Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Optimal distributed control for nonlocal Cahn-Hilliard/Navier-Stokes systems in 2D

Sergio Frigeri

Weierstrass Institute for Applied Analysis and Stochastics

$$\boldsymbol{u}_t - 2\mathrm{div}\,(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v} \tag{0.1}$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu) \tag{0.2}$$

$$\mu = a\varphi - K*\varphi + F'(\varphi) \tag{0.3}$$

$$\mathrm{div}(\boldsymbol{u}) = 0 \tag{0.4}$$

in $\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^d$, $d=2,3$ is a bounded smooth domain and $T>0$ is a prescribed final time. System (0.1)–(0.4) is subjected to

$$\boldsymbol{u}=0, \qquad m(\varphi)\nabla\mu\cdot\boldsymbol{n}=0 \qquad \text{on } \partial\Omega\times(0,T) \tag{0.5}$$

and to

$$\boldsymbol{u}(0)=\boldsymbol{u}_0, \qquad \varphi(0)=\varphi_0 \qquad \text{in } \Omega \tag{0.6}$$

$$(K*\varphi)(x) := \int_\Omega K(x-y)\varphi(y)dy \qquad a(x) := \int_\Omega K(x-y)dy$$

(K) $K \in W^{1,1}_{loc}(\mathbb{R}^d)$, $K(x) = K(-x)$, $a(x) := \int_\Omega K(x-y)dy \geq 0$, a.e.

$$a^* := \sup_{x\in\Omega} \int_\Omega |K(x-y)|dy < \infty, \quad b := \sup_{x\in\Omega} \int_\Omega |\nabla K(x-y)|dy < \infty$$

(V) ν is locally Lipschitz on $\mathbb{R}$ and $\exists\, \hat{\nu}_1, \hat{\nu}_2 > 0$ s.t.

$$\hat{\nu}_1 \leq \nu(s) \leq \hat{\nu}_2, \qquad \forall s \in \mathbb{R}$$

(M) $m \in C^1([-1,1])$, $m \geq 0$, $m(s) = 0$ iff $s = -1$ or $s = 1$. Moreover, $\exists\, \epsilon_0 > 0$ s.t. m is non-increasing in $[1-\epsilon_0, 1]$ and non-decreasing in $[-1, -1+\epsilon_0]$

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Furthermore, m and F are supposed to fulfill the condition

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We assume that F written in the form

$$F = F_1 + F_2$$

where the singular component F_1 and the regular component $F_2 \in C^2([-1,1])$ satisfy

(A2) $\exists\,\kappa > 4(a^* - a_* - b_*)$, where $b_* := \min_{[-1,1]} F_2''$, and $\epsilon_0 > 0$:

$$F_1''(s) \geq \kappa, \qquad \forall s \in (-1, -1+\epsilon_0] \cup [1-\epsilon_0, 1)$$

(A3) $\exists\,\epsilon_0 > 0$: F_1'' is non-decr. in $[1-\epsilon_0, 1)$ and non-incr. in $(-1, -1+\epsilon_0]$

(A4) $\exists\,c_0 > 0$: $F''(s) + a(x) \geq c_0, \quad \forall s \in (-1,1)$, a.e. $x \in \Omega$

(A5) $\exists\,\rho \in [0,1)$: $\rho F_1''(s) + F_2''(s) + a(x) \geq 0, \ \forall s \in (-1,1)$, a.e. $x \in \Omega$

(A6) $\exists\,\alpha_0 > 0$: $m(s) F_1''(s) \geq \alpha_0, \qquad \forall s \in [-1,1]$

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Typical situation

- $m(s) = k_1(1-s^2)$
- $F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\big((1+s)\log(1+s) + (1-s)\log(1-s)\big), \quad 0 < \theta < \theta_c$

  $\theta_c =$ given critical temperature below which the phase separation takes place

Set $H := L^2(\Omega), \quad V := H^1(\Omega)$

Let $\boldsymbol{u}_0 \in H_{div}, \varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$, $\boldsymbol{v} \in L^2(0,T;V'_{div})$ and $0 < T < +\infty$ be given. A pair $[\boldsymbol{u}, \varphi]$ is a weak sol. on $[0,T]$ if

