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# Equation and dynamic boundary condition of Cahn–Hilliard type

Takeshi FUKAO

Kyoto University of Education

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# 1. Motivation

- Dynamic b.c. (boundary condition) and conservation

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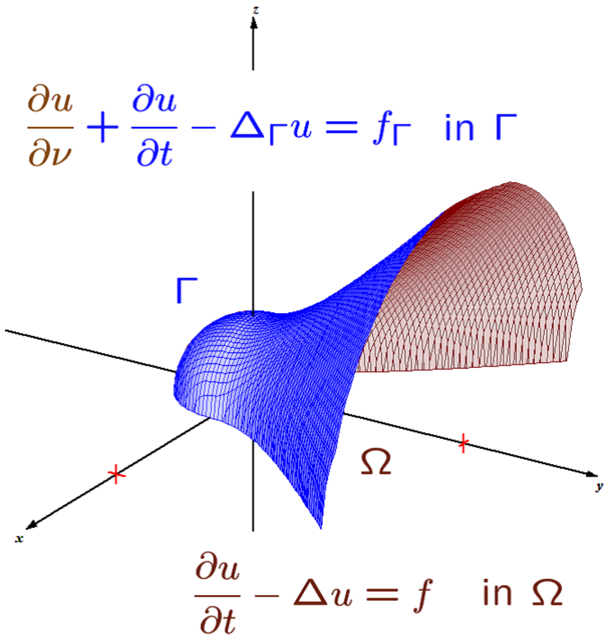
$\Gamma := \partial\Omega$: smooth boundary,
$\boldsymbol{\nu}$: unit normal vector outward from $\Gamma$.




$$\partial_{\boldsymbol{\nu}} u := \frac{\partial u}{\partial \boldsymbol{\nu}} = f_\Gamma \quad \text{in } \Gamma \quad \text{(Neumann b.c.)}$$

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Under the dynamic b.c. for parabolic PDEs (ex. heat eq.), we easily find the structure of some conservation with $f = f_\Gamma \equiv 0$:

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{in } Q := (0, T) \times \Omega, \qquad u(0) = u_0 \quad \text{on } \Omega.$$

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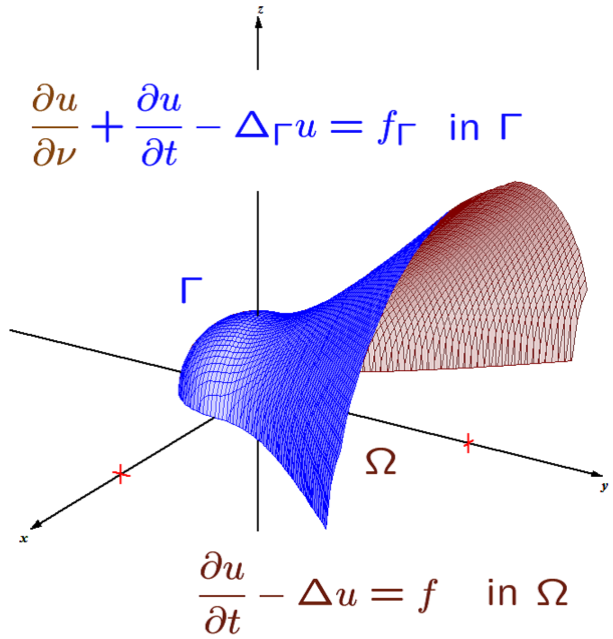


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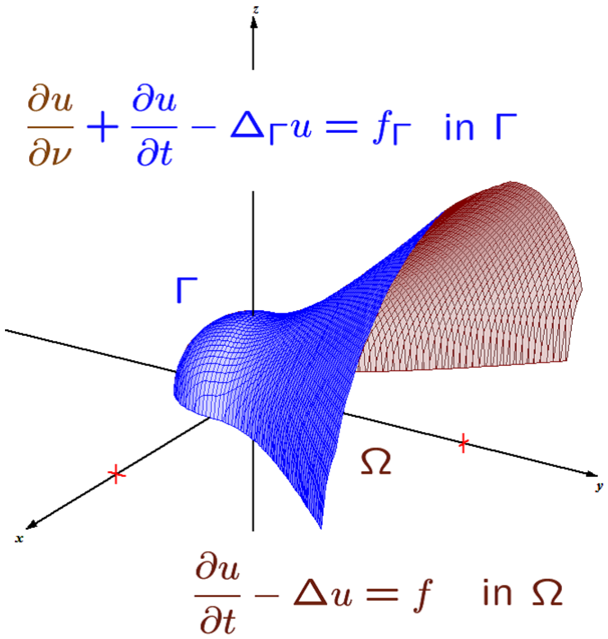


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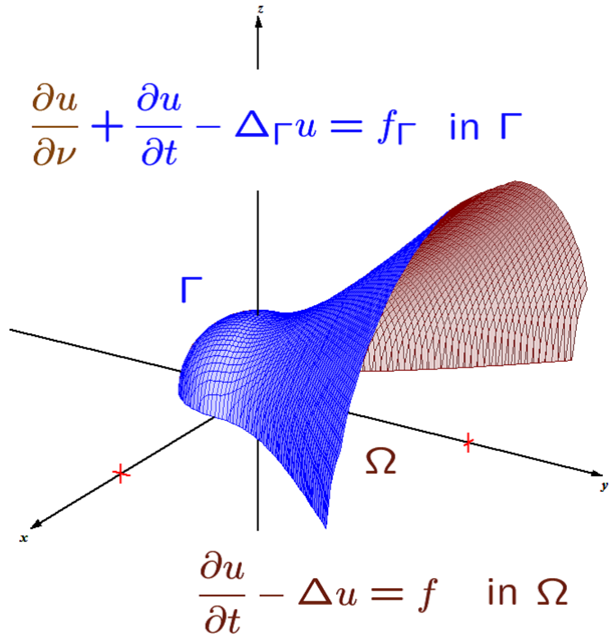


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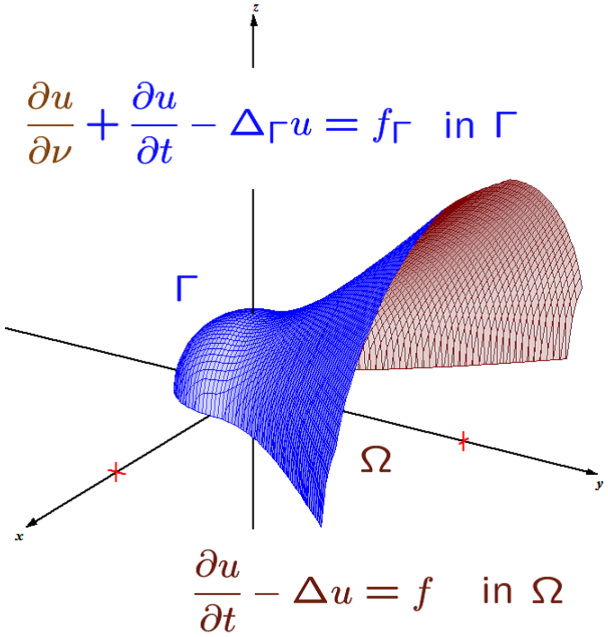


