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Optimal Control for Evolutionary PDEs and Related Topics

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On a nonstandard nonlocal phase field system: well-posedness and optimal control

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1 - The Cahn-Hilliard type system

The starting point:

- Model by Paolo Podio-Guidugli: Ricerche Mat. 2006
- Suitable modifications of the above model lead to the following Cahn-Hilliard type system with unknowns μ and ρ

$$(1+2g(\rho))\partial_t\mu + \mu\partial_t g(\rho) - \Delta\mu = 0 \quad \text{and} \quad \mu \geq 0$$
$$\partial_t\rho - \Delta\rho + f'(\rho) = \mu g'(\rho) \quad \text{with} \quad \rho \in \mathit{dom}\, f' \,.$$
$$+ \text{ (Neumann) BC } + \text{ IC for both } \mu \text{ and } \rho$$

The equations are meant in $\Omega \times (0, T)$, where Ω is a $3D$ domain; f is a double-well potential (roughly a W-shaped function) and $g : \mathbb{R} \to \mathbb{R}$ is nonnegative on $\mathit{dom}\, f$

2 - The Cahn-Hilliard type system (cont'd)

Typical double-well potentials with bounded domains

- The logarithmic double-well potential (where $C > 1$)

$$f(\rho) = \int_0^\rho \big(\ln(1+r) - \ln(1-r)\big)\, dr - C\rho^2\,, \quad \rho \in [-1,1]$$

- The double obstacle potential (where $C > 0$)

$$f(\rho) = I_S(\rho) - C\rho^2, \quad \rho \in \mathbb{R}, \quad \text{with (say)} \quad S = [-1,1]$$

where I_S is the indicator function of S, i.e.

$$f(s) = 0 \quad \text{if } s \in S \quad \text{and} \quad f(s) = +\infty \quad \text{if } s \notin S.$$

In this case, $f'(\rho)$ is replaced by $\partial I(\rho) - 2C\rho$ and the equation becomes a differential inclusion.

3 - The Cahn-Hilliard type system (cont'd)

Several papers by

- C Pierluigi Colli (Pavia)
- G myself
- K Pavel Krejčí (Prague)
- P Paolo Podio-Guidugli (Roma2)
- S Jürgen Sprekels (Berlin)

in some combinations (always containing CGS)
on a variety of problems

- generalizations in several directions
  (more general potentials, more general constituitive laws)
- long time behavior
- asymptotics with respect to some parameter
- time discretization
- optimal control problems

4 - Plan of the talk

The more recent cooperation with P. Colli and J. Sprekels led to three papers on

a nonlocal variant of the system

- CGS1 on well-posedness

  J. Differential Equations 2016

- CGS2 on a regular optimal control problem

  manuscript pp. 1-45

- CGS3 on a singular optimal control problem

  manuscript pp. 1-32

Next pages

- The nonlocal problem and few words on CGS1
- Some detail on CGS2 and CGS3

5 - The nonlocal problem

Recall

$$\begin{aligned}
&(1) \qquad (1+2g(\rho))\partial_t\mu + \mu\partial_t g(\rho) - \Delta\mu = 0 \quad \text{and} \quad \mu \geq 0 \\
&(2) \qquad \partial_t\rho - \Delta\rho + f'(\rho) = \mu g'(\rho) \quad \text{with} \quad \rho \in dom\, f' \\
&\qquad\qquad + \text{ BC for both } \mu \text{ and } \rho + \text{ IC for both } \mu \text{ and } \rho
\end{aligned}$$

We replace $-\Delta\rho$ in (2) by a different term $B[\rho]$
that does not involve space derivatives and is non-local, in general.
The new system reads

$$\begin{aligned}
&(1+2g(\rho))\partial_t\mu + \mu\partial_t g(\rho) - \Delta\mu = 0 \\
&\partial_t\rho + B[\rho] + f'(\rho) = \mu g'(\rho) \\
&\quad + \text{ BC for } \mu, \text{ only, } + \text{ IC for both } \mu \text{ and } \rho
\end{aligned}$$

6 - Partial lists of references

- Nonlocal versions of the Cahn-Hilliard equation (or related equations)

Giacomin-Lebowitz 1997-'98
Gajewski 2002
Gajewski-Zacharias 2003
Han 2004
Bates-Han 2005
Gajewski-Griepentrog 2006
Colli-Krejčí-Rocca-Sprekels 2007
Londen-Petzeltová 2011
Frigeri-Grasselli 2012
Gal-Grasselli 2014
Abels-Bosia-Grasselli 2015

