Nonlocal Cahn-Hilliard equations: the strict separation property and some consequences

Maurizio Grasselli

Dipartimento di Matematica, Politecnico di Milano, ITALY
maurizio.grasselli@polimi.it

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The Cahn-Hilliard equation (CHe): structure

- binary system (e.g., an alloy) occupying a bdd domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, for any $t \in (0, T)$
- $\varphi : \Omega \times (0, T) \to [-1, 1]$ relative concentration difference

$$\varphi_t = \operatorname{div}\left(m(\varphi)\nabla\mu\right)$$
$$\mu(\varphi) = \frac{\delta\mathcal{F}}{\delta\varphi}(\varphi)$$

- m mobility (e.g., $m > 0$ constant or $m(s) = (1 - s^2)$)
- $\mathcal{F}$ nonconvex free energy functional
- $m(\varphi)\partial_{\mathbf{n}}\mu = 0$ on $\partial\Omega \times (0, T)$ so the total mass is conserved

The free energy functionals

- the local functional (Cahn & Hilliard '58, Cahn '61)

$$\mathcal{L}(\varphi) = \int_\Omega \left(\frac{k}{2} |\nabla \varphi|^2 + F_0(\varphi) \right) \, dx$$

where $k > 0$ and

$$F_0(s) = \frac{\theta}{2} \left[(1+s) \log(1+s) + (1-s) \log(1-s) \right] - \frac{\theta_c}{2} s^2$$

- the nonlocal functional (Giacomin & Lebowitz '96, '97, '98)

$$\mathcal{E}(\varphi) = -\frac{1}{2} \int_\Omega \int_\Omega J(x-y) \varphi(x) \varphi(y) \, dxdy + \int_\Omega F(\varphi) \, dx$$

where $J : \mathbb{R}^d \to \mathbb{R}$ is s.t. $J(x) = J(-x)$ and

$$F(s) = \frac{\theta}{2} \left[(1+s) \log(1+s) + (1-s) \log(1-s) \right]$$

Rewriting the nonlocal free energy functional

Adding and subtracting $\frac{1}{2}(1 * J)(x)$ with $1 * J > 0$ yield

$$\mathcal{E}(\varphi) = \underbrace{\frac{1}{4}\int_\Omega\int_\Omega J(x-y)|\varphi(x)-\varphi(y)|^2\,dxdy}_{\approx\frac{k}{2}\int_\Omega|\nabla\varphi|^2dx} + \int_\Omega F_2(\varphi,x)\,dx$$

where

$$F_2(s,x) = F(s) - \frac{1}{2}(1 * J)(x)s^2$$

which is nonconvex provided that $1 * J$ is large enough

Remark

In many contributions the above functional is introduced with F_2 double well independent of x

The case of constant mobility

Suppose $m(s) = k = 1$ then we have

- local CHe

$$
\begin{aligned}
&\varphi_t = \Delta\left(-\Delta\varphi + F_0'(\phi)\right)\\
&\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\varphi = 0\\
&\varphi(0) = \varphi_0
\end{aligned}
$$

- nonlocal CHe

$$
\begin{aligned}
&\varphi_t = \Delta\left(-J * \varphi + F'(\varphi)\right)\\
&\partial_{\mathbf{n}}\mu = 0\\
&\varphi(0) = \varphi_0
\end{aligned}
$$

The local CHe: basic literature

- Well-posedness: Elliott & Luckhaus ’91; Debussche & Dettori ’95; Kenmochi, Niezgódka & Pawlow ’95
- Convergence to equilibrium: Abels & Wilke ’07
- Regularity and attractors: Miranville & Zelik ’05
- Dynamic boundary conditions: Gilardi, Miranville & Schimperna ’09
- Survey paper: Cherfils, Miranville & Zelik ’11

Basic assumptions

- $J \in W^{1,1}(\mathbb{R}^d)$ with $J(x) = J(-x)$
- $F \in C[-1,1] \cap C^2(-1,1)$ s.t.

