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Boundary control problems and Economic Applications

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Foreword

PDEs/SPDEs (e.g the deterministic/stochastic heat equation) with control at the boundary of a given domain $\mathcal{O}$.

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unbounded control operators arise in rewriting the equation as an ODE in a Hilbert space (e.g. $L^2(\mathcal{O})$).

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suitable conditions needed to give sense to the equation at least in mild form

Dynamic Programming approach to the control problem

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HJB equation has an unbounded term in the first derivative.

Up to now such control problems/HJB equations are treated only in special cases: by viscosity solutions or, in the stochastic case, by BSDEs, (with a structural restriction: the control acts only on the same direction of the noise).

Here new results on existence of regular solution for the HJB equations and of optimal feedback controls in the stochastic case.

OVERVIEW

- Introduction: the simplest case

- Motivating applied problems:
  - Boundary control of reaction-diffusion equations;
  - Boundary control of Burgers/Navier-Stokes equations;
  - Boundary control of first order PDEs/SPDEs (vintage capital);
  - Controlled SPDEs for the forward mortality force.

- A first regularity result:
  - setting of the problem;
  - solving the HJB equation;
  - verification theorem and feedback control strategies.

- Future extensions: deterministic case

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Introduction :

a simple boundary control problem

Simple controlled SPDE (where $0 \le t \le T < +\infty$, $\xi \in [0, \pi]$)

$$\begin{cases} dy(s,\xi) = \left[\Delta_\xi y(s,\xi)\right] + \sigma dW(s)(\xi) & \text{in } (t,T] \times (0,\pi) \\ y(t,\xi) = x(\xi) & \text{on } (0,\pi) \end{cases} \tag{1}$$

with the following boundary conditions. Dirichlet case:

$$y(s,0) = \alpha_1(s) \quad y(s,\pi) = \alpha_2(s) \tag{2}$$

Neumann case:

$$\frac{\partial y}{\partial n}(s,0) = \alpha_1(s) \quad \frac{\partial y}{\partial n}(s,\pi) = \alpha_2(s) \tag{3}$$

where

- Δ_ξ is the Laplace operator.
- W is a cylindrical Wiener process in $L^2(0,\pi)$.
- σ, is a given constant, possibly equal to zero.
- $x(\cdot) \in L^2(0,\pi)$ is the initial state (e.g. temperature distribution);

- $\alpha : [t,T] \times \Omega \longrightarrow \mathbb{R}^2$ is a prog. meas. control process.
- the "unique solution" is denoted by $y^{\alpha,t,x}$.

Goal: Optimal control. Minimize the objective functional

$$I(t,x;\alpha) := \mathbb{E}\left\{ \int_t^T \int_0^\pi \beta_1(y(s,\xi))d\xi + \beta_2(\alpha(s))ds + \int_0^\pi \gamma(y(T,\xi))d\xi \right\} \quad (4)$$

where $\beta_1, \beta_2, \gamma \in C^0(\mathbb{R})$ depend on the objective of the controller.

Many papers on this kind of problems by many authors both in deterministic and stochastic case For the deterministic case e.g. Cannarsa, Lasiecka, Soner, Tessitore, Triggiani and many others included various speakers here). For the stochastic case we recall Debussche-Fuhrman-Tessitore, Fabbri-Goldys, G.-Rouy-Swiech, Masiero.

Here: new results on the Goal in the stochastic case through Dynamic Programming.

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Motivating problem 1 :
Boundary control of a reaction diffusion equation

Let $\mathcal{O}$ to be an open, connected, bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$. State equation: the controlled SPDE on the time interval $[t, T]$, for $0 \le t \le T < +\infty$,

$$\begin{cases} dy(s,\xi) = \left[\Delta_\xi y(s,\xi) + f\left(y(s,\xi)\right)\right] + dW_Q(s)(\xi) & \text{in } (t,T] \times \mathcal{O} \\ y(t,\xi) = x(\xi) & \text{on } \mathcal{O} \\ y(s,\xi) = \alpha(s,\xi) & \text{on } (t,T] \times \partial\mathcal{O}, \end{cases} \tag{5}$$

where

- Δ_ξ is the Laplace operator. We consider the Dirichlet or Neumann boundary conditions.
- $f \in C^0(\mathbb{R})$ is a nonlinear function (a "reaction" term);
- W_Q is a Q-Wiener process with $Q \in \mathcal{L}^+(L^2(\mathcal{O}))$ and $(\mathcal{F}_s)_{s\in[t,T]}$ is the augmented filtration generated by W_Q.
- $x(\cdot) \in L^2(\mathcal{O})$ is the initial state (e.g. temperature distribution);
- given a control process α, the unique solution is $y^{\alpha,t,x}$.