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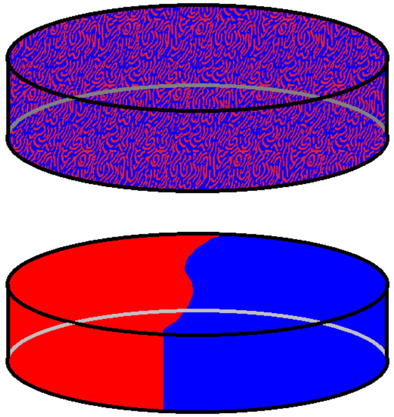
- Cahn–Hilliard equation with total mass conservation

$u$: order parameter, $\mu$: chemical potential

$$\frac{\partial u}{\partial t} - \Delta\mu = 0 \quad \text{in } Q := (0,T) \times \Omega,$$

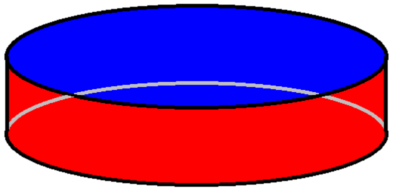
$$\mu = -\Delta u + u^3 - u - f \quad \text{in } Q,$$

$$\partial_{\boldsymbol{\nu}} u = \partial_{\boldsymbol{\nu}} \mu = 0 \quad \text{on } \Sigma := (0,T) \times \Gamma.$$



$$\int_\Omega u(t)dx = \int_\Omega u_0 dx.$$

Let $u = 1$ and $-1$ stand two different state, respectively. This describes the unti-diffusive phenomena, called Spinodal decomposition.



$$\int_\Omega u(t)dx + \int_\Gamma u(t)d\Gamma = \int_\Omega u_0 dx + \int_\Gamma u_0 d\Gamma.$$

Under the dynamic b.c. namely with the total mass conservation, the various situations of pattern formation may happen compared with one of Neumann b.c.

# 2. Equation and dynamic b.c. of Cahn–Hilliard type

• GMS(Goldstein–Miranville–Schimperna) problem

$u, \mu : Q := (0,T) \times \Omega \to \mathbb{R}$, $u_\Gamma, \mu_\Gamma : \Sigma := (0,T) \times \Gamma \to \mathbb{R}$,

$$\frac{\partial u}{\partial t} - \Delta \mu = 0, \quad \mu = -\Delta u + \xi + \pi(u) - f, \quad \xi = \beta(u) \quad \text{in } Q,$$

$$u_\Gamma = u_{|_\Gamma}, \quad \mu_\Gamma = \mu_{|_\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}} \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma,$$

$$\mu_\Gamma = \partial_{\boldsymbol{\nu}} u - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) - f_\Gamma, \quad \xi_\Gamma = \beta_\Gamma(u_\Gamma) \quad \text{on } \Sigma,$$

$$u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma,$$

Simbol "$|_\Gamma$" stands for the trace on $\Gamma$;

$\pi$, $\pi_\Gamma : \mathbb{R} \to \mathbb{R}$, Lipschitz continuous.

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◇ $\beta(r) = \beta_\Gamma(r) = r^3$, $\pi(r) = \pi_\Gamma(r) = -r$;

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$$\frac{\partial u}{\partial t} - \Delta \mu = 0, \quad \mu = -\Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q,$$

$$u_\Gamma = u_{|_\Gamma}, \quad \mu_\Gamma = \mu_{|_\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}} \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma,$$

$$\mu_\Gamma = \partial_{\boldsymbol{\nu}} u - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) - f_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{on } \Sigma,$$

$$u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma,$$

Simbol "$|_\Gamma$" stands for the trace on $\Gamma$; $\beta$, $\beta_\Gamma$: maximal monotone graphs on $\mathbb{R} \times \mathbb{R}$, ${}^\exists \widehat{\beta}$, ${}^\exists \widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty)$, proper l.s.c. convex, $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$ s.t. $\beta = \partial \widehat{\beta}$, $\beta_\Gamma = \partial \widehat{\beta}_\Gamma$; $\pi$, $\pi_\Gamma : \mathbb{R} \to \mathbb{R}$, Lipschitz continuous.

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◇ $\beta(r) = \beta_\Gamma(r) = \partial I_{[-1,1]}(r)$, $\pi(r) = \pi_\Gamma(r) = -r$.

- Total mass conservation

$$\int_\Omega u(t)dx + \int_\Gamma u_\Gamma(t)d\Gamma = \int_\Omega u_0 dx + \int_\Gamma u_{0\Gamma} d\Gamma \quad {}^\forall t \in [0,T].$$

- Known results with related to total mass conservation

◇ G. R. Goldstein, A. Miranville and G. Schimperna, A Cahn–Hilliard model in a domain with non-permeable walls, Physica D, **240** (2011), 754–766.

◇ L. Cherfils, S. Gatti and A. Miranville, A variational approach to a Cahn–Hilliard model in a domain with nonpermeable walls, J. Math. Sci. (N.Y.), **189**(2013), 604–636.

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◇ C. G. Gal and H. Wu, Asymptotic behavior of a Cahn–Hilliard equation with Wentzell boundary conditions and mass conservation, Discrete Contin. Dyn. Syst., **22** (2008), 1041–1063.

GMS problem with the case $\beta(r) = \beta_\Gamma(r) = r^3$, $\pi(r) = \pi_\Gamma(r) = -r$

$$\frac{\partial u}{\partial t} - \Delta \mu = 0, \quad \mu = -\Delta u + u^3 - u - f \quad \text{in } Q, \quad \text{(CH)}$$

$$u_\Gamma = u_{|_\Gamma}, \quad \mu_\Gamma = \mu_{|_\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}} \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma,$$

$$\mu_\Gamma = \partial_{\boldsymbol{\nu}} u - \Delta_\Gamma u_\Gamma + u_\Gamma^3 - u_\Gamma - f_\Gamma \quad \text{on } \Sigma. \quad \text{(CH)}$$

Popular Cahn–Hilliard equation with dynamic b.c.

$$\frac{\partial u}{\partial t} - \Delta \mu = 0, \quad \mu = -\Delta u + u^3 - u - f \quad \text{in } Q, \quad \text{(CH)}$$

$$u_\Gamma = u_{|_\Gamma}, \quad \partial_{\boldsymbol{\nu}} \mu = 0 \quad \text{on } \Sigma,$$

$$\frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}} u - \Delta_\Gamma u_\Gamma + u_\Gamma^3 - u_\Gamma = f_\Gamma \quad \text{on } \Sigma. \quad \text{(AC)}$$

Racke–Zheng (2003), Wu–Zheng (2004), Miranville–Zelik (2005), Gal (2007), Gilardi–Miranville–Schimperna (2009), Gal–Miranville (2009), Cavaterra–Gal–Grasselli (2011), Colli–Gilardi–Sprekels (2014), etc.