7 - Partial lists of references (cont'd)

- Control problems for systems related to Cahn-Hilliard

Hintermüller-Wegner 2012, 2 papers 2014
Zhao-Liu 2 papers 2013, 1 paper 2014
Colli-Farshbaf Shaker-Gilardi-Sprekels 2 papers 2015
Rocca-Sprekels 2015
Colli-Gilardi-Sprekels 2016

Colli-Gilardi-Sprekels to appear
Colli-Gilardi-Rocca-Sprekels in preparation (submitted)
Frigeri-Rocca-Sprekels to appear

8 - The non local operator

The operator B acts as follows:
if ρ is a function defined in $Q := \Omega \times (0, T)$, then $B[\rho]$ is a function defined in the same set Q, in order that

B is causal, i.e., for every $x \in \Omega$ and $t \in (0, T]$ it holds:

$B[\rho](x, t)$ is known whenever

$\rho(y, s)$ is known for every $y \in \Omega$ and every $s \in [0, t]$

that is, for every $t \in (0, T)$, setting $Q_t := \Omega \times (0, t)$, we have

$$\rho_1 = \rho_2 \text{ on } Q_t \quad \text{implies} \quad B[\rho_1] = B[\rho_2] \text{ on } Q_t$$

Moreover, B satisfies a number of properties. For instance

B maps $L^2(Q)$ into itself and is Lipschitz continuous

A typical example is an integral operator like

$$B[\rho](x, t) = \int_0^t \int_\Omega K\big(x, y, t, s, \rho(y, s)\big)\, dy\, ds$$

with a smooth $K : \Omega^2 \times \{(t, s) : 0 \le s \le t \le T\} \times \mathbb{R} \to \mathbb{R}$

9 - The non local operator (cont'd)

Another example is given by the non-local-in-space operator

$$B[\rho](x,t) = \int_\Omega K\big(x, y, \rho(y,t)\big)\, dy$$

where $K : \Omega^2 \times \mathbb{R} \to \mathbb{R}$ is smooth as needed. For instance

$$B[\rho](x,t) = \int_\Omega k(|x-y|)\rho(y,t)\, dy$$

for a given $k : [0, +\infty) \to \mathbb{R}$.

- In the latter case, sufficient conditions on k for our theory are

$$\begin{gathered} k \text{ is } C^1 \text{ on } (0,+\infty) \text{ and satisfies} \\ |k(r)| \le C r^{-\alpha} \quad \text{and} \quad |k'(r)| \le C r^{-\beta} \\ \text{with} \quad \alpha < 3/2 \quad \text{and} \quad \beta < 5/2. \end{gathered}$$

In particular, the Newtonian potential $k(r) = C/r$ is allowed.

10 - Our well-posedness result

Recall the system

$$(1) \qquad (1 + 2g(\rho))\partial_t \mu + \mu \partial_t g(\rho) - \Delta\mu = 0 \quad \text{with} \quad \mu \geq 0$$

$$(2) \qquad \partial_t \rho + B[\rho] + f'(\rho) = \mu g'(\rho) \quad \text{with} \quad \rho \in dom\, f'$$

$$+ \text{ Neumann BC for } \mu + \text{ IC for } \mu \text{ and } \rho$$

Some precise/rough assumptions (allowing f_{log} and f_{2obs}):

- $f : \mathbb{R} \to (-\infty, +\infty]$ is split as $f = f_{conv} + f_{pert}$ where f_{conv} is convex, proper, l.s.c. and $f_{conv}(0) = \min f_{conv}$ the perturbation f_{pert} is smooth enough
- $g : \overline{dom\, f} \to \mathbb{R}$ is nonnegative, concave and smooth enough
- B satisfies proper assumptions and L^p type estimates
- the initial data μ_0 and ρ_0 are smooth with $\mu_0 \geq 0$.

Theorem. *Read (2) as a differential inclusion with f' replaced by $\partial f_{conv} + f'_{pert}$ if necessary. Then, there exists a unique solution (μ, ρ) in a proper space.*

11 - Method for existence

Recall the system ($\mu \geq 0$ and BC and IC to be added)

$$(1) \qquad (1+2g(\rho))\partial_t\mu + \mu\partial_t g(\rho) - \Delta\mu = 0$$
$$(2) \qquad \partial_t\rho + B[\rho] + f'(\rho) = \mu g'(\rho)$$

- Construct the maps $\mathcal{F}_i$ (with proper domains) as follows

  for a given μ, $\mathcal{F}_1(\mu)$ is the solution ρ to (2)

  for a given ρ, $\mathcal{F}_2(\rho)$ is the solution μ to (1)