$$\lim_{s \to -1} F'(s) = -\infty, \quad \lim_{s \to 1} F'(s) = +\infty, \quad F''(s) \geq \alpha > 0$$

Remark

- *no sign assumption on J is required*
- $1 * J$ *need not to be positive*
- $F(s) = +\infty$ *for any* $s \notin [-1,1]$
- $\exists \xi \in (-1,1)$ *s.t.* $F'(\xi) = 0$ *and we can suppose* $F(\xi) = 0$ *so that* $F(s) \geq 0$ *for all* $s \in [-1,1]$

Definition

Let φ_0 be a measurable function s.t. $F(\varphi_0) \in L^1(\Omega)$ and $T > 0$ be given. A function φ is a weak solution on $[0, T]$ corresponding to φ_0 if

$$\varphi \in L^\infty(0, T; H) \cap L^2(0, T; V)$$
$$\varphi \in L^\infty(\Omega \times (0, T)) \text{ with } |\varphi(x, t)| < 1 \text{ a.e. } (x, t) \in \Omega \times (0, T)$$
$$\varphi_t \in L^2(0, T; V')$$
$$\mu = -J * \varphi + F'(\varphi) \in L^2(0, T; V)$$

and φ satisfies the identity

$$\langle \varphi_t, v \rangle_{V', V} + (\nabla \mu, \nabla v) = 0 \quad \forall\, v \in V, \text{ a.e. } t \in (0, T)$$

with $\varphi(0, \cdot) = \varphi_0$.

The nonlocal CHe: well-posedness 3

Theorem

Let φ_0 *be measurable and s.t.* $F(\varphi_0) \in L^1(\Omega)$, $|\overline{\varphi_0}| < 1$. *Let* $T > 0$ *be given. Then* $\exists$ *a unique weak sol* φ *which satisfies the dissipative inequality for all* $t \geq 0$,

$$\mathcal{E}(\varphi(t)) + \omega \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \, d\tau \\ \leq \mathcal{E}(\varphi_0)e^{-\omega t} + C\left[1 + F(\overline{\varphi_0})\right]$$

where ω *and* C *are positive constants independent of the initial condition. Moreover, for all* $t \in [0, T]$, *there holds*

$$\|\varphi_1(t) - \varphi_2(t)\|_{V'} \leq K\|\varphi_{01} - \varphi_{02}\|_{V'} e^{Ct}$$

where φ_1 *and* φ_2 *are two weak sols to on* $[0, T]$ *with initial data* φ_{01} *and* φ_{02}, *respectively, and* K *depends on* $\|F'(\varphi_i)\|_{L^1(0,T;L^1(\Omega))}$, $i = 1, 2$.

- the proof is given in Frigeri & G. ’12 (or Gal, Giorgini & G. ’16) for a proof (see also Colli, Krejčí, Rocca & Sprekels ’07 for a more abstract setting)
- $|\overline{\varphi}_0| < 1$ implies that the initial datum cannot be a pure phase

Remark

Some consequences are

$$t \mapsto \|\varphi(t)\|_{L^\infty(\Omega)} \text{ is measurable}$$
$$\sup_{t\geq 0} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1$$
$$\varphi \in C([0,T];L^p(\Omega)), \quad p \in [2,\infty)$$

Lemma (Gal, Giorgini & G '16)

$\exists\, C > 0$, *independent of the initial condition, s.t.*

$$\|\varphi_t\|_{L^\infty(1,t;V')} + \|\varphi_t\|_{L^2(t,t+1;H)} \leq C, \quad \forall\, t \geq 1$$

$$\|\nabla \mu\|_{L^\infty(1,t;H)} \leq C, \quad \forall\, t \geq 1$$

$$\sup_{t\geq 1} \|\varphi(t)\|_V \leq C.$$

The nonlocal CHe: regularity 2

Lemma (Gal, Giorgini & G '16)