• Approach from the abstract evolution equation

$H := L^2(\Omega)$, $V := H^1(\Omega)$, $H_\Gamma := L^2(\Gamma)$, $V_\Gamma := H^1(\Gamma)$, $\boldsymbol{H} := H \times H_\Gamma$,
$\boldsymbol{V} := \{(z, z_\Gamma) \in V \times V_\Gamma : z_\Gamma = z_{|_\Gamma}$ a.e. on $\Gamma\}$, $\boldsymbol{W} := (H^2(\Omega) \times H^2(\Gamma))$;
$a(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$

$$a(\boldsymbol{u}, \boldsymbol{z}) := \int_\Omega \nabla u \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma z_\Gamma d\Gamma \quad {}^\forall \boldsymbol{u}, \boldsymbol{z} \in \boldsymbol{V};$$

$\boldsymbol{H}_0 := \{\boldsymbol{z} \in \boldsymbol{H} : m(\boldsymbol{z}) = 0\}$

$$m(\boldsymbol{z}) := \frac{\int_\Omega z dx + \int_\Gamma z_\Gamma d\Gamma}{|\Omega| + |\Gamma|} \quad {}^\forall \boldsymbol{z} \in \boldsymbol{H}, \quad m_0 := m(\boldsymbol{u}_0) := m\big((u_0, u_{\Gamma 0})\big);$$

$\boldsymbol{V}_0 := \boldsymbol{V} \cap \boldsymbol{H}_0$

$$|\boldsymbol{z}|_{\boldsymbol{V}_0} := \left\{ \int_\Omega |\nabla z|^2 dx + \int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma \right\}^{1/2} \quad {}^\forall \boldsymbol{z} \in \boldsymbol{V}_0;$$

$\boldsymbol{F} : \boldsymbol{V}_0 \to \boldsymbol{V}_0^*$

$$\langle \boldsymbol{F}\boldsymbol{z}, \tilde{\boldsymbol{z}} \rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} := a(\boldsymbol{z}, \tilde{\boldsymbol{z}}) \quad {}^\forall \boldsymbol{z}, \tilde{\boldsymbol{z}} \in \boldsymbol{V}_0.$$

**Lemma 1.** ${}^{\exists}c_P > 0$ *s.t.*

$$c_P|\boldsymbol{z}|^2_{\boldsymbol{V}} \le \langle \boldsymbol{F}\boldsymbol{z}, \boldsymbol{z}\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} = |\boldsymbol{z}|^2_{\boldsymbol{V}_0} \quad {}^{\forall}\boldsymbol{z} \in \boldsymbol{V}_0.$$

$$(\boldsymbol{z}_1^*, \boldsymbol{z}_2^*)_{\boldsymbol{V}_0^*} := \langle \boldsymbol{z}_1^*, \boldsymbol{F}^{-1}\boldsymbol{z}_2^*\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} \quad {}^{\forall}\boldsymbol{z}_1^*, \boldsymbol{z}_2^* \in \boldsymbol{V}_0^*;$$

$\boldsymbol{H}_0 := \{\boldsymbol{z} \in \boldsymbol{H} : m(\boldsymbol{z}) = 0\}$, $\boldsymbol{V}_0 := \boldsymbol{V} \cap \boldsymbol{H}_0$, $\boldsymbol{V}_0 \hookrightarrow\hookrightarrow \boldsymbol{H}_0 \hookrightarrow\hookrightarrow \boldsymbol{V}_0^*$.

We can apply the approach of abstract evolution equation to the Cahn–Hilliard equation, Kenmochi–Niezgódka–Pawłow(1995), Kubo(2012)
$\boldsymbol{v} := \boldsymbol{u} - m_0\boldsymbol{1}$, $\boldsymbol{v}_0 := \boldsymbol{u}_0 - m_0\boldsymbol{1}$, namely $(v, v_\Gamma) := (u - m_0, u_\Gamma - m_0)$,
$(v_0, v_{0\Gamma}) := (u_0 - m_0, u_{0\Gamma} - m_0)$, where $m_0 := m(\boldsymbol{u}_0)$;
$\boldsymbol{f} := (f, f_\Gamma)$, $\boldsymbol{\beta}(\boldsymbol{z}) := (\beta(z), \beta_\Gamma(z_\Gamma))$, $\boldsymbol{\pi}(\boldsymbol{z}) := (\pi(z), \pi_\Gamma(z_\Gamma))$.

(A1) $\beta$, $\beta_\Gamma$: maximal monotone graphs, that is, $\beta = \partial\widehat{\beta}$, $\beta_\Gamma = \partial\widehat{\beta}_\Gamma$ by some proper l.s.c. convex $\widehat{\beta}$, $\widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty]$ with $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$;
(A2) $\pi$, $\pi_\Gamma : \mathbb{R} \to \mathbb{R}$, Lipschitz continuous;
(A3) $\boldsymbol{f} \in L^2(0, T; \boldsymbol{V})$ (or $\boldsymbol{f} \in W^{1,1}(0, T; \boldsymbol{H})$), $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in \boldsymbol{V}$;

(A4) $D(\beta_\Gamma) \subseteq D(\beta)$, $^\exists \varrho, c_0 > 0$ s.t.

$$\left|\beta^\circ(r)\right| \le \varrho\left|\beta_\Gamma^\circ(r)\right| + c_0 \quad {}^\forall r \in D(\beta_\Gamma);$$

(A5) $m_0 \in \mathrm{int} D(\beta_\Gamma)$, $\widehat{\beta}(v_0 + m_0) \in L^1(\Omega)$, $\widehat{\beta}_\Gamma(v_{0\Gamma} + m_0) \in L^1(\Gamma)$.

**Theorem 1.** *Under the assumptions (A1)–(A5), there exists $(\boldsymbol{v}, \boldsymbol{\mu}, \boldsymbol{\xi})$ such that*

$$\boldsymbol{v} \in H^1(0,T;\boldsymbol{V}_0^*) \cap L^\infty(0,T;\boldsymbol{V}_0) \cap L^2(0,T;\boldsymbol{W}),$$
$$\boldsymbol{\mu} \in L^2(0,T;\boldsymbol{V}), \quad \boldsymbol{\xi} := (\xi, \xi_\Gamma) \in L^2(0,T;\boldsymbol{H}),$$

$\xi \in \beta(v + m_0)$ *a.e. in* $Q$, $\xi_\Gamma \in \beta_\Gamma(v_\Gamma + m_0)$ *a.e. on* $\Sigma$, *and*

$$\left\langle \boldsymbol{v}'(t), \boldsymbol{z} \right\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} + a\big(\boldsymbol{\mu}(t), \boldsymbol{z}\big) = 0 \quad {}^\forall \boldsymbol{z} \in \boldsymbol{V}_0,$$

$$\big(\boldsymbol{\mu}(t), \boldsymbol{z}\big)_{\boldsymbol{H}} = a\big(\boldsymbol{v}(t), \boldsymbol{z}\big) + \big(\boldsymbol{\xi}(t) + \boldsymbol{\pi}(\boldsymbol{v}(t) + m_0\mathbf{1}) - \boldsymbol{f}(t), \boldsymbol{z}\big)_{\boldsymbol{H}} \quad {}^\forall \boldsymbol{z} \in \boldsymbol{V},$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 := \boldsymbol{u}_0 - m_0\mathbf{1} \quad \textit{in } \boldsymbol{H}_0.$$

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(A2) $\pi$, $\pi_\Gamma : \mathbb{R} \to \mathbb{R}$, Lipschitz continuous;
(A3)$'$ $\boldsymbol{f} \in L^2(0, T; \boldsymbol{V}^*)$, $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in \boldsymbol{V}$.