- Look for a fixed point of the map
$$\mathcal{F} := \mathcal{F}_2 \circ \mathcal{F}_1 : \quad \mu_{given} \longmapsto \mu_{constructed}$$
- Choose *dom* $\mathcal{F}$ carefully and use the Tikhonov theorem
- Our choice:

$$\mathcal{V} = L^{10/3}(Q) \cap L^2(0,T;H^1(\Omega)) \quad \text{with its weak topology}$$
$$dom\,\mathcal{F} = \{v \in \mathcal{V} : \ \|v\|_{\mathcal{V}} \leq C, \ v \geq 0\} \quad \text{suitable } C$$

12 - The distributed control problem

The modified system

$$\begin{array}{ll} (1) & (1+2g(\rho))\partial_t\mu + \mu\partial_t g(\rho) - \Delta\mu = \overset{\substack{\text{control}\\ \downarrow}}{u} \quad \text{with} \quad \mu \geq 0 \\ (2) & \partial_t\rho + B[\rho] + f'(\rho) = \mu g'(\rho) \quad \text{with} \quad \rho \in dom\, f' \\ (3) & \quad + \text{ Neumann BC for } \mu + \text{ IC for } \mu \text{ and } \rho \end{array}$$

The control box, where $\mathcal{V} := H^1(0,T;L^2(\Omega))$

$$\mathcal{U}_{ad} = \{v \in \mathcal{V} : \ 0 \leq u \leq u_{max}, \ \|v\|_{\mathcal{V}} \leq R\}$$

The cost functional, where $Q := \Omega \times (0,T)$

$$J\big(u,(\mu,\rho)\big) := \frac{\beta_1}{2}\int_Q |\rho-\rho_Q|^2 + \frac{\beta_2}{2}\int_Q |\mu-\mu_Q|^2 + \frac{\beta_3}{2}\int_Q |u|^2$$

The control problem

$$\min\left\{J\big(u,(\mu,\rho)\big) : \ u \in \mathcal{U}_{ad}, \ (u,\mu,\rho) \text{ satisfies (1)-(3)}\right\}$$

13 - The regular case

Less general framework with rispect to well-posedness. Namely

- just smooth logarithmic type potentials, i.e.,

$$f : (-1,1) \to \mathbb{R} \quad \text{is } C^3, \quad f'(0) = 0, \quad f'' \geq -M$$
$$\lim_{\rho \to -1} f'(\rho) = -\infty \quad \text{and} \quad \lim_{\rho \to +1} f'(\rho) = +\infty$$

- g and initial data as before, i.e.,

$g : [-1,1] \to \mathbb{R}$ is nonnegative, concave and smooth

the initial data μ_0 and ρ_0 are smooth with $\mu_0 \geq 0$

- B satisfies proper L^p type estimates as before; moreover

$B : L^2(Q) \to L^2(Q)$ is Fréchet differentiable

its Fréchet derivative $DB : L^2(Q) \to \mathcal{L}(L^2(Q); L^2(Q))$

satisfies proper L^p type estimates

14 - The regular case (cont'd)

Steps for the control problem:

- Well-posedness and construction of
  a) the control-to-state mapping $\mathcal{S} : u \mapsto (\mu, \rho)$
  b) the reduced cost functional $u \mapsto j(u) := J\big(u, \mathcal{S}(u)\big)$
- Existence of an optimal control $\overline{u}$
- NC (necessary conditions) for $\overline{u}$ to be an optimal control:
  this involves the directional derivative $\delta j(\overline{u}; h)$ of j
  in the directions $h := v - \overline{u}$ with $v \in \mathcal{U}_{ad}$
- Hence, we need directional differentiability of j, i.e.,
  at least directional differentiability of the map $\mathcal{S}$:
  this leads to a linearized system $LS(h)$ (to be solved!!!)
  whose solution $\delta\mathcal{S}(\overline{u}; h)$ is the derivative in the direction h
- For the above NC one has to solve $LS(h)$
  with infinitely many values of h ($h = v - \overline{u}$ with $v \in \mathcal{U}_{ad}$)
- Hence, one more step: elimination of such solutions in NC;
  this leads to an adjoint problem (to be solved!!!)