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- $\boldsymbol{u}$, φ satisfy

$$\boldsymbol{u} \in L^\infty(0,T;H_{div}) \cap L^2(0,T;V_{div}), \quad \boldsymbol{u}_t \in L^{4/d}(0,T;V'_{div}), \qquad d = 2,3$$

$$\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V), \quad \varphi_t \in L^2(0,T;V')$$

$$\varphi \in L^\infty(Q_T), \qquad |\varphi(x,t)| \leq 1 \quad \text{a.e. } (x,t) \in Q_T := \Omega \times (0,T)$$

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- $\forall \boldsymbol{w} \in V_{div}, \forall \psi \in V$ and for a.e. $t \in (0,T)$ we have

$$\langle \boldsymbol{u}_t, \boldsymbol{w} \rangle + 2\left(\nu\left(\varphi\right) D\boldsymbol{u}, D\boldsymbol{w}\right) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) = \Big((a\varphi - K * \varphi)\nabla\varphi, \boldsymbol{w}\Big) + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

$$\langle \varphi_t, \psi \rangle + \int_\Omega m(\varphi) F''(\varphi) \nabla\varphi \cdot \nabla\psi + \int_\Omega m(\varphi) a \nabla\varphi \cdot \nabla\psi$$

$$+ \int_\Omega m(\varphi)(\varphi\nabla a - \nabla K * \varphi) \cdot \nabla\psi = (\boldsymbol{u}\varphi, \nabla\psi)$$

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- the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\varphi(0) = \varphi_0$ hold

Theorem (F., Grasselli, Rocca '15 & F., Gal, Grasselli '15)

Assume ***(K)***, ***(V)***, ***(M)*** *and* ***(A1)–(A4)***. *Let* $\boldsymbol{u}_0 \in H_{div}$ *and* $\varphi_0 \in L^\infty(\Omega)$, $F(\varphi_0) \in L^1(\Omega)$ *and* $M(\varphi_0) \in L^1(\Omega)$, *where* $M \in C^2(-1,1)$ *s.t.* $m(s)M''(s) = 1\ \forall s \in (-1,1)$ *and* $M(0) = M'(0) = 0$. *Let also* $\boldsymbol{v} \in L^2_{loc}([0,\infty);V'_{div})$. *Then,* $\forall T > 0\ \exists$ *a weak sol* $[\boldsymbol{u},\varphi]$ *on* $[0,T]$. *In addition, if* $d = 2$ *the weak sol* $[\boldsymbol{u},\varphi]$ *satisfies the energy equation*

$$\frac{1}{2}\frac{d}{dt}\big(\|\boldsymbol{u}\|^2 + \|\varphi\|^2\big) + \int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 + \int_\Omega am(\varphi)|\nabla\varphi|^2 + \nu\|\nabla\boldsymbol{u}\|^2$$
$$= \int_\Omega m(\varphi)(\nabla K * \varphi - \varphi\nabla a)\cdot\nabla\varphi + \int_\Omega (a\varphi - K*\varphi)\boldsymbol{u}\cdot\nabla\varphi + \langle\boldsymbol{v},\boldsymbol{u}\rangle,$$

for a.a. $t > 0$, *and if* $d = 3$, $[\boldsymbol{u},\varphi]$ *satisfies the following energy inequality*

$$\frac{1}{2}\big(\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2\big) + \int_0^t\int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 + \int_0^t\int_\Omega am(\varphi)|\nabla\varphi|^2$$
$$+\nu\int_0^t\|\nabla\boldsymbol{u}\|^2 \le \frac{1}{2}\big(\|\boldsymbol{u}_0\|^2 + \|\varphi_0\|^2\big) + \int_0^t\int_\Omega m(\varphi)\big(\nabla K * \varphi - \varphi\nabla a\big)\cdot\nabla\varphi$$
$$+\int_0^t\int_\Omega (a\varphi - K*\varphi)\boldsymbol{u}\cdot\nabla\varphi + \int_0^t\langle\boldsymbol{v},\boldsymbol{u}\rangle d\tau, \qquad \forall t > 0$$

Theorem (F., Grasselli, Rocca '15 & F., Gal, Grasselli '15)