Consider the functional (writing $y(t,\xi)$ for $y^{\alpha,t,x}(s,\xi)$)

$$I(t,x;\alpha) := \mathbb{E}\left\{ \int_t^T\int_{\mathcal{O}} \beta_1(y(s,\xi))d\xi + \int_{\partial\mathcal{O}} \beta_2(\alpha(s,\xi))d\xi ds + \int_{\mathcal{O}} \gamma(y(T,\xi))d\xi \right\} \tag{6}$$

where $\beta_1, \beta_2, \gamma \in C^0(\mathbb{R})$ depend on the objective of the controller.

Example: minimizing the energy and the distance from a given temperature profile. In economics: spatial dynamics of economic growth (Boucekkine-Camacho-Fabbri 2013, deterministic case)

The goal of the controller here would be to minimize the functional I above, over all control strategies α which are progressively measurable with respect to the augmented filtration generated by W_Q, and such that the above integrals make sense.

Difficult problem studied only with viscosity solutions in [G.-Rouy-Swiech 00, Tran-Tran 06] (see also, for the deterministic case, e.g., [Cannarsa-G.-Soner 93] and [Cannarsa-Tessitore 96]).

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Motivating problem 2:

Boundary control of Burgers (Navier-Stokes) equations;

State equation: stochastic controlled Burgers equation ($n = 1$)

$$\begin{cases} dy(s,\xi) = \left[\dfrac{\partial^2 y(s,\xi)}{\partial \xi^2} + \dfrac{1}{2}\dfrac{\partial}{\partial \xi} y^2(s,\xi)\right] ds + dW_Q(s)(\xi), \quad s \in (t,T], \xi \in (0,1) \\ y(t,\xi) = x(\xi), \quad \xi \in [0,1], \\ y(s,0) = \alpha_0(s), \quad y(s,1) = \alpha_1(s), \quad s \in [t,T]. \end{cases}$$

Here:

- the function $y : [t,T] \times [0,1] \times \Omega \longrightarrow \mathbb{R}$, $(s,\xi,\omega) \mapsto y(s,\xi,\omega)$ describes the evolution of the velocity field of the fluid;

- the control $\alpha : [t,T] \times (0,1) \times \Omega \longrightarrow \mathbb{R}$, $(s,\xi) \mapsto \alpha(s,\xi,\omega)$ gives the dynamics of the external force acting at every point of $(0,1)$;

- W_Q is a Q-Wiener process with $Q \in \mathcal{L}_1^+(L^2(0,1))$ and $(\mathcal{F}_s^t)_{s\in[t,T]}$ is the augmented filtration generated by W_Q;

- $x(\cdot) \in L^2(0,1)$ gives the distribution of the initial velocity field;

- As before, the solution is denoted by $y^{\alpha,t,x}$.

Objective of the controller (see e.g. [Shritharan 89], [Choi et al, 93], [Da Prato-Debussche, 98-99]): minimize the functional

$$I(t,x;\alpha(\cdot)) = \mathbb{E}\left\{ \int_t^T \left[\int_0^1 g_0(y^{\alpha,t,x}(s,\xi))d\xi + h(\alpha(s)) \right] ds \right.$$
$$\left. + \int_0^1 \varphi_0(y^{\alpha,t,x}(T,\xi))d\xi \right\}. \qquad (7)$$

Example: to get the final velocity field to be close to a given $\bar{y}$ while minimizing the “vorticity" of the flow (measured here by the integral of the space derivative) and the energy spent controlling the system.

We then minimize the functional I over all control strategies α which are progressively measurable with respect to $\mathcal{F}_s^t$, and such that $\mathbb{E}\int_t^T |\alpha(s)|^2 d\xi ds < +\infty$.

Difficult problem studied, for the DP approach, only in the case of “distributed” control (see e.g. [Da Prato - Debussche 98-99-00], [Manca 08], [G.-Sritharan-Swiech 06-08]).