15 - The regular case (cont'd)

The control-to-state map $\mathcal{S}$. Recall

$$
\begin{array}{lll}
(1) & (1+2g(\rho))\partial_t\mu + \mu\partial_t g(\rho) - \Delta\mu = \overset{\text{control}}{\overset{\downarrow}{u}} & \text{with} \quad \mu \geq 0 \\
(2) & \partial_t\rho + B[\rho] + f'(\rho) = \mu g'(\rho) & \text{with} \quad \rho \in dom\, f' \\
(3) & \quad + \text{ Neumann BC for } \mu + \text{ IC for } \mu \text{ and } \rho &
\end{array}
$$

- Well-posedness result: the same as before (there was $u = 0$), with a remark: $u \in \mathcal{U}_{ad} \Rightarrow u \geq 0 \Rightarrow \mu \geq 0$ as required
- We need to choose Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ in order that

$$\mathcal{U}_{ad} \subset \mathcal{X} \quad \text{and} \quad \mathcal{S} : u \mapsto (\mu, \rho) \text{ maps } \mathcal{U}_{ad} \text{ into } \mathcal{Y}$$

- A good choice is (for differentiability)

$$
\begin{aligned}
&\mathcal{X} := H^1(0,T;H) \cap L^\infty(Q) \\
&\mathcal{Y} := \big(L^\infty(0,T;H) \cap L^2(0,T;V)\big) \times H^1(0,T;H) \\
&\text{where} \quad H := L^2(\Omega) \quad \text{and} \quad V := H^1(\Omega)
\end{aligned}
$$

16 - The linearized problem

For a given $\overline{u} \in \mathcal{U}_{ad}$, the pair $(\mu, \rho) := \mathcal{S}(\overline{u})$ solves

$$(1 + 2g(\overline{\rho}))\partial_t\overline{\mu} + \overline{\mu}g'(\overline{\rho})\partial_t\overline{\rho} - \Delta\overline{\mu} = \overline{u} \quad \text{with} \quad \overline{\mu} \geq 0$$
$$\partial_t\overline{\rho} + B[\overline{\rho}] + f'(\overline{\rho}) = \overline{\mu}g'(\overline{\rho}) \quad \text{with} \quad \overline{\rho} \in dom\, f'$$
$$\partial_n\overline{\mu} = 0, \quad \overline{\rho}(0) = \rho_0 \quad \text{and} \quad \overline{\mu}(0) = \mu_0$$

Given h, the linearized system $LS(h)$ reads: find (η, ξ) satisfying

$$(1 + 2g(\overline{\rho}))\,\partial_t\eta + 2g'(\overline{\rho})\,\partial_t\overline{\mu}\,\xi + g'(\overline{\rho})\,\partial_t\overline{\rho}\,\eta$$
$$+\,\overline{\mu}\,g''(\overline{\rho})\,\partial_t\overline{\rho}\,\xi + \overline{\mu}g'(\overline{\rho})\,\partial_t\xi - \Delta\eta = h$$
$$\partial_t\xi + f''(\overline{\rho})\,\xi + DB[\overline{\rho}](\xi) = \overline{\mu}\,g''(\overline{\rho})\,\xi + g'(\overline{\rho})\,\eta$$
$$\partial_n\eta = 0 \quad \text{and} \quad \eta(0) = \xi(0) = 0$$

Theorem. *For every $h \in \mathcal{X}$, system $LS(h)$ has a unique solution (η, ξ) in a proper space $\mathcal{Z} \subset \mathcal{Y}$, and, if $\overline{u} + \lambda h \in \mathcal{U}_{ad}$ $\forall\lambda \in [0, 1]$, then (η, ξ) is the directional derivative $\delta\mathcal{S}(\overline{u}; h) \in \mathcal{Y}$, i.e.,*

$$\|\mathcal{S}(\overline{u} + \lambda h) - \mathcal{S}(\overline{u}) - \lambda(\eta, \xi)\|_{\mathcal{Y}} = o(\lambda) \quad \text{as } \lambda \to 0^+.$$

17 - The linearized problem (cont'd)

Method for existence

- construct $LS_\varepsilon(h)$ approximation of $LS(h)$ by smoothing $(\overline{\mu}, \overline{\rho})$
- present $LS_\varepsilon(h)$ as a Cauchy problem for $y' + A_\varepsilon y = F$ with a nonlocal A_ε and apply an ad hoc result by Baiocchi (1967)
- a priori estimates
- limit as $\varepsilon \to 0$

Remark: the same strategy is used for the adjoint problem we introduce later on

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Method for directional differentiability

- the natural one, i.e., Taylor expansion of the nonlinearities and heavy estimates on $\mathcal{S}(\overline{u} + \lambda h) - \mathcal{S}(\overline{u}) - \lambda(\eta, \xi)$