*Let $d = 2$, ν constant, and suppose in addition that **(A5)** and **(A6)** are satisfied. Then, the weak sol is also unique. Moreover, let $[\boldsymbol{u}_i, \varphi_i]$ be two weak sols corresponding to two initial data $[\boldsymbol{u}_{0i}, \varphi_{0i}]$ and external force densities $\boldsymbol{v}_i$, with $\boldsymbol{u}_{0i} \in H_{div}$, $\varphi_{0i} \in L^\infty(\Omega)$ s.t. $F(\varphi_{0i}) \in L^1(\Omega)$, $M(\varphi_{0i}) \in L^1(\Omega)$ and $\boldsymbol{v}_i \in L^2_{loc}([0,\infty); V'_{div})$. Then, setting $\boldsymbol{u} := \boldsymbol{u}_2 - \boldsymbol{u}_1$, $\varphi := \varphi_2 - \varphi_1$ and $\boldsymbol{v} := \boldsymbol{v}_2 - \boldsymbol{v}_1$, the following continuous dependence estimate holds*

$$\begin{aligned}
&\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2_{V'} + \|\boldsymbol{u}\|^2_{L^2(0,t;V_{div})} + \|\varphi\|^2_{L^2(0,t;H)} \\
&\leq \left(\|\boldsymbol{u}(0)\|^2 + \|\varphi(0)\|^2_{V'}\right) \Lambda_0(t) \\
&+ |\overline{\varphi}(0)|^2 \Lambda_1(t) + \|\boldsymbol{v}\|^2_{L^2(0,T;V'_{div})} \Lambda_2(t),
\end{aligned}$$

for all $t \in [0,T]$, where Λ_0, Λ_1 and Λ_2 are continuous functions which depend on the norms of the two sols. The functions Λ_i also depend on F, K, ν and Ω

- We need stronger assumptions on K. In particular $K \in W^{2,1}_{loc}(\mathbb{R}^2)$ or K **admissible**

Definition (J. Bedrossian, N. Rodríguez & A. Bertozzi '11)

A kernel $K \in W^{1,1}_{loc}(\mathbb{R}^2)$ is admissible if the following conditions are satisfied:

(Ad1) $K \in C^3(\mathbb{R}^d \setminus \{0\})$

(Ad2) K *is radially symmetric,* $J(x) = \tilde{K}(|x|)$ *and* $\tilde{K}$ *is non-increasing*

(Ad3) $\tilde{K}''(r)$ *and* $\tilde{K}'(r)/r$ *are monotone on* $(0, r_0)$ *for some* $r_0 > 0$

(Ad4) $|D^3K(x)| \leq C_d|x|^{-d-1}$ *for some* $C_d > 0$

Newtonian and Bessel kernels are admissible for all $d \geq 2$

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Lemma (J. Bedrossian, N. Rodríguez & A. Bertozzi '11)

Let K be admissible. Then, $\forall p \in (1, \infty)$ $\exists C_p > 0$ s.t.

$$\|\nabla \chi\|_{L^p(\Omega)} \leq C_p \|\psi\|_{L^p(\Omega)}, \qquad \forall \psi \in L^p(\Omega)$$

where $\chi = \nabla K * \psi$

Theorem

*Suppose **(K)**, **(V)**, **(M)**, **(A1)**$_1$, **(A2)–(A6)** are satisfied and $K \in W^{2,1}_{loc}$ or K admissible. Let $\boldsymbol{u}_0 \in H_{div}$, $\varphi_0 \in V \cap L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$. Let also $\boldsymbol{v} \in L^2(0,T;H_{div})$. Then, $\forall T > 0$ $\exists$ a weak sol $[\boldsymbol{u},\varphi]$ on $[0,T]$ s.t.*

$$\boldsymbol{u} \in L^\infty(0,T;H_{div}) \cap L^2(0,T;V_{div}), \qquad \boldsymbol{u}_t \in L^2\left(0,T;V_{div}'\right)$$
$$\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \qquad \varphi_t \in L^2(0,T;H)$$