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Motivating problem 3:

Boundary control of first order SPDEs;

INVESTMENTS WITH VINTAGE CAPITAL (Barucci-Gozzi '99, Feichtinger-Hartl-Kort-Veliov '06, Faggian-Gozzi '09)

- State equation:

$$\begin{cases} \frac{\partial}{\partial t}x(t,s) = -\frac{\partial}{\partial s}x(t,s) - \delta x(t,s) + i_1(t,s) + \sigma \dot{W_Q}(t,s), & t > 0,\ s \in [0,\bar{s}] \\ \\ x(0,s) = x_0(s),\ s \in [0,\bar{s}], \quad x(t,0) = i_0(t), & t \geq 0 \end{cases}$$

At time $t \geq 0$, and vintage s, $i(t,s)$ is the investment rate (control) in capital of vintage s, $x(t,s)$ is the capital stock (state).

$x_{x_0(\cdot),i_0(\cdot),i_1(\cdot)}(t,s)$ denotes the unique solution.

New feature: age structure in the capital and investment:
$\longrightarrow$ **first order SPDE**.

- Constraints: for every $t \geq 0$, $s \in [0,\bar{s}]$, $x(t,s) \geq 0$

- The functional to maximize is the profit.

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Motivating problem 4:

Control of forward mortality models;

Forward mortality rates (Bauer-Benth-Kiesel 2012, Tappe-Weber '14)

Let $\mu_t(\tau, x)$ be the force of mortality for an x-year old individual at time t in τ years.

Following e.g. [Bauer-Benth-Kiesel, 2012] a good model for μ_t is the following

$$\begin{cases} d\mu_t = \left[\frac{\partial}{\partial \tau}\mu_t - \frac{\partial}{\partial x}\mu_t + \alpha_t\right] dt + \sigma dW_t, \quad t > 0,\ s \in [0, \bar{s}] \\ \mu_0 \text{ given for } (\tau, x) \in [0, +\infty)^2, \end{cases}$$

To take account of newborn cohort effects Bauer and Biffis (paper in preparation) propose to add the boundary condition

$$d\mu_t(\tau, 0) = B_t \mu_t(\tau, 0) dt + \Theta_t dZ_t.$$

where Z_t is another Brownian motion, possibly correlated with W_t.

Depending on the specific problem one is interested in controlling the coefficients B, Θ, α, σ.

Main features of the above problems

- The state equation is an SPDE which can be rephrased as an SDE in a Hilbert space: **infinite dimensional problem**.

- **Control/noise possibly at the boundary**.

- Possibly nonlinear dynamics.

- Driving semigroup: second order (first two cases) or first order (last two cases).

Goal: find the optimal strategies in feedback form by using the Dynamic Programming approach, solving the associated HJB equation.

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A first regularity result
for the solutions of the HJB equations in the stochastic case
(G.-Masiero, work in progress).

The setting.

Boundary control, internal noise

It is the initial problem with no boundary noise. Set $\mathcal{O} := (0,\pi)$. Let $0 \le t \le T < +\infty$.

$$\begin{cases} dy(s,\xi) = \left[\Delta_\xi y(s,\xi)\right] + \sigma dW(s)(\xi) & \text{in } (t,T] \times \mathcal{O} \\ y(t,\xi) = x(\xi) & \text{on } \mathcal{O} \\ y(s,\xi) = \alpha(s,\xi) \quad or \quad \dfrac{\partial y}{\partial n}(s,\xi) = \alpha(s,\xi) & \text{on } (t,T] \times \partial\mathcal{O}, \end{cases} \tag{8}$$

where

- Δ_ξ is the Laplace operator. We consider the Dirichlet or Neumann boundary conditions.
- W is a cylindrical Wiener process in $\mathcal{H} := L^2(0,\pi)$ and $\sigma \in \mathcal{L}(\mathcal{H})$;
- $x(\cdot) \in L^2(\mathcal{O})$ is the initial state;
- $\alpha : [t,T] \longrightarrow \Lambda \subset \mathbb{R}^2$ is a prog. meas. control process;
- the unique solution is denoted by $y^{\alpha,t,x}$. Well posedness always guaranteed in $\mathcal{H}$ if Λ is bounded: we assume this from now on.

The goal is the same as in the introductory problem.