18 - Necessary conditions for optimality

Recall the cost functional (where $Q := \Omega \times (0, T)$)

$$J\big(u, (\mu, \rho)\big) := \frac{\beta_1}{2} \int_Q |\rho - \rho_Q|^2 + \frac{\beta_2}{2} \int_Q |\mu - \mu_Q|^2 + \frac{\beta_3}{2} \int_Q |u|^2$$

Theorem. *Let $\overline{u} \in \mathcal{U}_{ad}$ be an optimal control and $(\overline{\rho}, \overline{\mu}) = \mathcal{S}(\overline{u})$. Then, we have for every $v \in \mathcal{U}_{ad}$*

$$\beta_1 \int_Q (\overline{\rho} - \rho_Q)\, \xi + \beta_2 \int_Q (\overline{\mu} - \mu_Q)\, \eta + \beta_3 \int_Q \overline{u}\, (v - \overline{u}) \geq 0$$

where $(\eta, \xi) = \delta\mathcal{S}(\overline{u}, v - \overline{u})$, the unique solution to $LS(v - \overline{u})$.

Next step: eliminate the solution (η, ξ) to $LS(v - \overline{u})$.

- Usual tool: find the proper adjoint problem

19 - The adjoint problem

The adjoint problem AP reads:
find (p, q) solving the backward problem

$$
\begin{aligned}
&-(1+2g(\overline{\rho}))\,\partial_t p - g'(\overline{\rho})\,\partial_t\overline{\rho}\,p - \Delta p - g'(\overline{\rho})\,q \\
&\qquad = \beta_2(\overline{\mu} - \mu_Q) \\
&-\partial_t q + f''(\overline{\rho})\,q - \overline{\mu}\,g''(\overline{\rho})\,q \\
&\qquad + g'(\overline{\rho})\big(\partial_t\overline{\mu}\,p - \overline{\mu}\,\partial_t p\big) + DB[\overline{\rho}]^*(q) \\
&\qquad = \beta_1(\overline{\rho} - \rho_Q) \\
&\partial_n p = 0 \quad \text{and} \quad p(T) = q(T) = 0
\end{aligned}
$$

where the adjoint operator $DB[\overline{\rho}]^*$ is backward-causal, i.e., $DB[\overline{\rho}]^*(q)$ known in $\Omega \times (t, T)$ whenever q known in $\Omega \times (t, T)$.

Theorem. *Let $\overline{u} \in \mathcal{U}_{ad}$ be an optimal control and $(\overline{\rho}, \overline{\mu}) = \mathcal{S}(\overline{u})$. Then, AP has a unique solution (p, q) in a proper space.*

20 - Necessary conditions for optimality revised

Recall the NC for optimality: for every $v \in \mathcal{U}_{ad}$, there holds

$$\beta_1 \int_Q (\overline{\rho} - \rho_Q)\, \xi + \beta_2 \int_Q (\overline{\mu} - \mu_Q)\, \eta + \beta_3 \int_Q \overline{u}\, (v - \overline{u}) \geq 0$$

where $(\eta, \xi) = \delta\mathcal{S}(\overline{u}, v - \overline{u})$, the unique solution to $LS(v - \overline{u})$.
On the other hand, AP has been constructed in order that $\beta_1 \int_Q (\overline{\rho} - \rho_Q)\, \xi + \beta_2 \int_Q (\overline{\mu} - \mu_Q)\, \eta = \int_Q p(v - \overline{u})$. Hence

Theorem. *Let $\overline{u} \in \mathcal{U}_{ad}$ be an optimal control with associated state $(\overline{\rho}, \overline{\mu}) = \mathcal{S}(\overline{u})$ and adjoint state (p, q). Then, $(\overline{u}, p)$ satisfies the variational inequality*

$$\int_Q (p + \beta_3\, \overline{u})(v - \overline{u}) \geq 0 \quad \textit{for every } v \in \mathcal{U}_{ad}.$$

In particular, if $\beta_3 > 0$, then

$$-\frac{p}{\beta_3} \ \textit{is the } L^2(Q) \textit{ orthogonal projection of } \overline{u} \textit{ onto } \mathcal{U}_{ad}.$$

21 - The singular control problem

Recall the state system ($\mu \geq 0$ and BC and IC to be added)

$$(1) \qquad (1 + 2g(\rho))\partial_t \mu + \mu \partial_t g(\rho) - \Delta \mu = u$$
$$(2) \qquad \partial_t \rho + B[\rho] + f'(\rho) = \mu g'(\rho)$$

where f was a logarithmic type potential.