Moreover, assume in addition $\boldsymbol{u}_0 \in V_{div}$ and $\varphi_0 \in V \cap C^\beta(\bar{\Omega})$, $\beta \in (0,1)$. Then, $\exists$ (unique) strong sol satisfying the above regularity and

$$\boldsymbol{u} \in L^\infty(0,T;V_{div}) \cap L^2(0,T;H^2(\Omega)^2), \qquad \boldsymbol{u}_t \in L^2(0,T;H_{div})$$

Finally, assume in addition $\varphi_0 \in H^2(\Omega)$ and that the following compatibility condition holds

$$\partial_{\boldsymbol{n}} B(\cdot,\varphi_0) = (\varphi_0)(\nabla K * \varphi_0)\cdot \boldsymbol{n} - N(\varphi_0)(\nabla a \cdot \boldsymbol{n}), \quad \text{on } \partial\Omega$$

Then, the strong sol also satisfies

$$\varphi \in L^\infty(0,T;H^2(\Omega)), \qquad \varphi_t \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

Strategy of the proof I

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- **Difficulty**: since ν **is nonconstant**, we cannot directly apply the classical regularity result for the incompressible NS system in 2D (as done in F., Grasselli & P. Krejčí) to deduce that $\boldsymbol{u} \in L^\infty(V_{div}) \cap L^2(H^2) \Longrightarrow$ **only rely on** $\boldsymbol{u} \in L^\infty(H_{div}) \cap L^2(V_{div})$

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- Rewrite the nonlocal CH as

$$\varphi_t + \boldsymbol{u} \cdot \nabla\varphi = \Delta B(\cdot,\varphi) + \mathrm{div}\big(N(\varphi)\nabla a\big) - \mathrm{div}\big(m(\varphi)(\nabla K * \varphi)\big)$$

where, $\forall s \in [-1,1]$ and a.e $x \in \Omega$, we have set

$$B(x,s) = \int_0^s \beta(x,\sigma)d\sigma, \qquad \beta(x,s) = m(s)\big(a(x) + F''(s)\big)$$

$$N(s) = sm(s) - M(s), \qquad M(s) = \int_0^s m(\sigma)d\sigma$$

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- the BC: $m(\varphi)\nabla\mu \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ can be rewritten as

$$\big[\nabla B(\cdot,\varphi) + N(\varphi)\nabla a - m(\varphi)(\nabla K * \varphi)\big] \cdot \boldsymbol{n} = 0, \qquad \text{on } \partial\Omega$$

Indeed $\nabla B(\cdot,\varphi) = M(\varphi)\nabla a + \beta(\cdot,\varphi)\nabla\varphi$

Strategy of the proof II

- Test nonlocal CH by $B(\cdot,\varphi)_t = \beta(\cdot,\varphi)\varphi_t$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla B(\cdot,\varphi)\|^2 + \int_\Omega \beta(\cdot,\varphi)\varphi_t^2 + \big(\boldsymbol{u}\cdot\nabla\varphi, \beta(\cdot,\varphi)\varphi_t\big)$$
$$= -\big(N(\varphi)\nabla a, \nabla B(\cdot,\varphi)_t\big) + \big(m(\varphi)(\nabla K * \varphi), \nabla B(\cdot,\varphi)_t\big)$$

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- Thanks to **(A5)** and **(A6)** we have

$$\beta(x,s) = m(s)(F_1''(s) + F_2''(s) + a(x)) \geq (1-\rho)m(s)F_1''(s) \geq (1-\rho)\alpha_0$$

  and thanks to **(A1)** we have

$$\beta(x,s) \leq k^*, \qquad k^* = \|mF''\|_{C([-1,1])} + \|m\|_{C([-1,1])}\|a\|_{L^\infty(\Omega)}$$

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- By means of *technical arguments* we are led to

$$\frac{d\Phi}{dt} + (1-\rho)\alpha_0\|\varphi_t\|^2 \leq C_{m,F,K}\left(1 + \|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2\right)\left(1 + \|\nabla\varphi\|^2\right)$$

$$\Phi := \|\nabla B(\cdot,\varphi)\|^2 + 2\big(N(\varphi)\nabla a, \beta(\cdot,\varphi)\nabla\varphi\big) - 2\big(m(\varphi)(\nabla K * \varphi), \beta(\cdot,\varphi)\nabla\varphi\big)$$