Let $\kappa \in (0,1)$. Then $\exists\, C = C(\kappa) > 0$ s.t., for any measurable initial data φ_0 with $F(\varphi_0) \in L^1(\Omega)$ and $\overline{\varphi}_0 \in [-1+\kappa, 1-\kappa]$, there holds, for all $t \geq 1$,

$$\|F'(\varphi)\|_{L^\infty(1,t;V)} + \|\mu(t)\|_{L^\infty(1,t;V)} + \|\mu\|_{L^2(t,t+1;H^2(\Omega))} \leq C$$

as well as

$$\|\nabla\mu\|_{L^q(t,t+1;L^p(\Omega))} + \|\nabla\varphi\|_{L^q(t,t+1;L^p(\Omega))} \leq C, \quad \frac{p-2}{p} = \frac{2}{q}, \; d = 2$$

$$\|\nabla\mu\|_{L^q(t,t+1;L^p(\Omega))} + \|\nabla\varphi\|_{L^q(t,t+1;L^p(\Omega))} \leq C, \quad \frac{3p-6}{2p} = \frac{2}{q}, \; d = 3$$

where $2 \leq p < \infty$ if $d = 2$ and $2 \leq p \leq 6$ if $d = 3$.

- For a given $\kappa \in (0,1)$ define

$$\mathcal{H}_\kappa = \left\{ \varphi \in H : F(\varphi) \in L^1(\Omega) \text{ and } \overline{\varphi} \in [-1+\kappa, 1-\kappa] \right\}$$

endowed with the metric

$$\mathbf{d}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\| + \left| \int_\Omega F(\varphi_1)\, dx - \int_\Omega F(\varphi_2)\, dx \right|^{1/2}$$

- $\mathcal{H}_\kappa$ is a complete metric space
- set $S(t) : \mathcal{H}_\kappa \to \mathcal{H}_\kappa$, $S(t)\varphi_0 = \varphi(t)$ for all $t \geq 0$

Theorem (Gal, Giorgini & G '16)

Let $\kappa \in (0,1)$. Then the dissipative dynamical system $(\mathcal{H}_\kappa, S(t))$ has a connected global attractor $\mathcal{A}_\kappa$ bdd in $\mathcal{H}_\kappa \cap V$.

The strict separation property

- **Question:** given φ_0, which may be pure (i.e. $+1$ or -1) on positive measure sets, are there $t_0 > 0$ and $\delta > 0$ depending on $\overline{\varphi}_0$ s.t.

$$|\varphi(x,t)| \le 1 - \delta$$

  for almost any $(x,t) \in \overline{\Omega} \times [t_0, \infty)$?
  If this happens we call it strict separation property (ssp)
- the ssp property holds for the local CHe in dimension two (Miranville & Zelik '05) with homogeneous Neumann b.c.
- **Main result:** the ssp property also holds for the nonlocal CHe in dimension two (Gal, Giorgini & G '16)
- in dimension three the validity of the ssp is open

The nonlocal CHe: the strict separation property

- $d = 2$
- additional assumption: $F \in C^3(-1,1)$ satisfies

$$F''(s) \le e^{C|F'(s)|+C}, \quad F'(s)F'''(s) \ge 0, \quad |F'''(s)| \le C|F''(s)|^2$$

Theorem (Gal, Giorgini & G '16)

Let $\kappa \in (0,1)$. Then $\exists\, t_0 > 1$ and $\delta(\kappa) > 0$ s.t., for any measurable initial datum φ_0 with $F(\varphi_0) \in L^1(\Omega)$ and $\overline{\varphi}_0 \in [-1+\kappa, 1-\kappa]$, there holds

$$\|\varphi(t)\|_{L^\infty(\Omega)} \le 1 - \delta(\kappa), \quad \forall\, t \ge t_0.$$

- Let $p \geq 1$, observe that

$$\int_\Omega F''(\varphi)^p \, dx \leq \int_\Omega e^{p[C_1|F'(\varphi)|+C_2]} \, dx = e^{pC_2} \int_\Omega e^{pC_1|F'(\varphi)|} \, dx$$