ASSOCIATED INFINITE-DIMENSIONAL PROBLEM

(see e.g. [Fabbri-G-Swiech-Fuhrman-Tessitore 16])

The operators

State space: $\mathcal{H} = L^2(\mathcal{O})$. Operator $A_D : D(A_D) \subset \mathcal{H} \to \mathcal{H}$:

$$\begin{cases} D(A_D) := H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}) \\ A_D \phi := \Delta \phi. \end{cases} \tag{9}$$

A_D is maximal dissipative, self-adjoint, invertible, $A^{-1} \in \mathcal{L}(L^2(\mathcal{O}))$, and the generated semigroup e^{tA_D} is analytic (see e.g. [Renardy-Rogers 04]).

Similarly $A_N : D(A_N) \subset \mathcal{H} \to \mathcal{H}$ is defined as

$$\begin{cases} D(A_N) := \left\{ u \in H^2(\mathcal{O}) \; : \; \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\mathcal{O} \right\} \\ A_N \phi := \Delta \phi. \end{cases} \tag{10}$$

If $\lambda \geq 0$ then $(A_N - \lambda I)$ is maximal dissipative, self-adjoint, and $e^{t(A_N - \lambda I)}$ is analytic. If $\lambda > 0$ then $(A_N - \lambda I)$ is invertible and $(A_N - \lambda I)^{-1} \in \mathcal{L}(L^2(\mathcal{O}))$ (see e.g. [Agmon 62]).

Dirichlet map. Consider the Laplace equation with Dirichlet boundary conditions

$$\begin{cases} \Delta_\xi y(\xi) = 0, & \xi \in \mathcal{O} \\ y(\xi) = \gamma(\xi), & \xi \in \partial\mathcal{O}. \end{cases} \tag{11}$$

For $\gamma \in H^s(\partial\mathcal{O})$, call the unique solution $D\gamma \in H^{s+1/2}(\mathcal{O})$. For all $s \geq 0$, the operator

$$\begin{cases} D : H^s(\partial\mathcal{O}) \longrightarrow H^{s+1/2}(\mathcal{O}) \\ \gamma \mapsto D\gamma \end{cases} \tag{12}$$

is continuous (see e.g. [Lions-Magenes 72].

Moreover (see e.g. [Lasiecka, 80]) D is continuous as a linear map between the spaces $L^2(\partial\mathcal{O})$ and $D((-A_D)^{1/4-\epsilon})$ for all $\epsilon > 0$:

$$D : L^2(\partial\mathcal{O}) \longrightarrow D((-A_D)^{1/4-\epsilon}). \tag{13}$$

Neumann map. Consider the Laplace equation with Neumann boundary conditions

$$\begin{cases} \Delta_\xi y(\xi) = \lambda y(\xi), & \xi \in \mathcal{O} \\ \frac{\partial}{\partial n} y(\xi) = \gamma(\xi), & \xi \in \partial\mathcal{O}, \end{cases} \tag{14}$$

where n is the outward unit normal vector and $\frac{\partial}{\partial n}$ is the normal derivative.

For $\gamma \in H^s(\partial\mathcal{O})$, call the unique solution $N_\lambda \gamma \in H^{s+3/2}(\mathcal{O})$. For all $s \geq 0$, the operator

$$\begin{cases} N_\lambda : H^s(\partial\mathcal{O}) \longrightarrow H^{s+3/2}(\mathcal{O}) \\ \gamma \mapsto N_\lambda \gamma \end{cases} \tag{15}$$

is continuous (see e.g. [Lions-Magenes 72].

Moreover (see e.g. [Lasiecka, 80]) N_λ is continuous as a linear map between the spaces $L^2(\partial\mathcal{O})$ and $D((-A_N + \lambda I)^{3/4-\epsilon})$ for all $\epsilon > 0$:

$$N_\lambda : L^2(\partial\mathcal{O}) \longrightarrow D((-A_N + \lambda I)^{3/4-\epsilon}). \tag{16}$$

The infinite dimensional state equation

Let $\Lambda \subseteq L^2(\partial\mathcal{O}) = \mathbb{R}^2$ be the(bounded) control space and $a : \Omega \times [t,T] \longrightarrow \Lambda$, $a(s) = \alpha(s,\cdot)$ be the control process.

Define the process $Y \in L^2_{\mathcal{P}}(\Omega \times [t,T], \mathcal{H})$ where $Y(s) = y(s,\cdot)$ where y is the solution of the original state equation.