Theorem 2. *Under the assumptions (A1)–(A3)$'$ for each $\boldsymbol{f}^{(i)}$ and $\boldsymbol{v}_0^{(i)}$, the corresponding solutions $(\boldsymbol{v}^{(i)}, \boldsymbol{\mu}^{(i)}, \boldsymbol{\xi}^{(i)})$ $(i = 1, 2)$ satisfy*

$$\left|\boldsymbol{v}^{(1)}(t) - \boldsymbol{v}^{(2)}(t)\right|_{\boldsymbol{V}_0^*}^2 + \int_0^t \left|\boldsymbol{v}^{(1)}(s) - \boldsymbol{v}^{(2)}(s)\right|_{\boldsymbol{V}_0}^2 ds$$
$$\le C\left\{\left|\boldsymbol{v}_0^{(1)} - \boldsymbol{v}_0^{(2)}\right|_{\boldsymbol{V}_0^*}^2 + \int_0^t \left|\boldsymbol{f}^{(1)}(s) - \boldsymbol{f}^{(2)}(s)\right|_{\boldsymbol{V}^*}^2 ds\right\} \quad {}^\forall t \in [0, T].$$

◇ P. Colli and T. Fukao, Equation and dynamic boundary condition of Cahn–Hilliard type with singular potentials, Nonlinear Anal., **127** (2015), 413–433.

(A6) $\boldsymbol{f} \in H^1(0,T;\boldsymbol{H})$, $\boldsymbol{f}(0) \in \boldsymbol{V}$, $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in (H^3(\Omega)\times H^3(\Gamma))\cap \boldsymbol{V}$, $\boldsymbol{\beta}^\circ(\boldsymbol{u}_0) \in \boldsymbol{V}$.

Theorem 3. *Under the assumptions (A1), (A2), (A4)–(A6), there exists $(\boldsymbol{u}, \boldsymbol{\mu}, \boldsymbol{\xi})$ such that*

$$\boldsymbol{u} \in W^{1,\infty}(0,T;\boldsymbol{V}^*) \cap H^1(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{W}),$$
$$\boldsymbol{\mu} \in L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W}), \quad \boldsymbol{\xi} \in L^\infty(0,T;\boldsymbol{H}),$$

$\xi \in \beta(u)$ *a.e. in* $Q$, $\xi_\Gamma \in \beta_\Gamma(u_\Gamma)$ *a.e. on* $\Sigma$, *and*

$$\frac{\partial u}{\partial t} - \Delta\mu = 0, \quad \mu = -\Delta u + \xi + \pi(u) - f \quad \text{a.e. in } Q,$$

$$u_\Gamma = u_{|_\Gamma}, \quad \mu_\Gamma = \mu_{|_\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}}\mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{a.e. on } \Sigma,$$

$$\mu_\Gamma = \partial_{\boldsymbol{\nu}} u - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) - f_\Gamma \quad \text{a.e. on } \Sigma,$$

$$u(0) = u_0 \quad \text{a.e. in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma.$$

- Key of the proof $\quad {}^\forall\lambda \in (0,1]$, $\boldsymbol{u}_\lambda = \boldsymbol{v}_\lambda + m_0\mathbf{1}$, $\boldsymbol{P} : \boldsymbol{H} \to \boldsymbol{H}_0$,

$\boldsymbol{P}\boldsymbol{z} := \boldsymbol{z} - m(\boldsymbol{z})\mathbf{1}$. Consider the following problem with viscosity (GMS;$\lambda$)

$$(\boldsymbol{F}^{-1} + \lambda \boldsymbol{I})\bigl(\boldsymbol{v}'_\lambda(t)\bigr) + \partial\varphi\bigl(\boldsymbol{v}_\lambda(t)\bigr) = \boldsymbol{P}\Bigl(-\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(t)\bigr) - \boldsymbol{\pi}\bigl(\boldsymbol{u}_\lambda(t)\bigr) + \boldsymbol{f}(t)\Bigr)$$

$$\text{for a.a. } t \in (0,T),$$

$$\boldsymbol{v}_\lambda(0) = \boldsymbol{v}_0 := \boldsymbol{u}_0 - m_0\mathbf{1} \quad \text{in } \boldsymbol{H}_0,$$

where $\boldsymbol{\beta}_\lambda := (\beta_\lambda, \beta_{\Gamma,\lambda})$, $\boldsymbol{\pi} := (\pi, \pi)$, $\varphi : \boldsymbol{H}_0 \to [0, +\infty]$

$$\varphi(\boldsymbol{z}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \dfrac{1}{2}\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma & \text{if } \boldsymbol{z} \in \boldsymbol{V}_0, \\ +\infty & \text{if } \boldsymbol{z} \in \boldsymbol{H}_0 \setminus \boldsymbol{V}_0. \end{cases}$$

$$\beta_\lambda(r) := \frac{1}{\lambda}\Bigl(r - (I + \lambda\beta)^{-1}(r)\Bigr),\ \beta_{\Gamma,\lambda}(r) := \frac{1}{\lambda\varrho}\Bigl(r - (I + \lambda\varrho\beta_\Gamma)^{-1}(r)\Bigr),$$

**Lemma 2.** [Calatroni–Colli (2013)]

$$\bigl|\beta_\lambda(r)\bigr| \le \varrho\bigl|\beta_{\Gamma,\lambda}(r)\bigr| + c_0 \quad {}^\forall r \in \mathbb{R}.$$

# 3. Degenerate parabolic eq. with dynamic b.c.

- Enthalpy formulation of the Stefan problem $\quad u, u_\Gamma$: unknown

$$\frac{\partial u}{\partial t} - \Delta\beta(u) = g \quad \text{in } Q,$$

$$\beta(u_\Gamma) = \beta(u)_{|\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}}\beta(u) - \Delta_\Gamma\beta(u_\Gamma) = g_\Gamma \quad \text{on } \Sigma,$$

