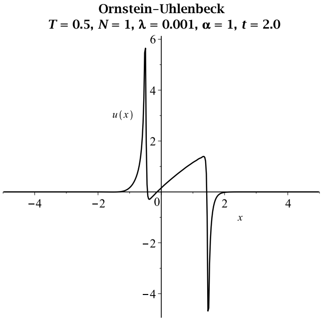

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 1.5
u(x)
x

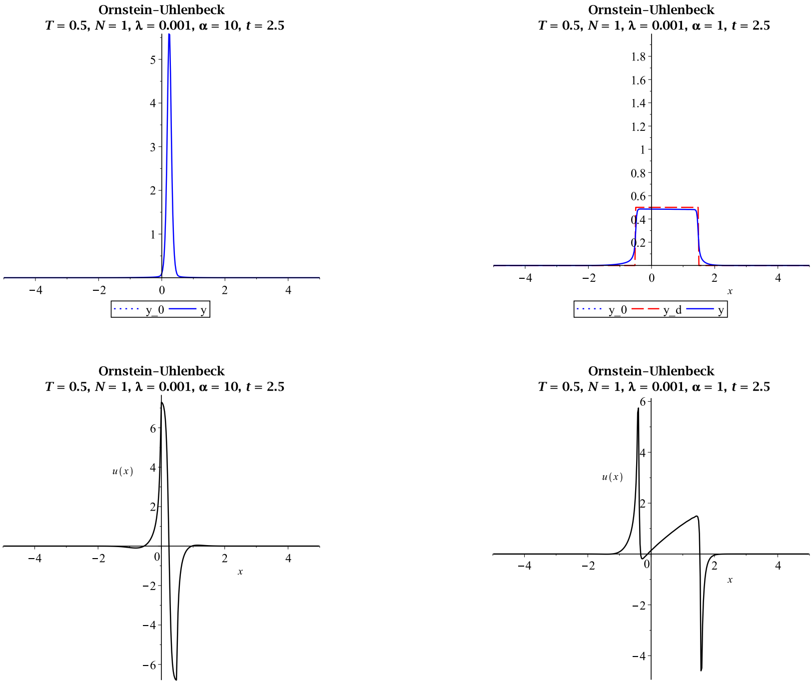

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
u(x)
x

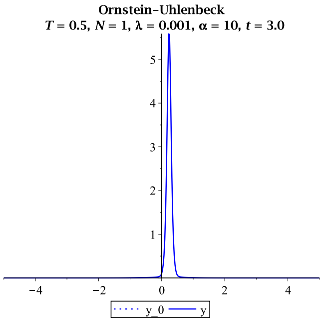

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 2.0
y_0
y

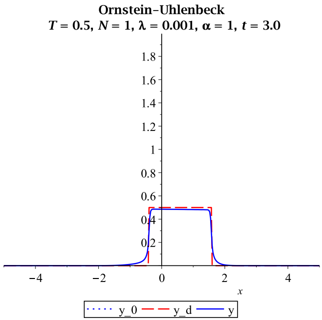

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
x
y_0
y_d
y

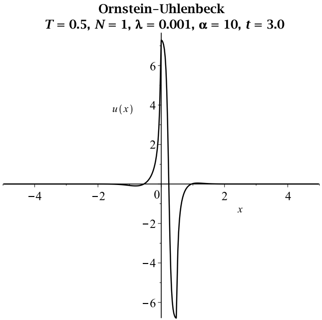

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 2.0
u(x)
x

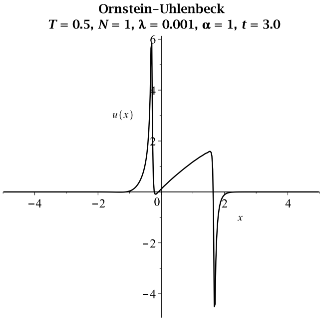

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
u(x)
x

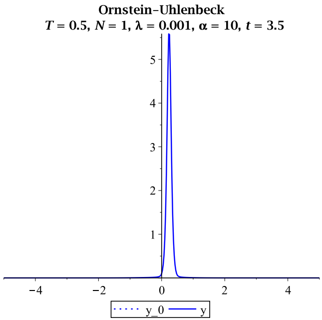

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 2.5
y_0
y
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.5
x
y_0
y_d
y
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 2.5
u(x)
x
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.5
u(x)
x