- applying the Trudinger-Moser inequality to $pC_1|F'(\varphi(t))| \in V$ for almost every $t \geq 1$, we get

$$\|F''(\varphi)\|^p_{L^p(\Omega)} \leq e^{pC_2} e^{C\left[p^2\|F'(\varphi)\|^2_V+1\right]}$$

- thus we find

$$\|F''(\varphi)\|_{L^\infty(1,t;L^p(\Omega))} \leq Ce^{pC}, \quad \forall\, t \geq 1$$

- further considerations allow us to deduce, for all $t \geq 1$,

$$\|F'(\varphi(t))\|_V \leq C, \; \|F''(\varphi(t))\|_{L^p(\Omega)} \leq C$$
$$\|F'''(\varphi(t))\|_{L^p(\Omega)} \leq Ce^{pC}$$

- we can then take the test function

$$v(t) = |F'(\varphi(t))|^{p-1}F'(\varphi(t))F''(\varphi(t))$$

- and after some efforts and a number of tricks we find . . .

ssp: sketch of the proof 3

- the following inequality

$$\frac{d}{dt}\|F'(\varphi)\|_{L^{p+1}(\Omega)}^{p+1} + \frac{\alpha p}{(p+1)}\int_\Omega |\nabla |F'(\varphi)|^{\frac{p+1}{2}}|^2\,dx$$
$$\leq Cp^2 + Cp^{2s}\left(\int_\Omega |F'(\varphi)|^{\frac{p+1}{2}}\,dx\right)^2$$
$$\leq Cp^{2s}\left(1+\left(\int_\Omega |F'(\varphi)|^{\frac{p+1}{2}}\,dx\right)^2\right)$$

for all $p \geq 1$, $s > 1$ and almost every $t \geq 1$

- we now fix some $s > 1$ and an iterative argument eventually yields

$$\sup_{t\geq t_0}\|F'(\varphi(t))\|_{L^\infty(\Omega)} \leq C\sup_{t\geq 1}\|F'(\varphi(t))\| \leq C$$

for some $t_0 > 1$ and the ssp property follows

Corollary

$\exists\, C = C(\kappa) > 0$ *s.t.* $\|\mu(t)\|_{L^\infty(\Omega)} \leq C$ *for all* $t \geq t_0$.

Corollary

$\exists\, C = C(\kappa) > 0$ *and* $\alpha \in (0,1)$ *s.t.*

$$|\varphi(x_1,t_1) - \varphi(x_2,t_2)| \leq C\left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}\right)$$
$$|\mu(x_1,t_1) - \mu(x_2,t_2)| \leq C\left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}\right)$$

for all $(x_1,t_1),(x_2,t_2) \in Q_t = [t,t+1] \times \overline{\Omega}$, $t > t_0$.

ssp: consequences 2

The ssp allows us to interpret the weak sol as a weak sol of a similar pb with a smooth potential (for t large enough) so that further regularity properties can be proven

Theorem (Gal, Giorgini & G '16)

$\exists\, C(\kappa) > 0$ and $t_1 > t_0$ s.t.

$$\|\varphi_t\|_{L^\infty(t_1,t;H)} + \|\varphi_t\|_{L^2(t,t+1;V)} + \|\nabla\mu_t\|_{L^2(t,t+1;H)} \leq C, \ \forall\, t \geq t_1.$$

Then, by comparison, we get, for all $t \geq t_1$,

$$\begin{aligned}&\|\nabla\mu\|_{L^\infty(t_1,t;L^p(\Omega))} + \|\mu\|_{L^\infty(t,t+1;H^2(\Omega))}\\&+ \|\nabla\varphi\|_{L^\infty(t_1,t;L^p(\Omega))} + \|\varphi_{tt}\|_{L^2(t,t+1;V')} \leq C.\end{aligned}$$