New aim: take the same control box $\mathcal{U}_{ad}$ and the same cost functional J as before, but replace the logarithmic type potential f in (2) by a double obstacle potential

$$f(\rho) = I(\rho) + f_{pert}(\rho), \quad \rho \in \mathbb{R}$$

where I is the indicator function of $[-1, 1]$ and f_{pert} is a smooth perturbation.

Then, replace $f'(\rho)$ in (2) by $\partial I(\rho) + f'_{pert}(\rho)$, read the equation as a differential inclusion, and solve the control problem

$$\min \left\{ J\big(u, (\mu, \rho)\big) : \ u \in \mathcal{U}_{ad}, \ (u, \mu, \rho) \text{ satisfies (1)-(2)} \right\}$$

22 - The singular control problem (cont'd)

The state system reads ($\mu \geq 0$ and BC and IC to be added)

$$
\begin{aligned}
&(1) \qquad (1+2g(\rho))\partial_t\mu + \mu\partial_t g(\rho) - \Delta\mu = u \\
&(2) \qquad \partial_t\rho + B[\rho] + \partial I(\rho) + f'_{pert}(\rho) \ni \mu g'(\rho)
\end{aligned}
$$

The control-to-state map $\mathcal{S}$ is well-defined (well-posedness result) and next points are

- Show that the singular control problem has a solution
- Fix an optimal control $\overline{u}$
  and look for necessary optimality conditions

- **Trouble:** f is too singular and cannot be linearized; thus
- no linearized problem is available for optimality conditions
- no adjoint problem in the usual sense is available

23 - Method for the singular control problem

We use the strategy adopted in the papers

- Colli-FS-Sprekels, *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary conditions and double obstacles*, Appl. Math. Optim. 2015
- Colli-FS-GG-Sprekels, *Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials*, SIAM J. Control Optim. 2015

where FS = M.H. Farshbaf-Shaker (Berlin)

Method: Approximate the double-obstacle potential with potentials of logarithmic type

24 - Method for the singular control problem (cont'd)

Precisely

$$
\begin{array}{ll}
(1)_\alpha & (1+2g(\rho))\partial_t\mu + \mu\partial_t g(\rho) - \Delta\mu = u \\
(2)_\alpha & \partial_t\rho + B[\rho] + f'_\alpha(\rho) + f'_{pert}(\rho) = \mu g'(\rho)
\end{array}
$$

where f_α is a multiple of the logarithmic potential, e.g.,

$$f_\alpha(\rho) := \alpha \int_0^\rho \big(\ln(1+r) - \ln(1-r)\big)\, dr, \quad \alpha \in (0,1)$$

Thus, for any α

- the control-to-state mapping $\mathcal{S}_\alpha$ is well-defined
- the regularized control problem has at least a solution
- necessary optimality conditions are known

Theorem. *If u_α is an optimal control for the regularized problem and $(\mu_\alpha, \rho_\alpha) = \mathcal{S}_\alpha(u_\alpha)$, then $(u_\alpha, \mu_\alpha, \rho_\alpha)$ converge to some $(\overline{u}, \overline{\mu}, \overline{\rho})$ in a proper topology (for a subsequence). Moreover, $\overline{u}$ is an optimal control for the singular problem and $(\overline{\mu}, \overline{\rho}) = \mathcal{S}(\overline{u})$.*

25 - Necessary optimality conditions

Fix an optimal control $\overline{u}$ and look for optimality conditions.

Try to approximate $\overline{u}$ by optimal controls u_α for the α-problem: we are not able. Thus

- *Barbu's trick:* contruct a modified α-control problem:

with the same control box $\mathcal{U}_{ad}$, solve

$$(*) \qquad \min_{u \in \mathcal{U}_{ad}} \Big(J\big(u, \mathcal{S}_\alpha(u)\big) + \frac{1}{2} \, \|u - \overline{u}\|^2_{L^2(Q)} \Big)$$

Thus, the same α-system but an ad hoc cost functional

This trick yields: if $\alpha_n \to 0$, then there are a subsequence $\{\alpha_{n_k}\}$ and optimal controls $u_{\alpha_{n_k}}$ for $(*)$ (read with $\alpha = \alpha_{n_k}$) such that

$$u_{\alpha_{n_k}} \to \overline{u} \quad \text{strongly in } L^2(Q)$$
$$\mathcal{S}_{\alpha_{n_k}}(u_{\alpha_{n_k}}) \to \mathcal{S}(\overline{u}) \quad \text{in a proper topology}$$