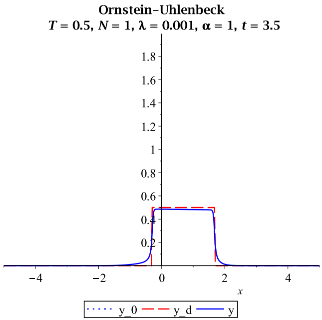

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 3.0
y_0
y

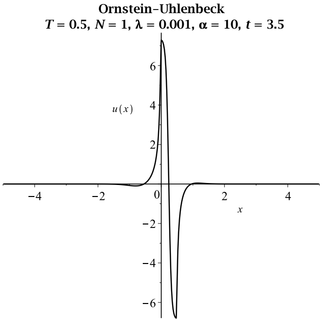

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
x
y_0
y_d
y

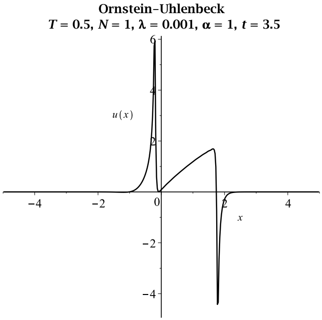

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 3.0
u(x)
x

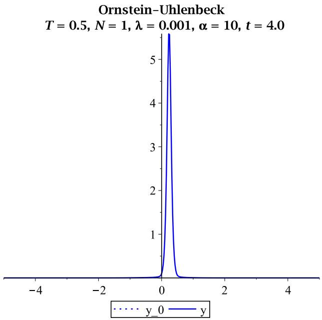

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
u(x)
x

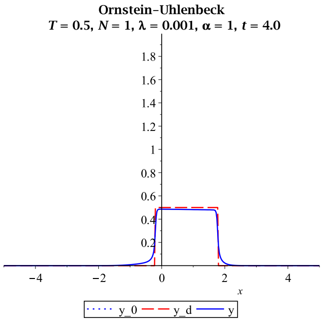

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 3.5
y_0
y

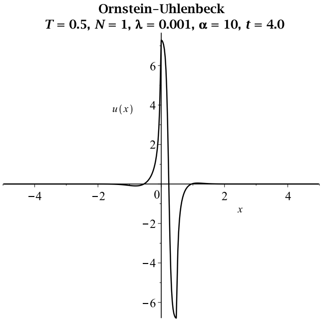

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
x
y_0
y_d
y

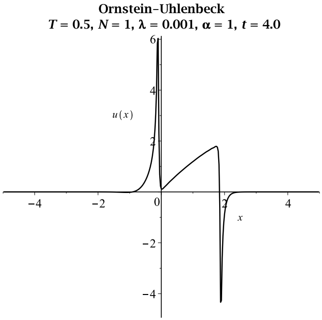

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 3.5
u(x)
x

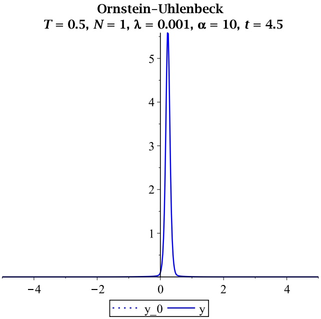

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
u(x)
x

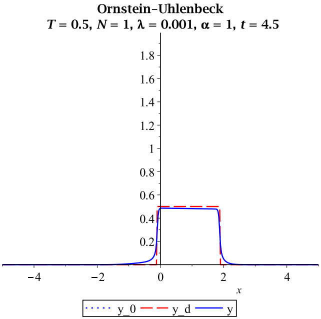

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 4.0
y_0
y

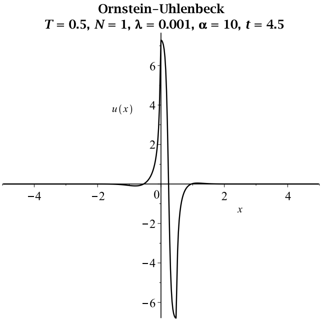

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
x
y_0
y_d
y

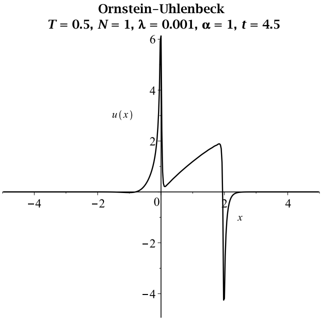

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 4.0
u(x)
x

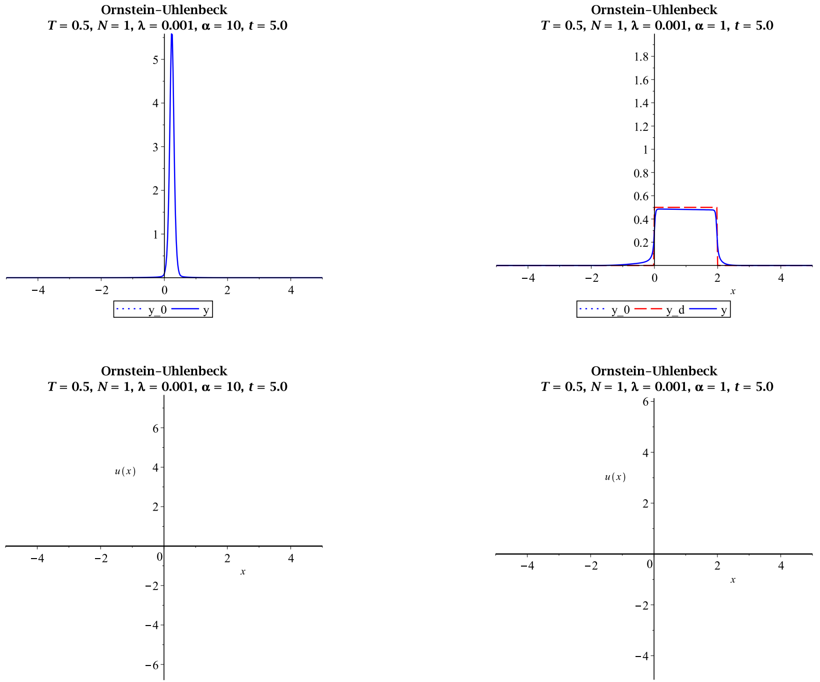

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
u(x)
x


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 4.5
y_0
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 4.5
u(x)
x


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
u(x)
x


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 5.0
y_0
y
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
x
y_0
y_d
y
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 5.0
u(x)
x
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
u(x)
x

Numerical Example (3)

Consider the two-dimensional stochastic process, modeling the dispersion of substance in shallow water [Heemink, 1990]

$$\sigma(x,t) := \begin{pmatrix} \sqrt{2D(x)} & 0 \\ 0 & \sqrt{2D(x)} \end{pmatrix}$$

with $D(x) := -\frac{1}{64}\left((x_1-4)^2+(x_2-4)^2\right)+\frac{3}{5} > 0$ in Ω and

$$b(x,t;u) := \begin{pmatrix} u_1 + \frac{\partial D}{\partial x_1} - \frac{1}{10} \\ u_2 + \frac{\partial D}{\partial x_2} - \frac{1}{10} \end{pmatrix} = \begin{pmatrix} u_1 - \frac{x_1}{32} + \frac{1}{40} \\ u_2 - \frac{x_2}{32} + \frac{1}{40} \end{pmatrix}$$

on $Q := \Omega \times [0,5]$ with $\Omega :=]0,8[\times]0,8[$.

Numerical Example (3)

- Initial distribution y_0: (smoothed) delta-Dirac located at $(4, 4)$.
- Target PDF is given by

$$y_d(x_1, x_2) = m \left[\frac{1}{x_1} \exp\left(\frac{2C_1}{\sigma^2} \log(x_1) \right) - \frac{2}{\sigma^2}(x_1 - 1) \right] \left[\frac{1}{x_2} \exp\left(\frac{2C_2}{\sigma^2} \log(x_2) \right) - \frac{2}{\sigma^2}(x_2 - 1) \right]$$

  with $C_1 = 2.625, C_2 = 2.125, \sigma = 0.5, m \approx 0.00004591595108$. (Equilibrium PDF of a stochastic Lotka-Volterra two-species prey-predator model [Yeung and Stewart, 2007])
- Other parameters: sampling time $T = 0.5$, regularization parameter $\lambda = 0.001$, and control bounds $u_a = -10$, $u_b = 10$.