Suppose $J \in W^{2,1}(\mathbb{R}^2)$ (it can be weakened to include Newtonian and Bessel potentials), then

Theorem (Gal, Giorgini & G '16)

$\exists\, C = C(\kappa, \rho) > 0$ *s.t.*

$$\sup_{t \geq t_1} \|\varphi(t)\|_{H^2(\Omega)} \leq C.$$

For any $p \in [2, \infty)$, $\exists\, C = C(\kappa, \delta, p) > 0$ *and* $t_2 > t_1$ *s.t.*

$$\|\varphi_t(t)\|_{L^p(\Omega)} + \|\mu(t)\|_{W^{2,p}(\Omega)} + \|\varphi(t)\|_{W^{2,p}(\Omega)} \leq C, \quad \forall\, t \geq t_2.$$

Theorem (Gal, Giorgini & G '16)

For every given $\kappa > 0$, the dynamical system $(\mathcal{H}_\kappa, S(t))$ has an exponential attractor $\mathcal{M}_\kappa$ bdd in $V \cap C^\alpha(\overline{\Omega})$.

Corollary

The global attractor $\mathcal{A}_\kappa$ is bdd in $V \cap C^\alpha(\overline{\Omega})$ and has finite fractal dimension, that is,

$$dim_F(\mathcal{A}_\kappa, C^\alpha(\overline{\Omega})) \leq C.$$

Corollary

If J is smoother then the global attractor and the exponential attractor are bdd in $\mathcal{H}_\kappa \cap W^{2,p}(\Omega)$, for any $p \in [2, \infty)$.

Corollary

If F is real analytic on $[-1+\delta(\kappa), 1-\delta(\kappa)]$. Then any weak sol φ is s.t.

$$\lim_{t\to\infty} \|\varphi(t) - \varphi_*\|_{L^\infty(\Omega)} = 0$$

where $\varphi_* \in V \cap C^\alpha(\overline{\Omega})$ solves $-J * \varphi_* + F'(\varphi_*) = C_*$ for some $C_* > 0$, and $\overline{\varphi}_0 = \overline{\varphi}_*$.

- mixture of two incompressible and immiscible fluids
- **u** average velocity
- density and viscosity are constants and equal to one

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} = \nabla p + \mu\nabla\varphi \\ \text{div}\, \mathbf{u} = 0 \\ \varphi_t + \mathbf{u}\cdot\nabla\varphi = \Delta\mu \\ \mu = -J * \varphi + F'(\varphi) \end{cases} \qquad \text{in } \Omega\times(0,T)$$

subject to the boundary and initial conditions

$$\mathbf{u} = \partial_{\mathbf{n}}\mu = 0, \quad \text{on } \partial\Omega\times(0,T), \quad \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \varphi(0,\cdot) = \varphi_0, \quad \text{in } \Omega$$

ssp: extension to (basic) CHNS systems

- the ssp also holds for CHNS systems in two dimensions
- the ssp for the local CHNS system is an open issue
- regularity results can thus be extended to CHNS systems
- $\exists$ of (smooth) attractors can be proven
- convergence to single equilibria can be shown
- optimal distributed control results can be possibly generalized (Frigeri, Rocca & Sprekels '16, smooth potential)

ssp: the degenerate case

- observe that

$$\varphi_t = \underbrace{\mathrm{div}\left(m(\varphi)F''(\varphi)\nabla\phi\right)}_{\text{diffusion}} - \mathrm{div}\left(m(\varphi)\nabla J * \varphi\right)$$

- ssp also holds for $d = 3$ (Londen & Petzeltová '11, $m(s)F''(s) = \text{constant}$)
- ssp still holds under more general assumptions on m and F with a different proof (Gal, Giorgini & G '16)
- further regularity of sols as well as more general results on the longtime behavior can be established (Gal, Giorgini & G '16)
- extensions to other systems (e.g. CHNS) are expected
- plenty of results for the nonlocal CHe (compare with the local CHe, Elliott & Garcke '96)