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$$\beta(x,s) = m(s)(F_1''(s) + F_2''(s) + a(x)) \geq (1-\rho)m(s)F_1''(s) \geq (1-\rho)\alpha_0$$

  and thanks to **(A1)** we have

$$\beta(x,s) \leq k^*, \qquad k^* = \|mF''\|_{C([-1,1])} + \|m\|_{C([-1,1])}\|a\|_{L^\infty(\Omega)}$$

- By means of *technical arguments* we are led to

$$\frac{d\Phi}{dt} + (1-\rho)\alpha_0\|\varphi_t\|^2 \leq C_{m,F,K}\left(1 + \|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2\right)\left(1 + \|\nabla\varphi\|^2\right)$$

$$\Phi := \|\nabla B(\cdot,\varphi)\|^2 + 2\big(N(\varphi)\nabla a, \beta(\cdot,\varphi)\nabla\varphi\big) - 2\big(m(\varphi)(\nabla K * \varphi), \beta(\cdot,\varphi)\nabla\varphi\big)$$

  We have $\quad K_1\left(\|\nabla\varphi(t)\|^2 - 1\right) \leq \Phi(t) \leq K_2\left(\|\nabla\varphi(t)\|^2 + 1\right)$

Strategy of the proof III

- **By using only** $\boldsymbol{u} \in L^{\infty}(0,T;H_{div}) \cap L^{2}(0,T;V_{div})$, Gronwall's lemma entails

$$\varphi \in L^{\infty}(0,T;V), \qquad \varphi_t \in L^{2}(0,T;H)$$

Strategy of the proof III

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$$\varphi \in L^\infty(0,T;V), \qquad \varphi_t \in L^2(0,T;H)$$

- By comparison in the nonlocal CH we obtain $B \in L^2(0,T;H^2(\Omega))$. On account of

$$\|\nabla B(\cdot,\varphi)\|_{L^4(\Omega)^2} \leq C\left(\|\varphi_t\|^{1/2} + \|\nabla \boldsymbol{u}\|^{1/2} + 1\right)$$

which yields $\varphi, \beta(\cdot,\varphi), B(\cdot,\varphi) \in L^4\left(0,T;W^{1,4}(\Omega)\right)$, and of

$$\partial^2_{ij}\varphi = \frac{1}{\beta}\partial^2_{ij}B - \frac{1}{\beta^2}\partial_i\beta\partial_j B - \frac{M(\varphi)}{\beta}\partial_i\left(\partial_j a\right) - \frac{m(\varphi)}{\beta}\partial_i\varphi\partial_j a + \frac{M(\varphi)}{\beta^2}\partial_i\beta\partial_j a$$

we get also $\varphi \in L^2(0,T;H^2(\Omega))$

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- *Step 2*: establish $\boldsymbol{u} \in L^\infty(0,T;V_{div}) \cap L^2(0,T;H^2(\Omega)^2)$ (assuming $\boldsymbol{u}_0 \in V_{div}$ and $\varphi_0 \in V \cap C^\beta(\overline{\Omega})$)

Strategy of the proof IV

- Test NS by $\boldsymbol{u}_t$. We need to control the H^2-norm of $\boldsymbol{u}$ by the L^2-norm of $\boldsymbol{u}_t$ The tool is a regularity result for the following inhomogeneous Stokes system in non-divergence form

$$\begin{cases} -\omega(x)\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}(x) & \text{in } \Omega \\ \text{div}\,(\boldsymbol{u}) = 0 & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{0.7}$$

Proposition (Y. Sun & Z. Zhang '13)

Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\omega \in C^\delta\left(\overline{\Omega}\right)$, for some $\delta \in (0,1)$, s.t. $0 < \lambda_0 \leq \omega(x) \leq \lambda_1 < \infty$ $\forall x \in \overline{\Omega}$. Then any sol. $[\boldsymbol{u}, \pi] \in H^2(\Omega)^2 \times H^1(\Omega)$ of (0.7) satisfies the estimate

$$\|\boldsymbol{u}\|_{H^2(\Omega)^2} + \|\pi\|_{H^1(\Omega)} \leq C\left(\|\boldsymbol{f}\|_{L^2(\Omega)^2} + \|\pi\|_{L^2(\Omega)}\right),$$

for some constant $C = C(\lambda_0, \lambda_1, \Omega, \|\omega\|_{C^\delta(\overline{\Omega})}) > 0$