Numerical Example (3)

- Initial distribution y_0: (smoothed) delta-Dirac located at $(4,4)$.
- Target PDF is given by

$$y_d(x_1,x_2) = m\left[\frac{1}{x_1}\exp\left(\frac{2C_1}{\sigma^2}\log(x_1)\right) - \frac{2}{\sigma^2}(x_1-1)\right] \left[\frac{1}{x_2}\exp\left(\frac{2C_2}{\sigma^2}\log(x_2)\right) - \frac{2}{\sigma^2}(x_2-1)\right]$$

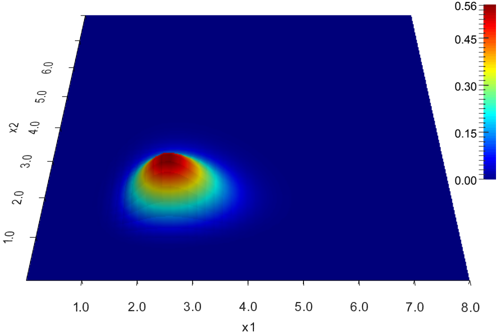
  with $C_1 = 2.625, C_2 = 2.125, \sigma = 0.5, m \approx 0.00004591595108$. (Equilibrium PDF of a stochastic Lotka-Volterra two-species prey-predator model [Yeung and Stewart, 2007])
- Other parameters: sampling time $T = 0.5$, regularization parameter $\lambda = 0.001$, and control bounds $u_a = -10$, $u_b = 10$.

Numerical Example (3)

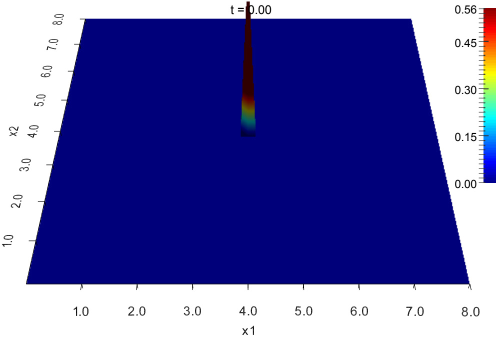
- Initial distribution y_0: (smoothed) delta-Dirac located at $(4,4)$.
- Target PDF is given by

$$y_d(x_1,x_2) = m\left[\frac{1}{x_1}\exp\left(\frac{2C_1}{\sigma^2}\log(x_1)\right) - \frac{2}{\sigma^2}(x_1-1)\right]$$
$$\left[\frac{1}{x_2}\exp\left(\frac{2C_2}{\sigma^2}\log(x_2)\right) - \frac{2}{\sigma^2}(x_2-1)\right]$$

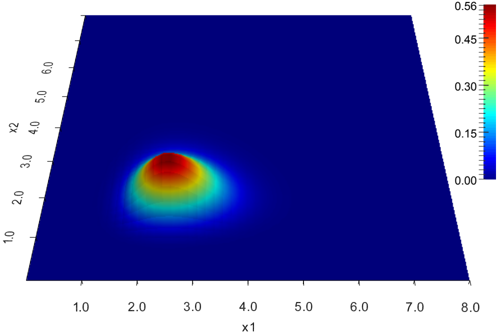
  with $C_1 = 2.625, C_2 = 2.125, \sigma = 0.5, m \approx 0.00004591595108$. (Equilibrium PDF of a stochastic Lotka-Volterra two-species prey-predator model [Yeung and Stewart, 2007])
- Other parameters: sampling time $T = 0.5$, regularization parameter $\lambda = 0.001$, and control bounds $u_a = -10$, $u_b = 10$.

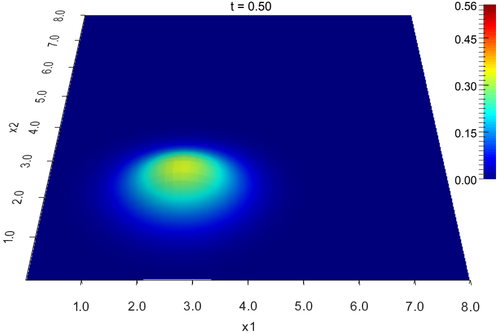


$u \in \mathbb{R}^2$:

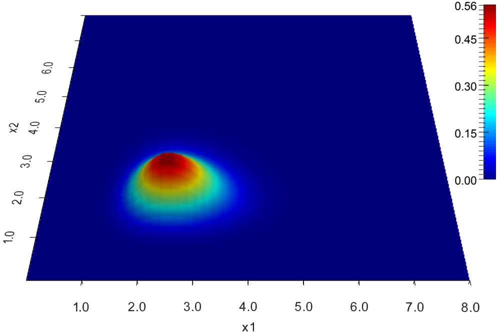


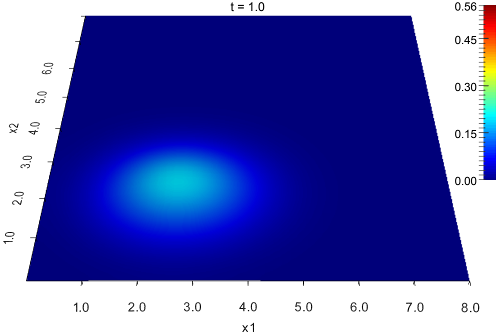
$u \in \mathbb{R}^2$:

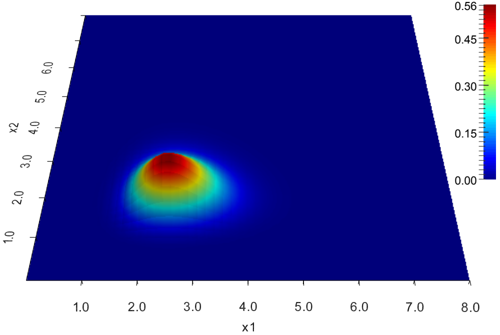




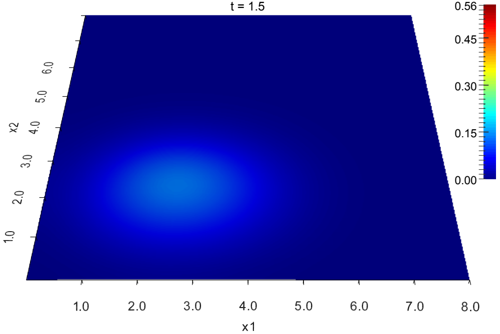
$u \in \mathbb{R}^2$:



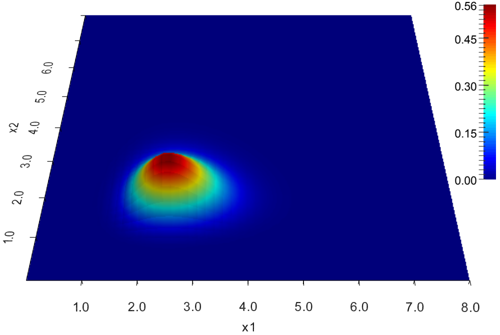


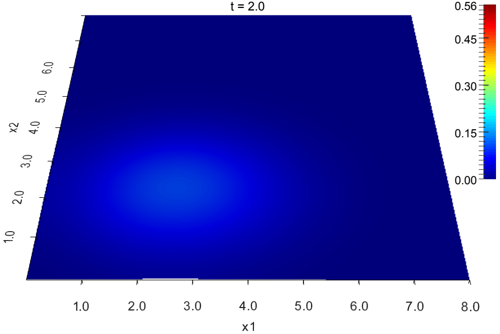


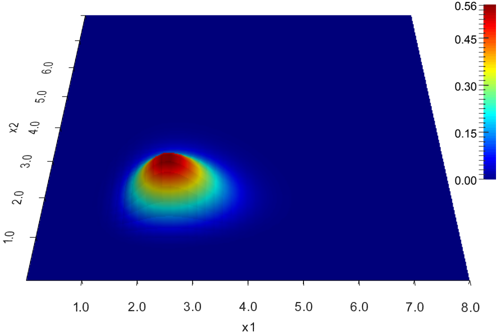
$u \in \mathbb{R}^2$:



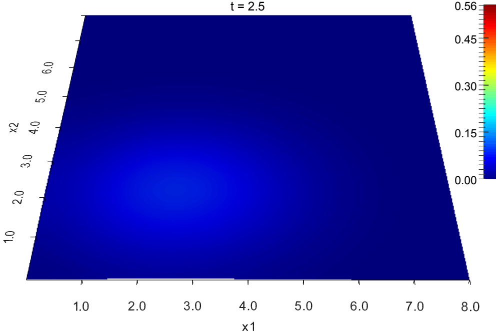
$u \in \mathbb{R}^2$:

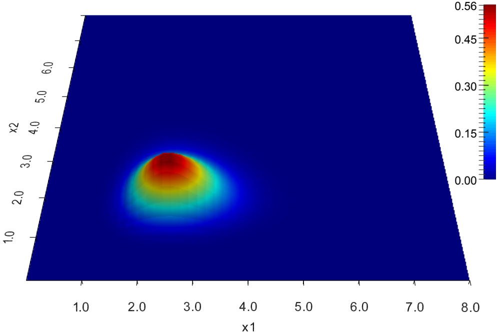




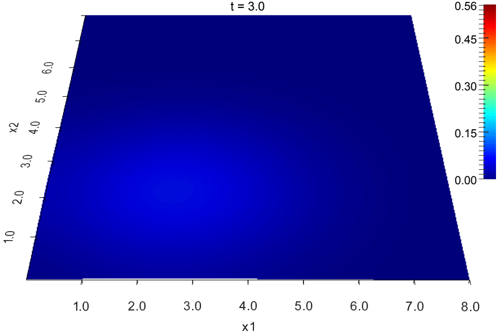


$u \in \mathbb{R}^2$:

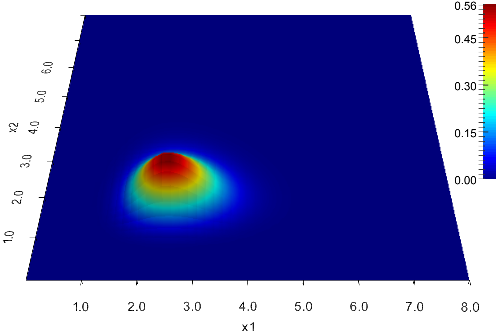


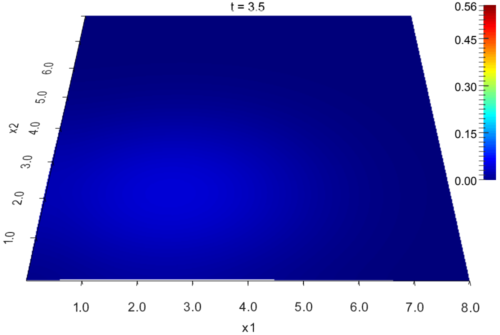


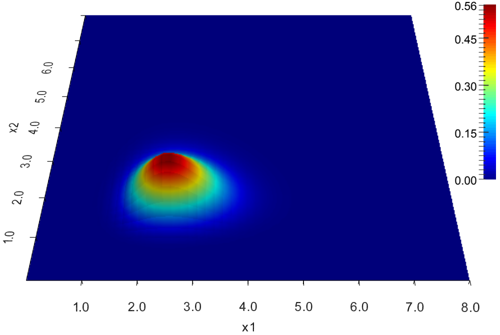
$u \in \mathbb{R}^2$:



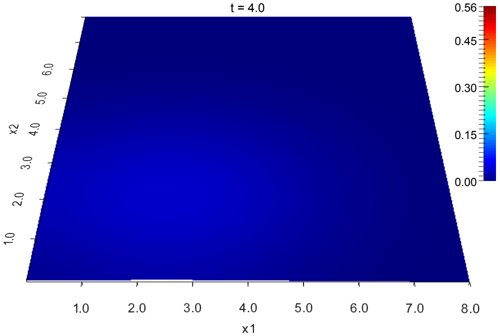
$u \in \mathbb{R}^2$:



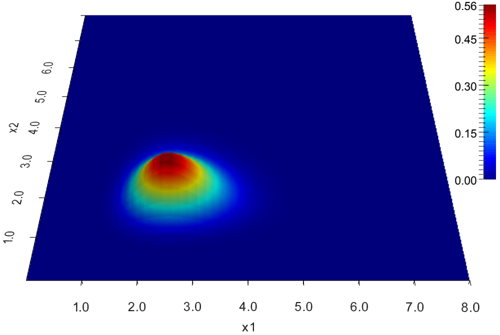


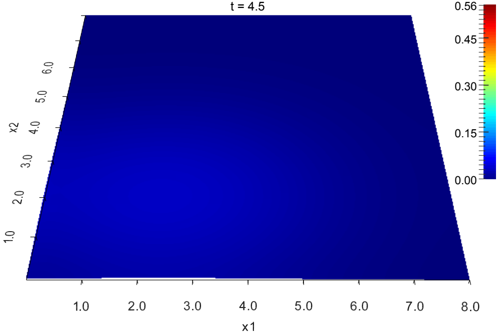


$u \in \mathbb{R}^2$:

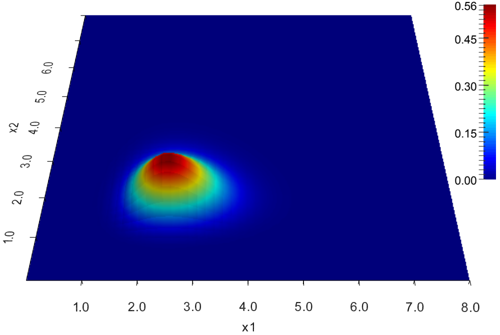


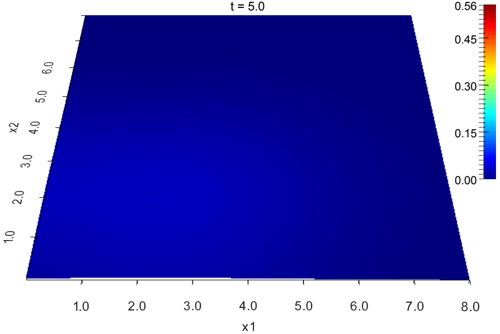
$u \in \mathbb{R}^2$:

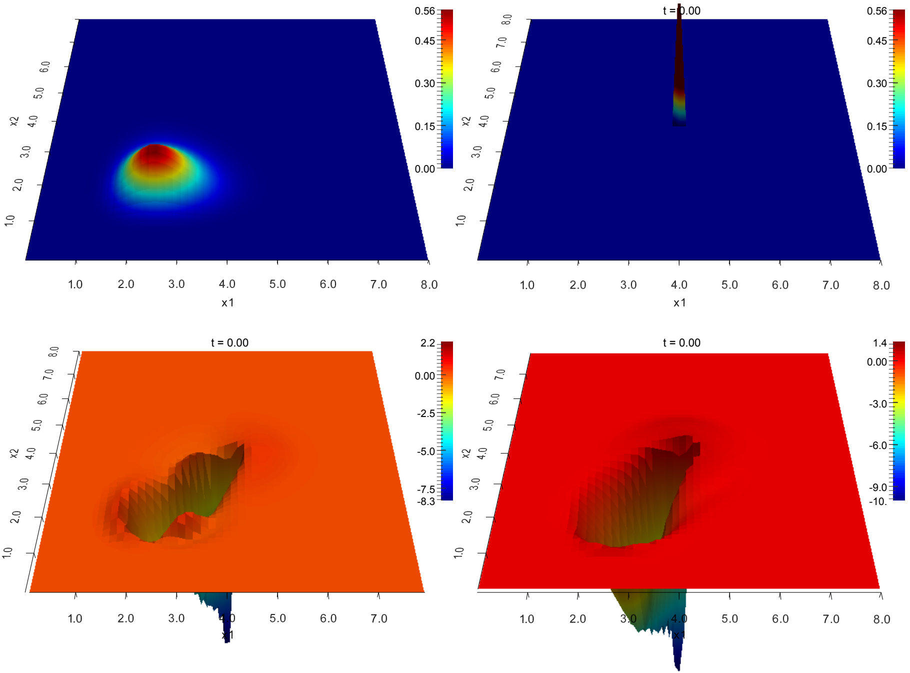




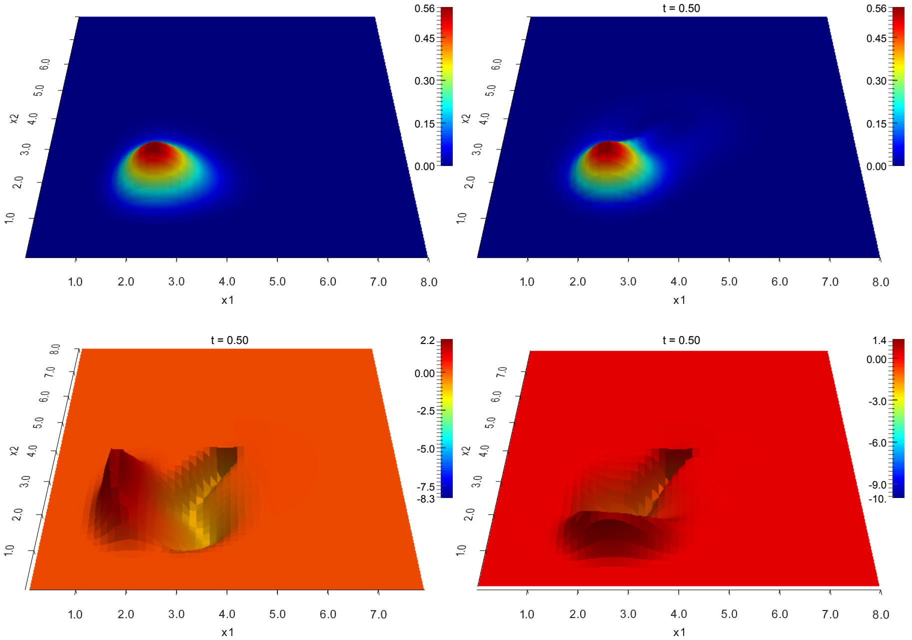
$u \in \mathbb{R}^2$:



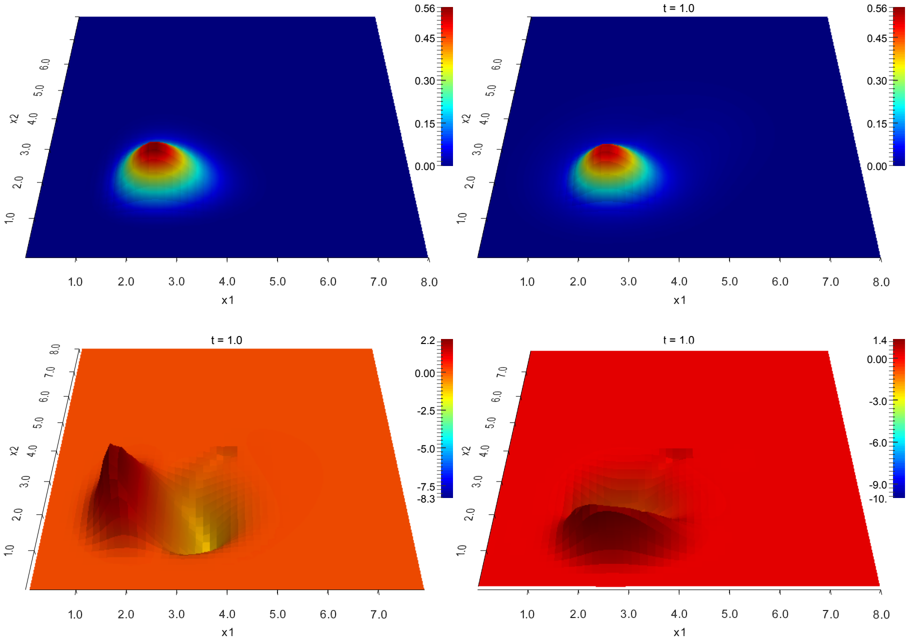


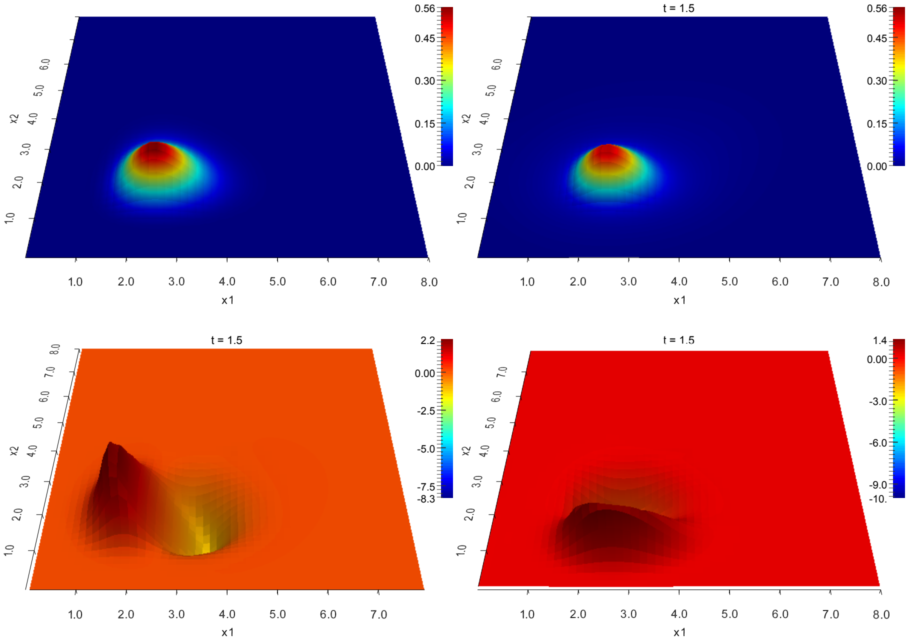


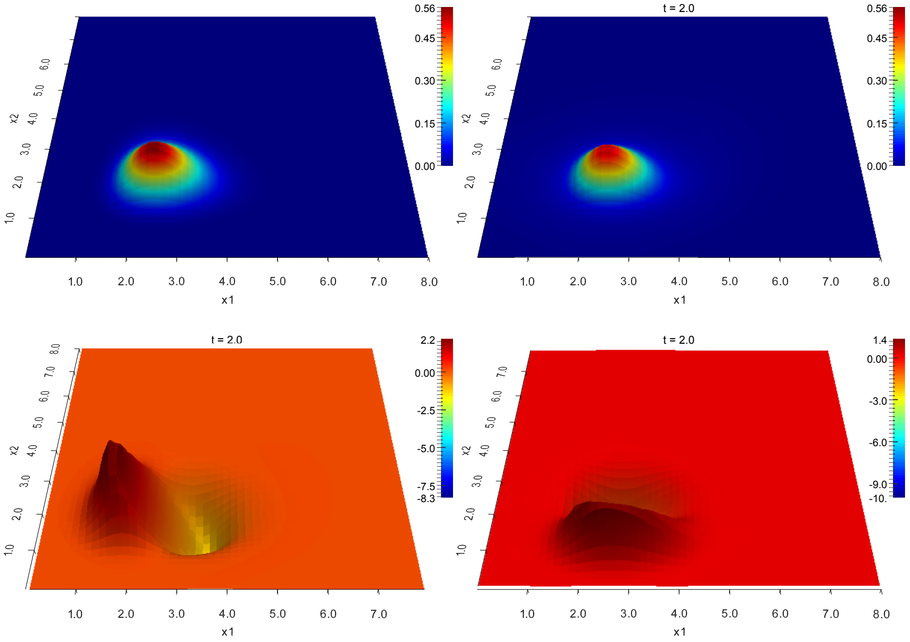
0.56
0.45
0.30
0.15
0.00
1.0 2.0 3.0 4.0 5.0 6.0
x2
1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0
x1
t = 0.00
0.56
0.45
0.30
0.15
0.00
1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0
x2
1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0
x1
t = 0.00
2.2
0.00
-2.5
-5.0
-7.5
-8.3
1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0
x2
1.0 2.0 3.0 4.0 5.0 6.0 7.0
x1
t = 0.00
1.4
0.00
-3.0
-6.0
-9.0
-10.
1.0 2.0 3.0 4.0 5.0 6.0 7.0
x2
1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0