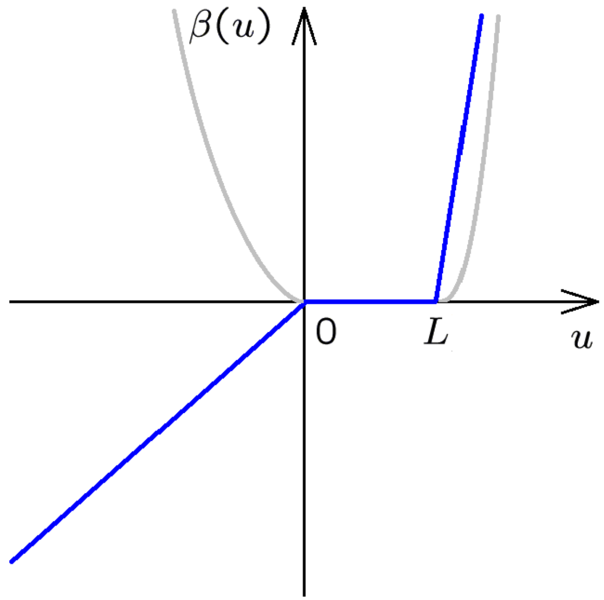
$$u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma.$$

$\beta : \mathbb{R} \to \mathbb{R}$ is a piecewise linear function

$$\beta(r) = \begin{cases} k_s r & \text{if } r < 0, \\ 0 & \text{if } 0 \le r \le L, \\ k_\ell(r - L) & \text{if } r > L, \end{cases}$$

$k_s, k_\ell, L$: positive constants,

$g, g_\Gamma, u_0, u_{0\Gamma}$: given functions.



- Known results

◇ T. Aiki, Multi-dimensional Stefan problems with dynamic boundary conditions, Appl. Anal., **56**(1995), 71–94.

$L^2$-framework

$$\frac{\partial u}{\partial t} - \Delta\beta(u) = g \quad \text{in } Q,$$

$$\theta_\Gamma = \beta(u)_{|_\Gamma}, \quad \frac{\partial \theta_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}}\beta(u) = g_\Gamma \quad \text{on } \Sigma.$$

◇ F. Andreu, N. Igbida, J. M. Mazón and J. Toledo, A degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions, Interfaces Free Bound., **8**(2006), 447–479.

$L^1$-framework

$$\frac{\partial u}{\partial t} - \Delta\beta(u) = g \quad \text{in } Q,$$

$$\beta(u_\Gamma) = \beta(u)_{|_\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}}\beta(u) - \Delta_\Gamma \beta_\Gamma(u_\Gamma) = g_\Gamma \quad \text{on } \Sigma.$$

- Asymptotic limit of Cahn–Hilliard equation (GMS)$_\varepsilon$: $^\forall\varepsilon \in (0,1]$

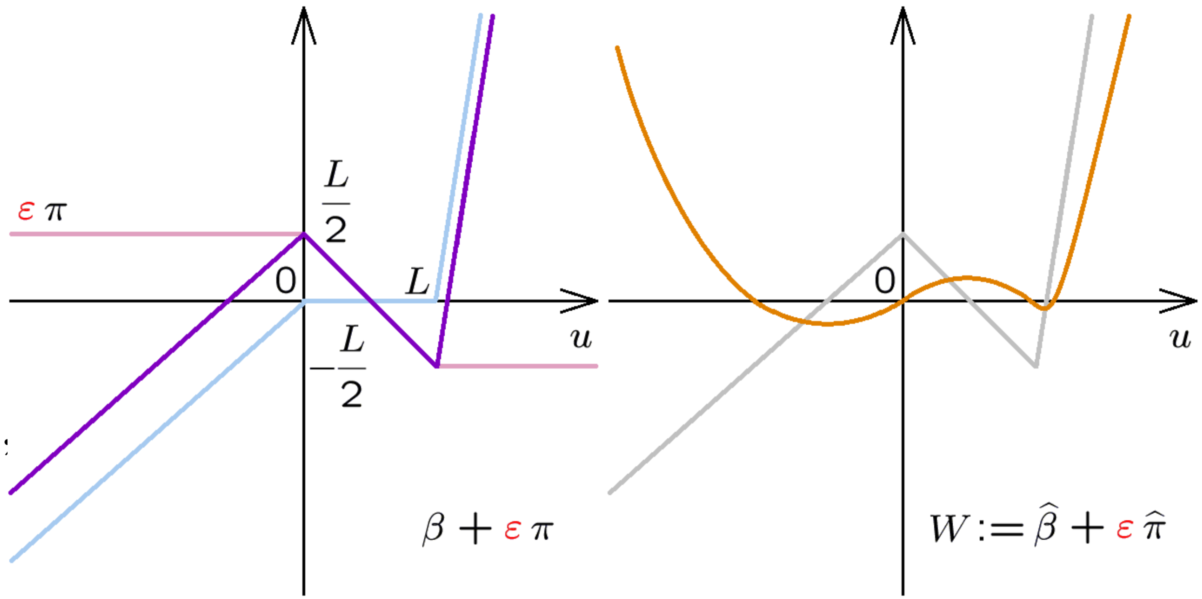
$$\frac{\partial u_\varepsilon}{\partial t} - \Delta\mu_\varepsilon = 0, \quad \mu_\varepsilon = -\varepsilon\Delta u_\varepsilon + \beta(u_\varepsilon) + \varepsilon\pi(u_\varepsilon) - f \quad \text{in } Q,$$

$$u_{\Gamma,\varepsilon} = (u_\varepsilon)_{|\Gamma}, \quad \mu_{\Gamma,\varepsilon} = (\mu_\varepsilon)_{|\Gamma}, \quad \frac{\partial u_{\Gamma,\varepsilon}}{\partial t} + \partial_{\boldsymbol{\nu}}\mu_\varepsilon - \Delta_\Gamma\mu_{\Gamma,\varepsilon} = 0 \quad \text{on } \Sigma,$$

$$\mu_{\Gamma,\varepsilon} = \varepsilon\partial_{\boldsymbol{\nu}} u_\varepsilon - \varepsilon\Delta_\Gamma u_{\Gamma,\varepsilon} + \beta(u_{\Gamma,\varepsilon}) + \varepsilon\pi(u_{\Gamma,\varepsilon}) - f_\Gamma \quad \text{on } \Sigma.$$

$\beta(=\beta_\Gamma) : \mathbb{R} \to \mathbb{R}$ is maximal monotone,
$\pi(=\pi_\Gamma) : \mathbb{R} \to \mathbb{R}$,

$$\pi(r) := \begin{cases} \dfrac{L}{2} & \text{if } r < 0, \\ \dfrac{L}{2} - r & \text{if } 0 \le r \le L, \\ -\dfrac{L}{2} & \text{if } r > L. \end{cases}$$



- Asymptotic limit of Cahn–Hilliard equation $\quad (\mathrm{GMS})_\varepsilon$: ${}^\forall\varepsilon \in (0,1]$