This suggests to take the limit in the necessary conditions. Thus

26 - Necessary optimality conditions (cont'd)

Write the necessary optimality cond's for the modified control pbl. They are similar to the ones for the original α control pbl, namely

- same adjoint problem
- a slightly different variational inequality:
  $$\int_Q \big(p_\alpha + \beta_3\,\overline{u}_\alpha + (\overline{u}_\alpha - \overline{u})\big)(v - \overline{u}) \geq 0$$
  for every $v \in \mathcal{U}_{ad}$

As $\overline{u}_\alpha \to \overline{u}$ strongly in $L^2(Q)$ (along the subsequence), the latter yields in the limit

$$\int_Q (p + \beta_3\,\overline{u})(v - \overline{u}) \geq 0 \qquad \longleftarrow \text{very good}$$

provided that $p := \lim p_\alpha$ actually exists in $L^2(Q)$

27 - The generalized adjoint problem

The α-adjoint problem reads

$$
\begin{aligned}
&-(1+2g(\overline{\rho}_\alpha))\,\partial_t p_\alpha - g'(\overline{\rho}_\alpha)\,\partial_t\overline{\rho}_\alpha\, p_\alpha - \Delta p_\alpha - g'(\overline{\rho}_\alpha)\, q_\alpha \\
&\quad = \beta_2(\overline{\mu}_\alpha - \mu_Q) \\
&-\partial_t q_\alpha + f''_\alpha(\overline{\rho}_\alpha) q_\alpha + f''_{pert}(\overline{\rho}_\alpha)\, q_\alpha - \overline{\mu}_\alpha\, g''(\overline{\rho}_\alpha)\, q_\alpha \\
&\quad + g'(\overline{\rho}_\alpha)\big(\partial_t\overline{\mu}_\alpha\, p_\alpha - \overline{\mu}_\alpha\,\partial_t p_\alpha\big) + DB[\overline{\rho}_\alpha]^*(q_\alpha) \\
&\quad = \beta_1(\overline{\rho}_\alpha - \rho_Q) \\
&\partial_n p_\alpha = 0 \quad \text{and} \quad p_\alpha(T) = q_\alpha(T) = 0
\end{aligned}
$$

and one expects a limit system.
Thus, a priori estimates. Namely

$$
\begin{aligned}
&\|p_\alpha\|_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} \le C \\
&\|q_\alpha\|_{L^\infty(0,T;L^2(\Omega))} \le C \quad (\text{nothing more is expected})
\end{aligned}
$$

As $(\overline{\mu}_\alpha, \overline{\rho}_\alpha) \to (\overline{\mu}, \overline{\rho})$ in a rather good topology, this is sufficient to take the limit of every term but

$$
\partial_t q_\alpha \quad \text{and} \quad f''_\alpha(\overline{\rho}_\alpha) q_\alpha = \frac{\alpha}{\overline{\rho}_\alpha(1-\overline{\rho}_\alpha)} q_\alpha
$$

28 - The generalized adjoint problem (cont'd)

Recall

$$
\begin{aligned}
(1)\qquad & -\big(1+2g(\overline{\rho}_\alpha)\big)\,\partial_t p_\alpha - g'(\overline{\rho}_\alpha)\,\partial_t\overline{\rho}_\alpha\, p_\alpha - \Delta p_\alpha - g'(\overline{\rho}_\alpha)\, q_\alpha \\
& \quad = \beta_2(\overline{\mu}_\alpha - \mu_Q) \\
(2)\qquad & -\color{red}{\partial_t q_\alpha} + \color{green}{f''_\alpha(\overline{\rho}_\alpha) q_\alpha} + f''_{pert}(\overline{\rho}_\alpha)\, q_\alpha - \overline{\mu}_\alpha\, g''(\overline{\rho}_\alpha)\, q_\alpha \\
& \quad + g'(\overline{\rho}_\alpha)\big(\partial_t\overline{\mu}_\alpha\, p_\alpha - \overline{\mu}_\alpha\,\partial_t p_\alpha\big) + DB[\overline{\rho}_\alpha]^*(q_\alpha) \\
& \quad = \beta_1(\overline{\rho}_\alpha - \rho_Q) \\
& \partial_n p_\alpha = 0 \quad \text{and} \quad p_\alpha(T) = q_\alpha(T) = 0
\end{aligned}
$$