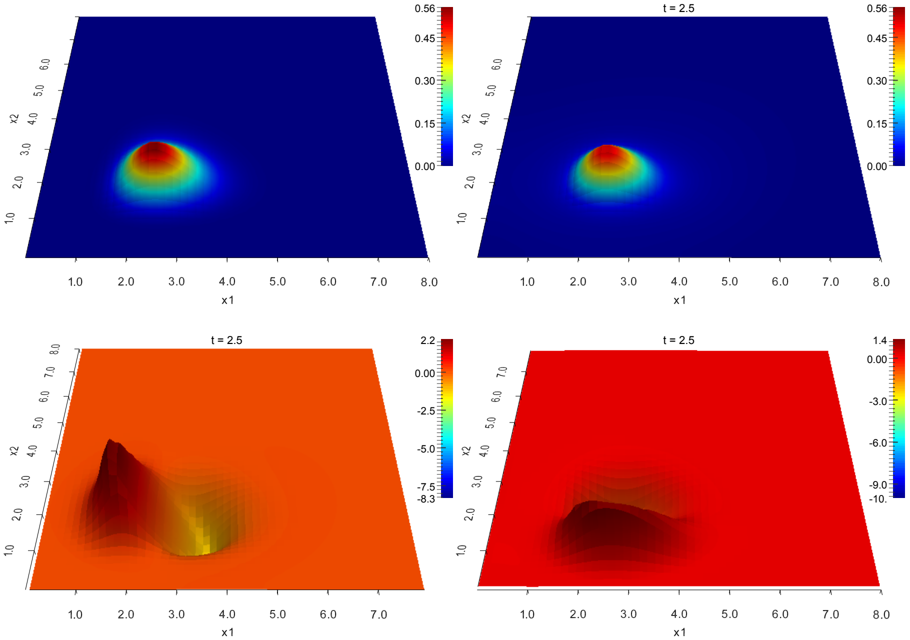


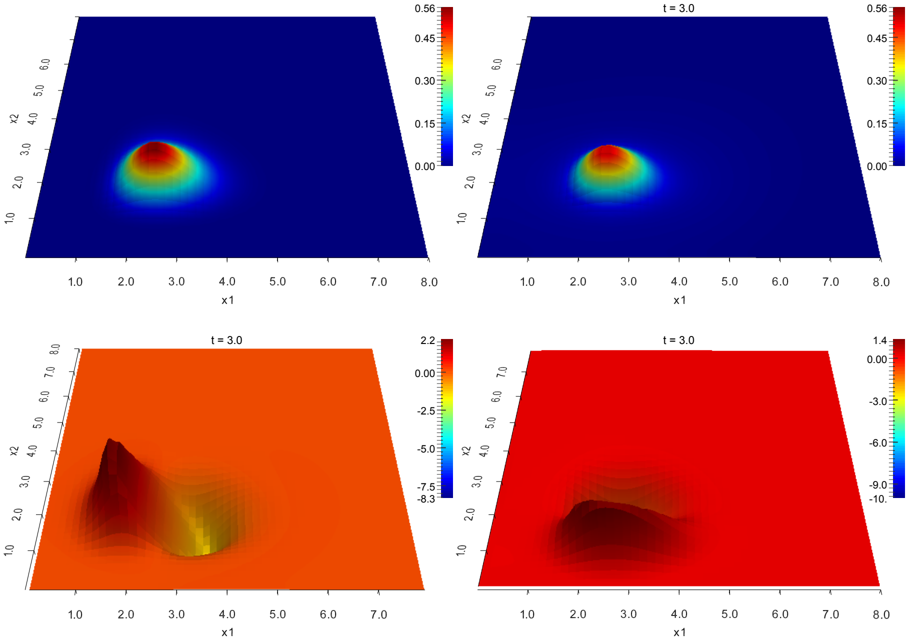
t = 0.50
t = 0.50
t = 0.50
x1
x2
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
0.56
0.45
0.30
0.15
0.00
2.2
0.00
-2.5
-5.0
-7.5
-8.3
1.4
0.00
-3.0
-6.0
-9.0
-10.


0.56
0.45
0.30
0.15
0.00
x2
1.0
2.0
3.0
4.0
5.0
6.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
x1
t = 1.0
0.56
0.45
0.30
0.15
0.00
x2
1.0
2.0
3.0
4.0
5.0
6.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
x1
t = 1.0
2.2
0.00
-2.5
-5.0
-7.5
-8.3
x2
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
x1
t = 1.0
1.4
0.00
-3.0
-6.0
-9.0
-10.
x2
1.0
2.0
3.0
4.0
5.0
6.0
7.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
x1

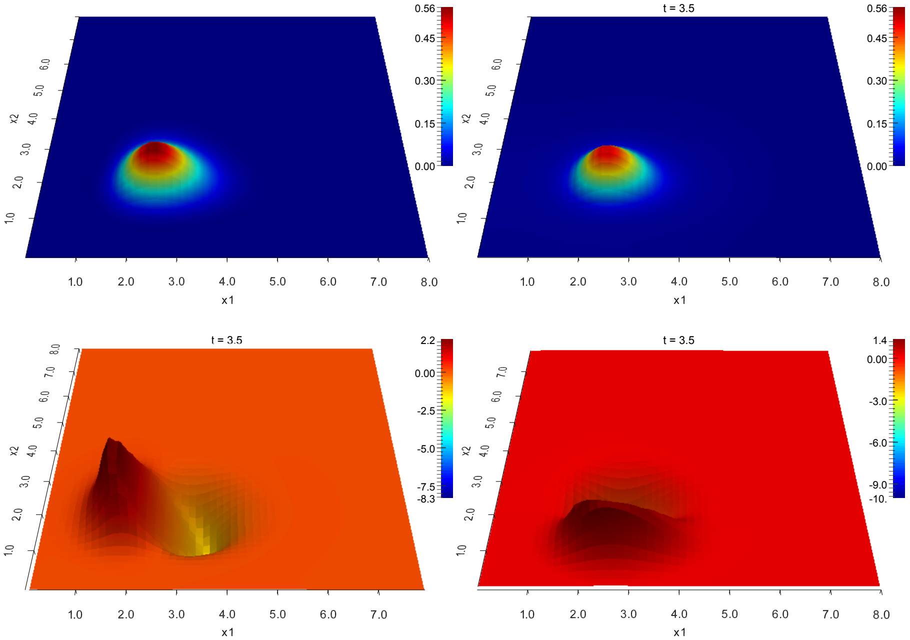

t = 1.5
t = 1.5
t = 1.5
x1
x2
0.56
0.45
0.30
0.15
0.00
2.2
0.00
-2.5
-5.0
-7.5
-8.3
1.4
0.00
-3.0
-6.0
-9.0
-10.

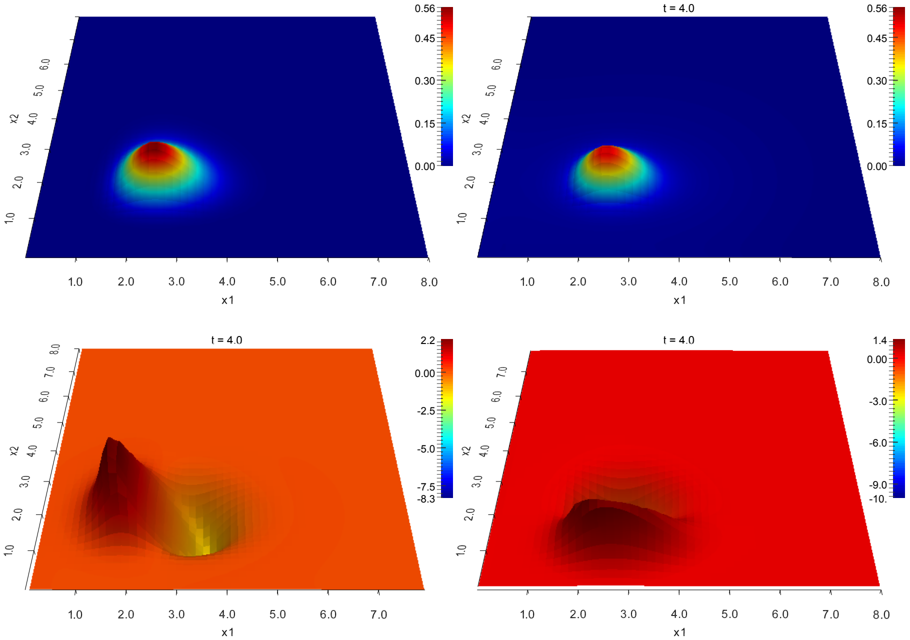

t = 2.0
t = 2.0
t = 2.0
x1
x2
0.56
0.45
0.30
0.15
0.00
2.2
0.00
-2.5
-5.0
-7.5
-8.3
1.4
0.00
-3.0
-6.0
-9.0
-10.
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0