By the known theory (see e.g. [FGS-FT 16]) the process Y is the unique solution of the abstract evolution equation in $\mathcal{H}$

$$\begin{cases} dY(s) = AY(s)dt + Ba(s)dt + \sigma dW(s), & s \in [t,T] \\ Y(t) = x \in \mathcal{H}, \end{cases} \tag{17}$$

where, in the Dirichlet case

$$A = A_D \qquad and \qquad B = -A_D D = (-A_D)^{3/4+\epsilon} C, \quad C \text{ continuous}$$

while in the Neumann case

$$A = A_N \qquad and \qquad B = (\lambda I - A_N) N_\lambda = (\lambda I - A_N)^{1/4+\epsilon} C.$$

This make sense in integral (or mild) form as:

$$Y(s) = e^{(s-t)A}x + \int_t^s (\eta - A)^\beta e^{(s-r)A} Ca(r)dr + \int_t^s e^{(s-r)A}\sigma dW(r). \quad (18)$$

where, in the Dirichlet case, $\eta = 0$ and $\beta = \frac{3}{4} + \epsilon$ while, in the Neumann case $\eta = \lambda$ and $\beta = \frac{1}{4} + \epsilon$.

The objective functional and the value function

The functional to minimize is easily rewritten as

$$J(t, x; a(\cdot)) := \mathbb{E} \int_t^T (g(s, Y(s)) + h(a(s)))\ ds + \mathbb{E}\varphi(Y(T))$$

for suitable continuous functions g, h, φ written depending on the given data β_1, β_2, γ.

The value function is defined as ($\mathcal{U}$ is the set of prog meas controls)

$$V(t, x) := \inf_{u(\cdot)\in\mathcal{U}} J(t, x; a(\cdot)), \qquad t \in [0, T],\ x \in \mathcal{H}.$$

The HJB equation for the value function

The associated Hamilton-Jacobi-Bellman (HJB) equation (whose candidate solution is the value function) is (here we set $Q := \sigma\sigma^*$)

$$\begin{cases} -\frac{\partial v(t,x)}{\partial t} = \frac{1}{2} Tr\ QD^2 v(t,x) + \langle Ax, Dv(t,x)\rangle + H_{min}(Dv(t,x)) + g(t,x), \\ \qquad\qquad t \in [0,T],\ x \in \mathcal{H}, \\ v(T,x) = \varphi(x), \end{cases} \tag{19}$$

where we have

$$H_{min}(p) := \inf_{u \in \mathbb{R}} H_{CV}(p\,;a) := \inf_{a \in \Lambda} \{< p, Ba >_{\mathcal{H}} + h(a)\} \tag{20}$$

Setting

$$\overline{H}_{min}(q) := \inf_{a \in \Lambda} \{\langle B^* p, a\rangle + h(a)\}, \quad \text{we have } H_{min}(p) = \overline{H}_{min}(B^* p).$$

Goal: find a solution with "enough" regularity to prove a verification theorem and to find an optimal feedback map of the type

$$a^*(s) = G(s, Y^*(s)), \qquad s \in [t,T]$$

where (a^*, Y^*) is an optimal couple. "Enough" means that $argmin$ in (20) make sense, i.e. that $B^* Dv(t,x)$ exists.

MAIN PROBLEM

The operator B is unbounded, hence is more difficult to give sense to $B^* Dv(t, x)$.

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Solving the HJB equation.

HJB EQUATION: WHAT IS KNOWN - 1

- If the value function is smooth (say C^1 in time and C^2 in space), then it solves the HJB equation. However this argument is only formal: in general the value function is not smooth.
- Even if the value function is smooth, it is difficult to prove directly regularity results for the value function going beyond the continuity.
- A good concept of solution in the context of HJB equations seems to be the concept of ***viscosity solution*** (Crandall and Lions, '80). It gives quite general existence and uniqueness results, it does not require regularity (classical or generalized) for the definition of solution and it can be used also in infinite dimension.
- Problem: Regularity results are very rare in this context. We only know some partial result for the first order deterministic case ([Federico-Goldys-FG '11, Federico-Tacconi '13])
- However we want regularity........