We have to find bounds for

$$
\color{red}{\partial_t q_\alpha} \quad \text{and} \quad \color{green}{f''_\alpha(\overline{\rho}_\alpha) q_\alpha}
$$

Remark: estimate for ■ yields estimate for ■ by comparison in (2)

29 - The generalized adjoint problem (cont'd)

Let $v \in H^1(0,T;L^2(\Omega))$. Then

$$\int_Q \partial_t q_\alpha \, v = -\int_Q q_\alpha \, \partial_t v + \int_\Omega \big(q_\alpha(T)v(T) - q_\alpha(0)v(0)\big)$$
$$= -\int_Q q_\alpha \, \partial_t v - \int_\Omega q_\alpha(0)v(0) \qquad \text{since } q_\alpha(T) = 0$$

Thus, if $v(0) = 0$, we obtain

$$\Big|\int_Q \partial_t q_\alpha \, v\Big| = \Big|\int_Q q_\alpha \, \partial_t v\Big| \le \|q_\alpha\|_{L^2(Q)} \|\partial_t v\|_{L^2(Q)}$$

By the estimate for q_α already obtained and the comparison argument for $f''_\alpha(\overline{\rho}_\alpha)$ we deduce that both $\partial_t q_\alpha$ and $f''_\alpha(\overline{\rho}_\alpha)$ are bounded in the dual space Y^* of

$$Y := \big\{ v \in H^1(0,T;L^2(\Omega)) : \; v(0) = 0 \big\}$$

Thus, $\partial_t q_\alpha \to \partial_t q$ and $f''_\alpha(\overline{\rho}_\alpha) q_\alpha \to$ something weakly in Y^*

30 - The generalized adjoint problem (cont'd)

Theorem. *For any sequence $\{\alpha_n\} \to 0$, there exist*

- *a subsequence $\{\alpha_{n_k}\}$*
- *corresponding optimal controls and states for the modified problem $(\overline{u}_\alpha, \overline{\mu}_\alpha, \overline{\rho}_\alpha, p_\alpha, q_\alpha)$ (with $\alpha = \alpha_{n_k}$)*
- *a triple (p, q, λ) such that:*

i) $(\overline{u}_\alpha, \overline{\mu}_\alpha, \overline{\rho}_\alpha, p_\alpha, q_\alpha, f''_\alpha(\overline{\rho}_\alpha) q_\alpha)$ *converges to* $(\overline{u}, \overline{\mu}, \overline{\rho}, p, q, \lambda)$ *in a proper topology;*

ii) (p, q, λ) *satisfies the following variational inequality and generalized adjoint problem*

- $\displaystyle\int_Q (p + \beta_3\, \overline{u})(v - \overline{u}) \geq 0$ *for every* $v \in \mathcal{U}_{ad}$
- $p \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$ *and*
  $-(1 + 2g(\overline{\rho}))\, \partial_t p - g'(\overline{\rho})\, \partial_t \overline{\rho}\, p - \Delta p - g'(\overline{\rho})\, q$
  $= \beta_2(\overline{\mu}_\alpha - \mu_Q)$ *with* $\partial_n p = 0$ *and* $p(T) = 0$
- 2^{nd} *equation and* $q(T) = 0$ *in a generalized sense.*

31 - The generalized adjoint problem (cont'd)

Detail. A part of the α-AP is

$$-\partial_t q_\alpha + f''_\alpha(\overline{\rho}_\alpha) q_\alpha + w_\alpha = \beta_1(\overline{\rho} - \rho_Q) \quad \text{and} \quad q_\alpha(T) = 0$$

where w_α stands for

$$(*) \qquad \underbrace{f''_{pert}(\overline{\rho})\, q - \overline{\mu}\, g''(\overline{\rho})\, q + g'(\overline{\rho})\big(\partial_t \overline{\mu}\, p - \overline{\mu}\, \partial_t p\big) + DB[\overline{\rho}]^*(q)}_{everything\ \ with\ \ index\ \ \alpha}$$

Now, we recall

$$Y := \big\{ v \in H^1(0,T; L^2(\Omega)) : \ v(0) = 0 \big\}$$

Then, we have in the limit

$$q \in L^\infty(0,T; L^2(\Omega)) \quad \text{and} \quad \lambda \in Y^*$$

$$\int_Q q\, \partial_t v + {}_{Y^*}\langle \lambda, v \rangle_Y + \int_Q w\, v = \beta_1 \int_Q (\overline{\rho}_\alpha - \rho_Q) v$$

for every $v \in Y$

where w is $(*)$ written as it is, i.e., without indices.

Thank you for your attention

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