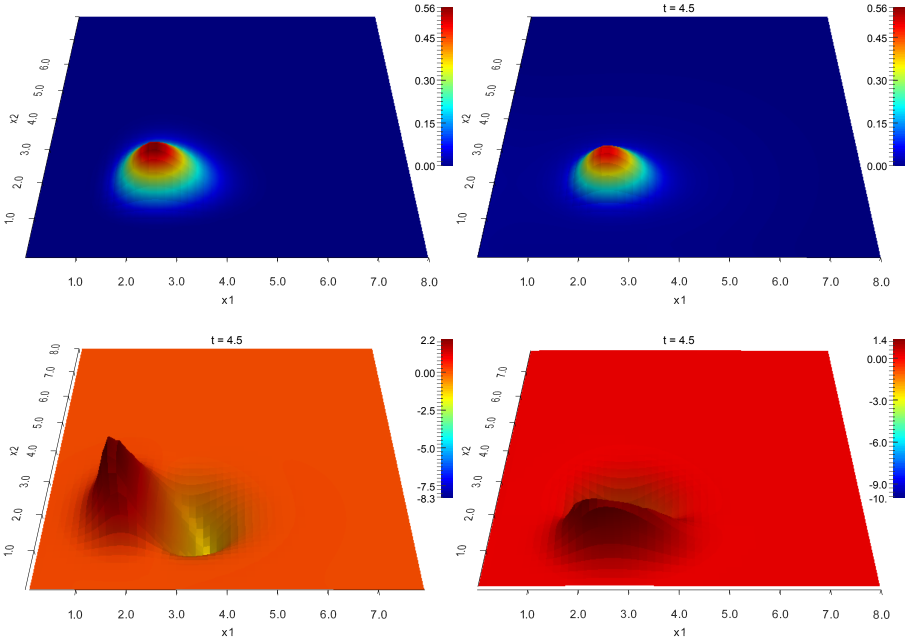

t = 2.5
t = 2.5
t = 2.5
x1
x2
0.56
0.45
0.30
0.15
0.00
2.2
0.00
-2.5
-5.0
-7.5
-8.3
1.4
0.00
-3.0
-6.0
-9.0
-10.
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0



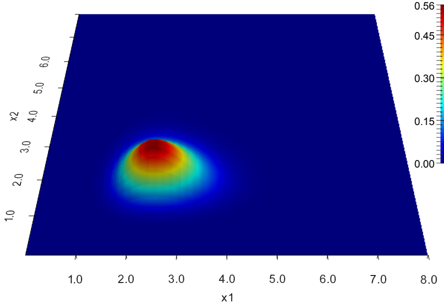
t = 3.0
t = 3.0
t = 3.0
x1
x2
0.56
0.45
0.30
0.15
0.00
2.2
0.00
-2.5
-5.0
-7.5
-8.3
1.4
0.00
-3.0
-6.0
-9.0
-10.

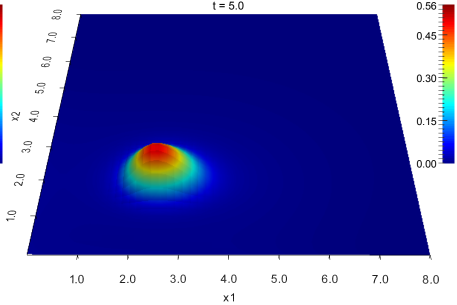

0.56
0.45
0.30
0.15
0.00
x2
1.0
2.0
3.0
4.0
5.0
6.0
1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0
x1
t = 3.5
0.56
0.45
0.30
0.15
0.00
x2
1.0
2.0
3.0
4.0
5.0
6.0
1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0
x1
t = 3.5
2.2
0.00
-2.5
-5.0
-7.5
-8.3
x2
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
1.0 2.0 3.0 4.0 5.0 6.0 7.0
x1
t = 3.5
1.4
0.00
-3.0
-6.0
-9.0
-10.
x2
1.0
2.0
3.0
4.0
5.0
6.0
7.0
1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0
x1


t = 4.0
t = 4.0
t = 4.0
x1
x2
0.56
0.45
0.30
0.15
0.00
2.2
0.00
-2.5
-5.0
-7.5
-8.3
1.4
0.00
-3.0
-6.0
-9.0
-10.
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0



t = 4.5
t = 4.5
t = 4.5
x1
x2
0.56
0.45
0.30
0.15
0.00
2.2
0.00
-2.5
-5.0
-7.5
-8.3
1.4
0.00
-3.0
-6.0
-9.0
-10.


0.56
0.45
0.30
0.15
0.00
x2
1.0
2.0
3.0
4.0
5.0
6.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
x1


t = 5.0
0.56
0.45
0.30
0.15
0.00
x2
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
x1

Some remarks

- The computed optimal control of the FPE is then applied to the stochastic process
- Right boundary conditions of Robin type
  (already embedded in the numerical scheme)
- Annunziato, Borzì et al. have applied the same Fokker-Planck Optimal Control framework to
  - the class of piecewise deterministic processes
  - optimal control of open quantum systems
  - subdiffusion processes
- estimates for the minimal horizon N that guarantees stability of the MPC closed-loop system by Fleig & Grüne, 2016

Some remarks

- The computed optimal control of the FPE is then applied to the stochastic process
- Right boundary conditions of Robin type
  (already embedded in the numerical scheme)
- Annunziato, Borzì et al. have applied the same Fokker-Planck Optimal Control framework to
  - the class of piecewise deterministic processes
  - optimal control of open quantum systems
  - subdiffusion processes
- estimates for the minimal horizon N that guarantees stability of the MPC closed-loop system by Fleig & Grüne, 2016

Some remarks

- The computed optimal control of the FPE is then applied to the stochastic process
- Right boundary conditions of Robin type
  (already embedded in the numerical scheme)
- Annunziato, Borzì et al. have applied the same Fokker-Planck Optimal Control framework to
  - the class of piecewise deterministic processes
  - optimal control of open quantum systems
  - subdiffusion processes
- estimates for the minimal horizon N that guarantees stability of the MPC closed-loop system by Fleig & Grüne, 2016

Some remarks

- The computed optimal control of the FPE is then applied to the stochastic process
- Right boundary conditions of Robin type
  (already embedded in the numerical scheme)
- Annunziato, Borzì et al. have applied the same Fokker-Planck Optimal Control framework to
  - the class of piecewise deterministic processes
  - optimal control of open quantum systems
  - subdiffusion processes
- estimates for the minimal horizon N that guarantees stability of the MPC closed-loop system by Fleig & Grüne, 2016

Some remarks

- The computed optimal control of the FPE is then applied to the stochastic process
- Right boundary conditions of Robin type
  (already embedded in the numerical scheme)
- Annunziato, Borzì et al. have applied the same Fokker-Planck Optimal Control framework to
  - the class of piecewise deterministic processes
  - optimal control of open quantum systems
  - subdiffusion processes
- estimates for the minimal horizon N that guarantees stability of the MPC closed-loop system by Fleig & Grüne, 2016

Thank you for your attention!