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta\mu_\varepsilon = 0, \quad \mu_\varepsilon = -\varepsilon\Delta u_\varepsilon + \beta(u_\varepsilon) + \varepsilon\pi(u_\varepsilon) - f \quad \text{in } Q,$$

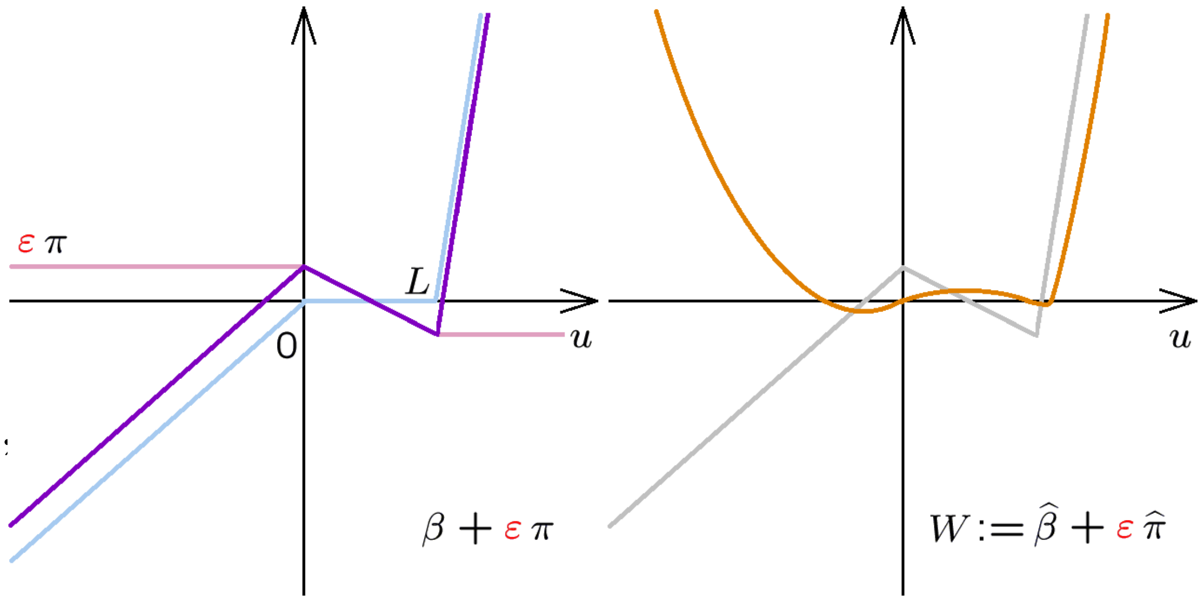
$$u_{\Gamma,\varepsilon} = (u_\varepsilon)_{|\Gamma}, \quad \mu_{\Gamma,\varepsilon} = (\mu_\varepsilon)_{|\Gamma}, \quad \frac{\partial u_{\Gamma,\varepsilon}}{\partial t} + \partial_{\boldsymbol{\nu}}\mu_\varepsilon - \Delta_\Gamma \mu_{\Gamma,\varepsilon} = 0 \quad \text{on } \Sigma,$$

$$\mu_{\Gamma,\varepsilon} = \varepsilon\partial_{\boldsymbol{\nu}} u_\varepsilon - \varepsilon\Delta_\Gamma u_{\Gamma,\varepsilon} + \beta(u_{\Gamma,\varepsilon}) + \varepsilon\pi(u_{\Gamma,\varepsilon}) - f_\Gamma \quad \text{on } \Sigma.$$

$\beta(=\beta_\Gamma) : \mathbb{R} \to \mathbb{R}$ is maximal monotone,

$\pi(=\pi_\Gamma) : \mathbb{R} \to \mathbb{R}$,

$$\pi(r) := \begin{cases} \dfrac{L}{2} & \text{if } r < 0, \\[2mm] \dfrac{L}{2} - r & \text{if } 0 \le r \le L, \\[2mm] -\dfrac{L}{2} & \text{if } r > L. \end{cases}$$



- Asymptotic limit of Cahn–Hilliard equation (GMS)$_\varepsilon$: $^\forall\varepsilon \in (0,1]$

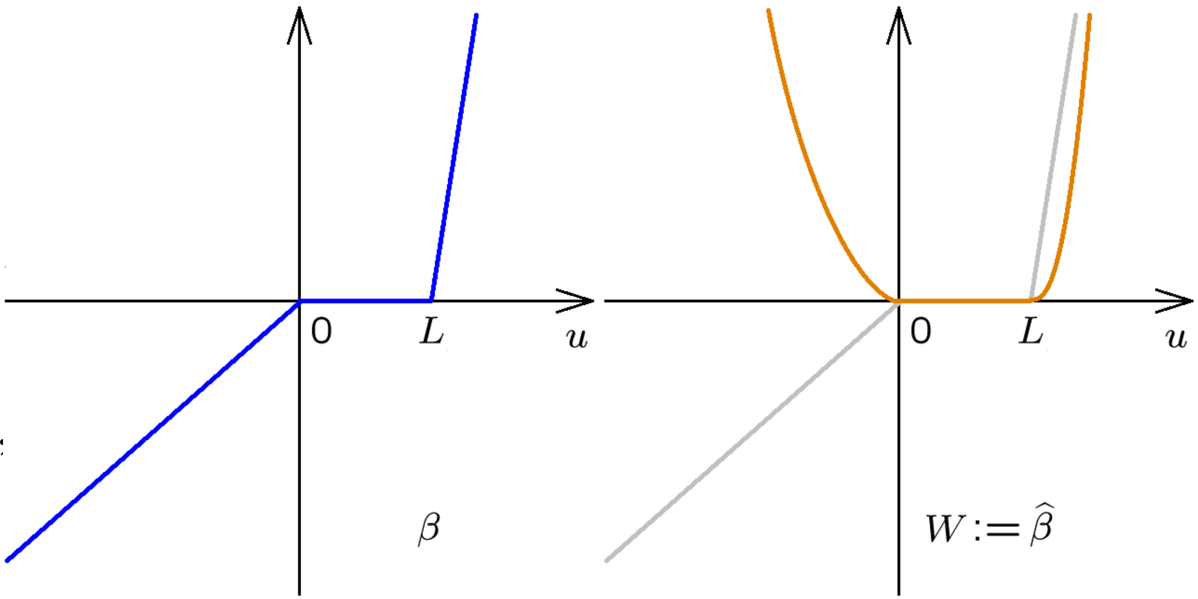
$$\frac{\partial u_\varepsilon}{\partial t} - \Delta \mu_\varepsilon = 0, \quad \mu_\varepsilon = \beta(u_\varepsilon) - f \quad \text{in } Q,$$

$$u_{\Gamma,\varepsilon} = (u_\varepsilon)_{|_\Gamma}, \quad \mu_{\Gamma,\varepsilon} = (\mu_\varepsilon)_{|_\Gamma}, \quad \frac{\partial u_{\Gamma,\varepsilon}}{\partial t} + \partial_{\boldsymbol{\nu}}\mu_\varepsilon - \Delta_\Gamma \mu_{\Gamma,\varepsilon} = 0 \quad \text{on } \Sigma,$$

$$\mu_{\Gamma,\varepsilon} = \beta(u_{\Gamma,\varepsilon}) - f_\Gamma \quad \text{on } \Sigma.$$