Strategy of the proof V

- Hölder continuity for φ can be proven (cf. Ladyženskaja, Solonnikov & Ural'ceva) $\Rightarrow$ $\nu(\varphi) \in C^{\delta,\delta/2}(\bar{\Omega} \times [0,T])$ for some $\delta > 0$

Strategy of the proof V

- Hölder continuity for φ can be proven (cf. Ladyženskaja, Solonnikov & Ural'ceva) $\Rightarrow$ $\nu(\varphi) \in C^{\delta,\delta/2}(\bar{\Omega} \times [0,T])$ for some $\delta > 0$
- The previous result can be applied to the NS system after writing it as

$$-\nu(\varphi)\Delta \boldsymbol{u} + \nabla\hat{\pi} = \boldsymbol{f}$$

where

$$\boldsymbol{f} := (a\varphi - K * \varphi)\nabla\varphi + \boldsymbol{v} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{u}_t + 2\nu'(\varphi)D\boldsymbol{u}\nabla\varphi, \quad \hat{\pi} := \pi - F(\varphi)$$

We get

$$\frac{d}{dt}\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + \frac{1}{8}\|\boldsymbol{u}_t\|^2 \le C\left(\|\boldsymbol{l}\|^2 + \|\boldsymbol{v}\|^2 + \|\nabla\varphi\|^2\right)$$

$$+ C(\|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2 + \|\nabla\varphi\|_{L^4}^4 + \|\varphi_t\|^2)\|D\boldsymbol{u}\|^2, \quad \boldsymbol{l} := -\frac{\varphi^2}{2}\nabla a - (J*\varphi)\nabla\varphi + \boldsymbol{v}$$

Strategy of the proof V

- Hölder continuity for φ can be proven (cf. Ladyženskaja, Solonnikov & Ural'ceva) $\Rightarrow$ $\nu(\varphi) \in C^{\delta,\delta/2}(\bar{\Omega} \times [0,T])$ for some $\delta > 0$
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Hence, on account of the improved regularity for φ in Step 1

$$\boldsymbol{u} \in L^\infty(0,T;V_{div}) \cap L^2(0,T;H^2(\Omega)^2), \quad \boldsymbol{u}_t \in L^2(0,T;H_{div})$$

Strategy of the proof VI

- *Step 3* (Nonlocal CH)$_t \times \varphi_t$ to get

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + (\nabla B(\cdot,\varphi)_t, \nabla\varphi_t) = -(\boldsymbol{u}_t \cdot \nabla\varphi, \varphi_t) - \big(\varphi\, m'(\varphi)\varphi_t \nabla a, \nabla\varphi_t\big)$$
$$+ \big(m'(\varphi)\varphi_t\,(\nabla K * \varphi)\,, \nabla\varphi_t\big) + (m(\varphi)\,(\nabla K * \varphi_t)\,, \nabla\varphi_t)$$

Strategy of the proof VI

- *Step 3* (Nonlocal CH)$_t \times \varphi_t$ to get

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + (\nabla B(\cdot,\varphi)_t, \nabla\varphi_t) = -(\boldsymbol{u}_t\cdot\nabla\varphi,\varphi_t) - \big(\varphi\, m'(\varphi)\varphi_t\nabla a, \nabla\varphi_t\big)$$
$$+\big(m'(\varphi)\varphi_t\,(\nabla K*\varphi)\,,\nabla\varphi_t\big) + (m(\varphi)\,(\nabla K*\varphi_t)\,,\nabla\varphi_t)$$

- By means of some technical arguments and the improved regularities obtained in Step 1 and Step 2 we get

$$\frac{d}{dt}\|\varphi_t\|^2 + \frac{1}{4}\alpha_0(1-\rho)\|\nabla\varphi_t\|^2 \le C\left(\|\varphi_t\|^4 + \|\varphi_t\|^2 + \|\boldsymbol{u}_t\|^2 + 1\right)$$