HJB EQUATION: WHAT IS KNOWN - 2

Results on existence of "smooth" solutions to infinite dimensional second order HJB equation are proven with four kind of techniques:

- by suitable fixed point arguments using smoothing properties of the transition semigroup associated to the uncontrolled problem (see e.g. [Cannarsa-Da Prato 1991] and many others);
- using a suitable convex regularization procedure (see e.g. [Barbu-Da Prato '81]);
- by representing the solution with a suitable Backward SDE (see e.g. [Fuhrman-Tessitore '02]);
- using suitable convex regualrization procedure (see e.g. [Barbu-Da Prato '81]);
- by suitable ad hoc change of variables or explicit representation formulae (see e.g. [Da Prato-Debussche '99]).

None of these approaches has been proved to work for our problem, up to now.

More precisely...

....the last two cannot be applied, up to now. Indeed:

- BSDE approach, up to now, (but see the previous talk of M. Fuhrman) needs always to assume $ImB \subset Im\sigma$ (the "structure condition") which is false here (true if we have also noise at the boundary);

- representation formulae are known only in quadratic cases (see e.g. Lasiecka-Triggiani) or in very special cases (see e.g. [Barucci-G 01].

The second method needs convexity of data. It has been used also recently ([Faggian-G. 10]) in the deterministic case but we found technical difficulties to extend it to the stochastic case.

Concerning the first method we extend and improve the present theory showing that we indeed have a "good enough" smoothing property for the uncontrolled problem. We have a twofold situation due to the unboudedness of B:

- in the Neumann case the red method is "easier" since the unboundedness of B is $(\lambda I - A)^{1/4+\epsilon}$;
- in the Dirichlet case the red method is "more difficult" since the unboundedness of B is $(-A)^{3/4+\epsilon}$.

Method and results

- We take the operator associated to the linear part of the HJB equation:

$$\mathcal{L}\psi(x) := \frac{1}{2} Tr\; QD^2\psi + \langle Ax, D\psi\rangle;$$

$\mathcal{L}$ generates (in the so-called π sense or $\mathcal{K}$ sense) the Ornstein-Uhlenbeck (O-U) semigroup defined as ($t \geq 0$)

$$R_t[\psi](x) = \mathbb{E}\left[\psi\left(e^{tA}x + \int_0^t e^{(t-s)A} dW(s)\right)\right]$$

$$= \int_{\mathcal{H}} \psi(e^{tA}x + y)\mathcal{N}_{Q_t}(dy)$$

where $\mathcal{N}_{Q_t}$ is the centered Gaussian measure in $\mathcal{H}$ with covariance

$$Q_t := \int_0^t e^{sA} Q e^{sA^*} ds.$$

which we assume to be nuclear from now on. Ok if $Q = I$.

- We use R_t to rewrite the HJB equation in the usual mild form:

$$v(t,x) = R_{T-t}[\varphi](x) + \int_t^T R_{s-t}[H_{min}(Dv(s,\cdot)) + g(s,\cdot)](x)\, ds \tag{21}$$

$$= R_{T-t}[\varphi](x) + \int_t^T R_{s-t}[\overline{H}_{min}(B^* Dv(s,\cdot)) + g(s,\cdot)](x)\, ds,$$

$(t \in [0,T]$, $x \in \mathcal{H})$.

- To solve this equation with a fixed point theorem we need, at least for any $\phi \in C_b(\mathcal{H})$ and $t > 0$, two things:
  - $B^* D R_t[\phi]$ is well defined;
  - $\|B^* D R_t[\phi]\|_\infty \leq \gamma(t) \|\phi\|_\infty$, where $\gamma \in L^1(0,T)$.

- The second works (up to now) only in the Neumann case.

Statements of results

Theorem 1 *Let $\phi \in C_b(\mathcal{H})$ and $t > 0$. Then $R_t[\phi] \in C_b^1(\mathcal{H})$ and we have, for every $h \in \mathbb{R}^2$,*

$$\langle B^* D(R_t\,[\phi])(x), h\rangle_{\mathbb{R}^2} = \tag{22}$$

$$\int_{\mathcal{H}} \phi\left(z + e^{tA}x\right) \langle (Q_t)^{-1/2}(\eta - A)^{\beta} e^{tA} Ch, (Q_t)^{-1/2} z\rangle \mathcal{N}(0, Q_t)(dz).$$

where (Dirichlet case) $\eta = 0$ and $\beta = \frac{3}{4} + \epsilon$ or (Neumann case), $\eta = \lambda$ and $\beta = \frac{1}{4} + \epsilon$.