$\beta(=\beta_\Gamma) : \mathbb{R} \to \mathbb{R}$ is maximal monotone,
$\pi(=\pi_\Gamma) : \mathbb{R} \to \mathbb{R}$,

$$\pi(r) := \begin{cases} \dfrac{L}{2} & \text{if } r < 0, \\ \dfrac{L}{2} - r & \text{if } 0 \le r \le L, \\ -\dfrac{L}{2} & \text{if } r > L. \end{cases}$$



(A7) $\beta$: maximal monotone graph, that is, $\beta = \partial\widehat{\beta}$, by some proper l.s.c. convex $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ with $\widehat{\beta}(0) = 0$;

(A8) $\boldsymbol{g} \in L^2(0, T; \boldsymbol{H}_0)$;

(A9) $\boldsymbol{u}_0 \in \boldsymbol{V}$, $m_0 \in \mathrm{int}D(\beta)$, $\widehat{\beta}(u_0) \in L^1(\Omega)$, $\widehat{\beta}(u_{0\Gamma}) \in L^1(\Gamma)$.

**Theorem 4.** *Under the assumptions (A7)–(A9),* $^{\exists 1}\boldsymbol{u} \in H^1(0, T; \boldsymbol{V}^*) \cap L^\infty(0, T; \boldsymbol{H})$ *and* $^{\exists}\boldsymbol{\xi} \in L^2(0, T; \boldsymbol{V})$ *s.t.*

$$\xi \in \beta(u) \quad \text{a.e. in } Q, \quad \xi_\Gamma \in \beta(u_\Gamma), \quad \xi_\Gamma = \xi_{|_\Gamma} \quad \text{a.e. on } \Sigma,$$

$$\left\langle u'(t), z \right\rangle_{V^*,V} + \left\langle u'_\Gamma(t), z_\Gamma \right\rangle_{V_\Gamma^*,V_\Gamma} + \int_\Omega \nabla\xi(t) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma \xi_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma$$

$$= \int_\Omega g(t) z dx + \int_\Gamma g_\Gamma(t) z_\Gamma d\Gamma \quad {}^\forall \boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{V}, \text{ for a.a. } t \in (0, T),$$

*and* $u(0) = u_0$ *a.e. in* $\Omega$, $u_\Gamma(0) = u_{0\Gamma}$ *a.e. on* $\Gamma$ *hold.*

◇ T. Fukao, Convergence of Cahn–Hilliard systems to the Stefan problem with dynamic boundary conditions, to appear in Asymptot. Anal.

• Remarks (i) The idea of asymptotic limit of Cahn–Hilliard equation is independent of the boundary condition.

◇ P. Colli and T. Fukao, Nonlinear diffusion equations as asymptotic limits of Cahn–Hilliard systems, J. Differential Equations, **260** (2016), 6930–6959.

(ii) We can choose very general $\beta$ for various degenerate parabolic equations, that is, porous media equation by $\beta(r) = |r|^{q-1}r$ $(q > 1)$, Hele-Shaw problem by $\beta(r) = \partial I_{[0,1]}(r)$. We can also treat fast diffusion equation by $\beta(r) = |r|^{q-1}r$ $(0 < q < 1)$.
(iii) We can find some physical interpretation:

◇ R. L. Pego, Front migration in the nonlinear Cahn–Hilliard equation, Proc. R. Soc. Lond. A, **422** (1989), 261–278.

(iv) Others types of asymptotic limit to the degenerate parabolic equations: Phase field to Stefan, reaction diffusion to Stefan [H. Murakawa (2007)], etc. They need some growth condition for $\widehat{\beta}$.

# 4. Boundary control problem of (GMS)

- Optimal control problem (GMS;$f_\Gamma$):

$$\frac{\partial u}{\partial t} - \Delta\mu = 0, \quad \mu = -\Delta u + \mathcal{W}'(u), \quad \text{in } Q,$$

$$u_\Gamma = u_{|_\Gamma}, \quad \mu_\Gamma = \mu_{|_\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}}\mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma,$$

$$\mu_\Gamma = \partial_{\boldsymbol{\nu}} u - \Delta_\Gamma u_\Gamma + \mathcal{W}_\Gamma'(u_\Gamma) - f_\Gamma \quad \text{on } \Sigma,$$

$$u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma,$$

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(A10) $-\infty \le r_- < 0 < r_+ \le +\infty$;
(A11) $\beta, \beta_\Gamma \in C^2(r_-, r_+)$; monotone, $\pi, \pi_\Gamma \in C^2(r_-, r_+)$; Lipschitz,
$\beta(0) = \beta_\Gamma(0) = 0$, $\beta', \beta_\Gamma'$ are bounded from below,
$\lim_{r \searrow r_-} \beta(r) = \lim_{r \searrow r_-} \beta_\Gamma(r) = -\infty$,
$\lim_{r \nearrow r_+} \beta(r) = \lim_{r \nearrow r_+} \beta_\Gamma(r) = +\infty$;
(A12) $\mathcal{W}' = \beta + \pi, \mathcal{W}_\Gamma' = \beta_\Gamma + \pi_\Gamma$ and $\mathcal{W}, \mathcal{W}_\Gamma \in C^3(r_-, r_+)$, $\mathcal{W}, \mathcal{W}_\Gamma \ge 0$,
$\mathcal{W}(0) = \mathcal{W}_\Gamma(0) = 0$, $\mathcal{W}'', \mathcal{W}_\Gamma''$ are bounded from below,
$\lim_{r \searrow r_-} \mathcal{W}'(r) = \lim_{r \searrow r_-} \mathcal{W}_\Gamma'(r) = -\infty$,
$\lim_{r \nearrow r_+} \mathcal{W}'(r) = \lim_{r \nearrow r_+} \mathcal{W}_\Gamma'(r) = +\infty$;

$f_\Gamma \in \mathcal{U}_{\rm ad}$: boundary control in an admissible set:

$$\mathcal{U}_{\rm ad} := \Big\{ \zeta_\Gamma \in \mathcal{X} := H^1(0,T;H_\Gamma) : |\zeta_\Gamma|_{\mathcal{X}} \le M_0 \Big\}.$$

---

(OP): Find a minimizer $((\bar{u}, \bar{u}_\Gamma), \bar{f}_\Gamma)$ of the cost

$$J\big((u,u_\Gamma), f_\Gamma\big) := \frac{b_Q}{2}\int_Q |u - u^*|^2 dx dt + \frac{b_\Sigma}{2}\int_\Sigma |u_\Gamma - u_\Gamma^*|^2 d\Gamma dt + \frac{b_0}{2}\int_\Sigma |f_\Gamma|^2 d\Gamma dt,$$

where $u^* \in L^2(0,T;H)$ and $u_\Gamma^* \in L^2(0,T;H_\Gamma)$ are target states, subject to the constraint $\bar{f}_\Gamma \in \mathcal{U}_{\rm ad}$ and the corresponding state

$$\bar{\boldsymbol{u}} = (\bar{u}, \bar{u}_\Gamma) = S(\bar{f}_\Gamma), \quad \text{solution } (\bar{\boldsymbol{u}}, \bar{\boldsymbol{\mu}}) \text{ of (GMS;} \bar{f}_\Gamma).$$

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This kind of boundary control problem for the Cahn–Hilliard equation was treated in the series of paper by Colli–Gilardi–Sprekels.