Strategy of the proof VI

- *Step 3* (Nonlocal CH)$_t \times \varphi_t$ to get

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + (\nabla B(\cdot,\varphi)_t, \nabla\varphi_t) = -(\boldsymbol{u}_t\cdot\nabla\varphi, \varphi_t) - \big(\varphi\, m'(\varphi)\varphi_t\nabla a, \nabla\varphi_t\big)$$
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which entails ($\varphi_t(0)\in H$ is ensured by the condition $\varphi_0\in H^2(\Omega)$)

$$\varphi_t \in L^\infty(0,T;H)\cap L^2(0,T;V)$$

and by comparison in the nonlocal CH we get $B(\cdot,\varphi)\in L^\infty(0,T;H^2(\Omega))$ and also

$$\varphi\in L^\infty(0,T;H^2(\Omega))$$

(Jointly with M. Grasselli and J. Sprekels)

Problem **(CP)**: minimize the **cost functional**

$$J(y,\boldsymbol{v}) := \frac{\beta_1}{2}\|\boldsymbol{u}-\boldsymbol{u}_Q\|^2_{L^2(Q)^2} + \frac{\beta_2}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)} + \frac{\beta_3}{2}\|\boldsymbol{u}(T)-\boldsymbol{u}_\Omega\|^2 + \frac{\beta_4}{2}\|\varphi(T)-\varphi_\Omega\|^2 + \frac{\gamma}{2}\|\boldsymbol{v}\|^2_{L^2(Q)^2}$$

where $y := [\boldsymbol{u},\varphi]$ solves

$$\textbf{(nloc CHNS)}\quad \begin{cases} \boldsymbol{u}_t - 2\text{div}\,(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v} \\ \varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \text{div}(m(\varphi)\nabla\mu) \\ \mu = a\varphi - K*\varphi + F'(\varphi) \\ \text{div}(\boldsymbol{u}) = 0 \qquad \text{in } \Omega\times(0,T) \\ m(\varphi)\nabla\mu\cdot\boldsymbol{n} = 0\,, \qquad \boldsymbol{u} = 0 \qquad \text{on } \partial\Omega \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\,, \qquad \varphi(0) = \varphi_0 \end{cases}$$

and the external body force density $\boldsymbol{v}$, which plays the role of the **control**, belongs to a suitable closed, bounded and convex subset of the **space of controls** $\mathcal{V} := L^2(0,T;H_{div})$

- Introducing the space

$$\mathcal{H} := \left[H^1(0,T;H_{div}) \cap C^0([0,T];V_{div}) \cap L^2(0,T;H^2(\Omega)^2)\right] \\ \times \left[C^1([0,T];H) \cap H^1(0,T;V) \cap C^0([0,T];H^2(\Omega))\right]$$

then, the **control-to-state map**

$$S : \mathcal{V} \to \mathcal{H}, \qquad \boldsymbol{v} \in \mathcal{V} \mapsto S(\boldsymbol{v}) := y := [\boldsymbol{u}, \varphi] \in \mathcal{H}$$

where $y := [\boldsymbol{u}, \varphi]$ is the unique strong sol to Problem **(nloc CHNS)** corresponding to $\boldsymbol{v} \in \mathcal{V}$ and to fixed initial data $\boldsymbol{u}_0 \in V_{div}$, $\varphi_0 \in H^2(\Omega)$, $F(\varphi_0) \in L^1(\Omega)$, $M(\varphi_0) \in L^1(\Omega)$ is well defined

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- With $\boldsymbol{v}_a, \boldsymbol{v}_b \in \mathcal{V} \cap L^\infty(Q)^2$ prescribed, set of **admissible controls**

$$\mathcal{V}_{ad} := \left\{\boldsymbol{v} \in \mathcal{V} : \; v_{a,i}(x,t) \le v_i(x,t) \le v_{b,i}(x,t), \; \text{a.e. } (x,t) \in Q, \; i = 1,2\right\}$$

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- Introducing the **reduced cost functional** $f(\boldsymbol{v}) := J\big(S(\boldsymbol{v}), \boldsymbol{v}\big)$, for all $\boldsymbol{v} \in \mathcal{V}$, then

$$\textbf{(CP)} \iff \min_{\boldsymbol{v} \in \mathcal{V}_{ad}} f(\boldsymbol{v})$$