Moreover for every $h \in \mathbb{R}^2$ we have the estimate

$$|B^* D(R_t\,[\phi])(x)| \leq \left\| (Q_t)^{-1/2} A^{\beta} e^{tA} C \right\| .\|\phi\|_\infty$$

hence, if $Q = Identity$,

$$|B^* D R_t\,[\phi]\,(x)| \leq C t^{-\frac{1}{2} - \beta} \|\phi\|_\infty. \tag{23}$$

Modification of a known theorem of [Da Prato - Zabczyk 91].

Note 1: control theoretic interpretation of this fact using null controllability.

Note 2: We see that the singularity in the last estimate is integrable only in the Neumann case. Even with different choices of Q (such that Q_t is nuclear) the Dirichlet case does not have integrable singularity in the above result.

Theorem 1 allows to solve satisfactorily the control problem in the Neumann case.

Now we go to Dirichlet case.

EXTENSION TO THE DIRICHLET CASE.

In the Dirichlet case we can do better. Indeed, writing $B = (-A)^{\beta} C$ we look only at the unboundedness generated by $(-A)^{\beta}$. **Indeed B is simpler since its image is two dimensional (but in a space of distributions larger than $\mathcal{H}$).**

This means that, heuristically, $B^* Dv$ is done by two directional derivatives, but in "unbounded" directions. Hence we substitute the term $(-A)^{\beta} e^{tA} C$ in (22) with $-Ae^{tA}D$ and we do all computations with the Fourier coefficients.

We get the same statement of Theorem 1 but with the following estimate instead of (23)

$$|B^* D R_t [\phi] (x)| \leq C t^{-1} \|\phi\|_{\infty}. \tag{24}$$

This is not yet enough but,

if we ask that the data φ and g are continuous with respect to a slightly weaker topology (e.g. $\varphi(x) = \varphi_0((-A)^\epsilon x)$ for some $\epsilon > 0$ and continuous φ_0 and similarly for g),

then, for these data, the singularity becomes integrable and the HJB equation can be solved using ideas from a recent paper [G.-Masiero] (in a different context).

Results on the HJB equation

For $\alpha \in (0,1)$ and $T>0$, $C^{0,1}_{\alpha,B}([0,T]\times\mathcal{H})$ is the space of functions $f \in C_b([0,T]\times\mathcal{H}) \cap C^{0,1}([0,T)\times\mathcal{H})$ such that for all $t\in(0,T]$, $x\in\mathcal{H}$ the map

$$(t,x) \mapsto (T-t)^\alpha B^* Df(t,x)$$

is bounded and continuous as a map from $[0,T)\times\mathcal{H}$ with values in $\mathcal{H}^*$. It is a Banach space endowed with the norm

$$\|f\|_{C^{0,1}_{\alpha,B}([0,T]\times\mathcal{H})} = \sup_{(t,x)\in[0,T]\times\mathcal{H}} |f(t,x)| + \sup_{(t,x)\in[0,T)\times\mathcal{H}} (T-t)^\alpha \left\|B^*Df\,(t,x)\right\|_{\mathbb{R}}.$$

Definition 1 *We say that a function $v:[0,T]\times\mathcal{H}\to\mathbb{R}$ is a mild solution of the HJB equation (19) if the following are satisfied:*

1. $v \in C^{0,1}_{\alpha,B}([0,T]\times\mathcal{H},\mathbb{R})$ *for some* $\alpha\in(0,1)$*;*
2. *the integral equality (21) holds.*

Theorem 2 *Assume that* $\overline{H}_{min}(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ *is Lipschitz continuous: there exists a constant* $K > 0$ *such that* $\forall\, p_1,\, p_2, \in \mathbb{R}$

$$|\overline{H}_{min}(p_1) - \overline{H}_{min}(p_2)| \leq K\left(|p_1 - p_2|\right).$$

Then the HJB equation (19) admits a unique mild solution according to Definition 1.

Proof Apply the contraction mapping principle using the smoothing property of R_t and careful estimates on the integral convolution term.

Then prove uniqueness using a Gronwall-like estimate. ∎

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Verification Theorem

and

Optimal Feedback Controls

The above Theorem 2 allows to solve the optimal control problem. But some nontrivial work is needed.