◇ P. Colli, G. Gilardi and J. Sprekels, A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions, Adv. Nonlinear Anal., **4** (2015), 311–325.

The strategy of the proof is completely same as their study.

▷ Problem with viscosity (GMS;$\lambda, f_\Gamma$), $S_\lambda : f_\Gamma \mapsto \boldsymbol{u}_\lambda$, (sol. $(\boldsymbol{u}_\lambda, \boldsymbol{\mu}_\lambda)$)

▷ Optimal control problem (OP;$\lambda$) for (GMS;$\lambda, f_\Gamma$).

▷ Linearized problem (to obtain the differentiability of $S_\lambda$) .

**Theorem 5.** *Assume (A4),(A5), (A10)–(A12). $^\forall\{\bar{f}_{\Gamma,\lambda}\}_{\lambda\in(0,1]}$; optimal controls of* (OP;$\lambda$), $^\exists\{\lambda_k\}_{k\in\mathbb{N}}$; $\lambda_k \searrow 0$ *s.t.* $\bar{f}_{\Gamma,\lambda_k} \to \bar{f}^*_\Gamma$ *weakly in* $H^1(0,T;H_\Gamma)$ *as* $k \to +\infty$, *where* $\bar{f}^*_\Gamma$ *is an optimal control of* (OP).

• Necessary optimality condition $\bar{f}_\Gamma$: target optimal control of (OP).

◇ V. Barbu, Necessary conditions for nonconvex distributed control problems governed by elliptic variational inequalities, J. Math. Anal. Appl., **80** (1981), 566–597.

(OP;$\lambda$, $\bar{f}_\Gamma$): Find a minimizer $((\tilde{u}_\lambda, \tilde{u}_{\Gamma,\lambda}), \tilde{f}_{\Gamma,\lambda})$ of the cost

$$\tilde{J}_{\bar{f}_\Gamma}\Big((u,u_\Gamma), f_\Gamma\Big) := J\Big((u,u_\Gamma), f_\Gamma\Big) + \frac{1}{2}\int_\Sigma |f_\Gamma - \bar{f}_\Gamma|^2 d\Gamma dt,$$

subject to the constraint $\tilde{f}_{\Gamma,\lambda} \in \mathcal{U}_{\mathrm{ad}}$ and the corresponding state $\tilde{\boldsymbol{u}}_\lambda = S_\lambda(\tilde{f}_{\Gamma,\lambda})$, solution of (GMS;$\lambda, \tilde{f}_{\Gamma,\lambda}$).

$\mathcal{Z} := H^1(0,T;\boldsymbol{V}_0^*) \cap L^2(0,T;\boldsymbol{V}_0)$ and $\mathcal{Z}_0 := \{\boldsymbol{\zeta} \in \mathcal{Z} : \boldsymbol{\zeta}(0) = \boldsymbol{0}\}$.

Theorem 6. *Assume (A4),(A5), (A10)–(A12).* $^\forall \bar{f}_\Gamma$*; optimal control of* (OP), $^\forall\{\tilde{f}_{\Gamma,\lambda}\}_{\lambda\in(0,1]}$ *of* (OP;$\lambda, \bar{f}_\Gamma$), $^\exists\{\lambda_k\}_{k\in\mathbb{N}}$*;* $\lambda_k \searrow 0$ *s.t.*

$$\tilde{f}_{\Gamma,\lambda_k} \to \bar{f}_\Gamma \quad \textit{in } L^2(0,T;H_\Gamma) \quad \textit{as } k \to +\infty.$$

*Moreover, there exist two elements*

$$\Lambda \in (H^1(0,T;V^*) \cap L^2(0,T;V))^*, \ \Lambda_\Gamma \in (H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma))^*$$

*and* $\tilde{\boldsymbol{q}} \in L^\infty(0,T;\boldsymbol{V}_0^*) \cap L^2(0,T;\boldsymbol{V}_0)$ *satisfying an adjoint system*

$$\int_0^T \langle \boldsymbol{\zeta}'(t), \boldsymbol{F}^{-1}\tilde{\boldsymbol{q}}(t)\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} dt + \int_0^T a\bigl(\tilde{\boldsymbol{q}}(t), \boldsymbol{\zeta}(t)\bigr)dt + \langle \Lambda, \zeta\rangle_Q + \langle \Lambda_\Gamma, \zeta_\Gamma\rangle_\Sigma$$

$$= b_Q \int_0^T \bigl(\bar{u}(t) - u^*(t), \zeta(t)\bigr)_H dt + b_\Sigma \int_0^T \bigl(\bar{u}_\Gamma(t) - u_\Gamma^*(t), \zeta_\Gamma(t)\bigr)_{H_\Gamma} dt$$

*for all* $\boldsymbol{\zeta} \in \mathcal{Z}_0$*, s.t.*

$$\int_0^T \bigl(\tilde{q}_\Gamma(t) + b_0 \bar{f}_\Gamma(t), f_\Gamma(t) - \bar{f}_\Gamma(t)\bigr)_{H_\Gamma} dt \geq 0 \quad {}^\forall f_\Gamma \in \mathcal{U}_{\mathrm{ad}}.$$

# Conclusion

☑ Setting suitable function spaces, we extend the well-posedness result Goldstein–Miranville–Schimperna (2011) of (GMS) to the case that $\beta$ and $\beta_\Gamma$ are multivalued. Theorems 1,2,3

◇ P. Colli and T. Fukao, Equation and dynamic boundary condition of Cahn–Hilliard type with singular potentials, Nonlinear Anal., **127** (2015), 413–433.

☑ Considering the asymptotic limit of Cahn–Hilliard equation, we treat the degenerate parabolic equation with dynamic b.c. Theorem 4

◇ T. Fukao, Convergence of Cahn–Hilliard systems to the Stefan problem with dynamic boundary conditions, to appear in Asymptot. Anal.

☑ Based on the known result by Colli–Gilardi–Sprekels (2015), we obtain two characterizations of the optimal control (OP), with some necessary optimality condition. Theorems 5, 6