- First we prove that the mild solution can be seen as the limit of classical solutions. The limit cannot be taken uniform, so we use the so-called π or $\mathcal{K}$-convergence.

- Second we need apply Ito formula to the approximating solutions and to pass to the limit to prove the Verification Theorem (sufficient condition for optimality) through the so-called Fundamental Identity.

- Third we need to solve the Closed Loop Equation (CLE), and for this we need to prove some further regularity of the solutions under some further assumptions.

Theorem 3 (Verification) *Assume the same as Theorem 2. Let* v *be the only mild solution of the HJB equation. Fix* $(t,x) \in [0,T] \times \mathcal{H}$*.*

Then, for every admissible control $a(\cdot)$*, we have the fundamental identity*

$$v(t,x) = J(t,x;a(\cdot)) -$$

$$-\mathbb{E} \int_t^T \left[\overline{H}_{CV}(B^* Dv(s,Y(s)); a(s)) - \overline{H}_{min}(B^* Dv(s,Y(s))) \right] ds$$

Hence if $\bar{a}(\cdot)$ *is admissible and, for all* $s \in [t,T]$*, we have*

$$\bar{a}(s) \in \arg\min_{a \in \Lambda} \overline{H}_{CV}(B^* Dv(s,Y(s)); a)$$

then $\bar{a}(\cdot)$ *is optimal and, in such case (recall that* V *is the value function,*

$$v(t,x) = V(t,x).$$

Theorem 4 (Optimal Feedbacks) *Assume the same as Theorem 2. Let v be the only mild solution of the HJB equation.*

Assume moreover that $\overline{H}_{min}$ has Lipschitz continuous derivative and that there exists a Lipschitz continuous selection θ of the map

$$p \mapsto \arg\min_{a\in\Lambda} \overline{H}_{CV}(p;a)$$

Fix $(t,x) \in [0,T] \times \mathcal{H}$.

Then, the closed loop equation

$$\begin{cases} dY(s) = AY(s)dt + B\theta(B^*Dv(s,Y(s)))dt + GdW(s), & s \in [0,T] \\ Y(0) = x, \end{cases} \tag{25}$$

has a unique solution Y^*. Setting

$$a^*(s) = \theta(B^*Dv(s,Y^*(s)))$$

the couple $(a^*(\cdot), Y^*(\cdot))$ is optimal.

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Extensions and future directions of research

Deterministic boundary control

Extend the method of convex regularization beyond the case treated in [Faggian-Gozzi 10].

Use the new regularity results in the stochastic case to get results on the deterministic control problems (see e.g. [Cannarsa-Da Prato 91]

Extend regularity results for viscosity solutions like the ones of [Federico-Goldys-FG '11, Federico-Tacconi '13, Rosestolato-Swiech '16])

Neumann boundary control and noise

This case has been treated in [Debussche-Fuhrman-Tessitore 07] using the BSDE approach.

Using the same ideas we exposed for the Dirichlet case it seems that we can show that $B^* D R_t \phi$ is well defined and has integrable singularity, hence allowing to solve the HJB equation.

The advantage with respect to this paper is the fact the less regular data can be treated.

Other Neumann extensions

- Control of Reaction - Diffusion equations when $f \neq 0$ using results of [Cerrai 99] to prove the analogous of Theorem 1.

- Control of Burgers (Navier-Stokes) equations using results of [Da Prato - Debussche 98-99] to prove the analogous of Theorem 1.

- Case when the diffusion coefficient depend on the state variable and is invertible.

In all this cases one proves smoothing properties generalizing the so called Bismut-Elworthy-Li formula. Case $n = 1$ ok. Case $n > 1$ to be worked out.

Other Dirichlet extensions

Dirichlet problem with boundary noise can be treated in weighted L^2 spaces.

Our approach seems possible to extend to this setting.

Other extensions

- Control in subdomains.
- Models with age structure or delay (first results in [G.-Masiero 15]).
- Maximum principle (on the line e.g. of what done in [Fuhrman-Hu-Tessitore]).
- Regularization of viscosity solutions (on the line of what is done in [Rosestolato-Swiech 15] in a special case with delay equations).
- Boundary control in the path dependent case using recent results on Path dependent PDE's (see e.g. [Ekren-Touzi-Zhang '13], [Flandoli-Zanco '13] and, in infinite dimension, our case, the paper [Cosso-Federico-FG-Rosestolato-Touzi '15].

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