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$$\textbf{(CP)} \Longleftrightarrow \min_{\boldsymbol{v} \in \mathcal{V}_{ad}} f(\boldsymbol{v})$$

Theorem

*Problem **(CP)** admits a sol* $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$, *with associated state* $\overline{y} := [\overline{\boldsymbol{u}}, \overline{\varphi}] := S(\overline{\boldsymbol{v}})$

- **Aim**: deduce **first order necessary conditions for existence of the optimal control**

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- To this purpose we consider the **linearized system** at $\overline{y} := [\overline{\boldsymbol{u}}, \overline{\varphi}] := S(\overline{\boldsymbol{v}})$

$$\boldsymbol{\xi}_t - 2\text{div}\big(\nu(\bar\varphi)D\boldsymbol{\xi}\big) - 2\text{div}\big(\nu'(\bar\varphi)\eta D\bar{\boldsymbol{u}}\big) + (\bar{\boldsymbol{u}}\cdot\nabla)\,\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\nabla)\,\bar{\boldsymbol{u}} + \nabla\pi^*$$

$$= -\bar\varphi\eta\nabla a + \eta\,(\nabla K * \bar\varphi) + \bar\varphi\,(\nabla K * \eta) + \boldsymbol{h} \qquad \text{in } Q$$

$$\eta_t + \bar{\boldsymbol{u}}\cdot\nabla\eta = -\boldsymbol{\xi}\cdot\nabla\bar\varphi + \text{div}\big(\beta(\cdot,\bar\varphi)\nabla\eta\big) + \text{div}\big(m'(\bar\varphi)\eta\nabla\,(a\bar\varphi - K*\bar\varphi)\,\big)$$

$$+ \text{div}\big(m(\bar\varphi)\,(\eta\nabla a - \nabla K * \eta)\,\big) + \text{div}\big(\eta\lambda'(\bar\varphi)\nabla\bar\varphi\big) \qquad \text{in } Q$$

$$\text{div}(\boldsymbol{\xi}) = 0 \qquad \text{in } Q$$

$$\boldsymbol{\xi} = [0,0]^T \qquad \text{on } \Sigma$$

$$\big[\beta(\cdot,\bar\varphi)\nabla\eta + m'(\bar\varphi)\eta\nabla\big(a\bar\varphi - K*\bar\varphi\big) + m(\bar\varphi)\big(\eta\nabla a - \nabla K*\eta\big) + \eta\lambda'(\bar\varphi)\nabla\bar\varphi\big]\cdot\boldsymbol{n} = 0$$

$$\boldsymbol{\xi}(0) = [0,0]^T, \qquad \eta(0) = 0 \qquad \text{in } \Omega \qquad\qquad \text{on } \Sigma$$

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$$= -\bar\varphi\eta\nabla a + \eta\,(\nabla K * \bar\varphi) + \bar\varphi\,(\nabla K * \eta) + \boldsymbol{h} \qquad \text{in } Q$$
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Proposition

For every $\boldsymbol{h} \in \mathcal{V}$ the linearized problem above has a unique sol satisfying

$$\boldsymbol{\xi} \in C\big([0,T];H_{div}\big) \cap L^2\big(0,T;V_{div}\big), \quad \eta \in C([0,T];H) \cap L^2(0,T;V)$$

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$$
\begin{aligned}
&\boldsymbol{\xi}_t - 2\mathrm{div}\big(\nu(\bar\varphi)D\boldsymbol{\xi}\big) - 2\mathrm{div}\big(\nu'(\bar\varphi)\eta D\bar{\boldsymbol{u}}\big) + (\bar{\boldsymbol{u}}\cdot\nabla)\,\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\nabla)\,\bar{\boldsymbol{u}} + \nabla\pi^* \\
&\ = -\bar\varphi\eta\nabla a + \eta\,(\nabla K * \bar\varphi) + \bar\varphi\,(\nabla K * \eta) + \boldsymbol{h} \qquad \text{in } Q \\
&\eta_t + \bar{\boldsymbol{u}}\cdot\nabla\eta = -\boldsymbol{\xi}\cdot\nabla\bar\varphi + \mathrm{div}\big(\beta(\cdot,\bar\varphi)\nabla\eta\big) + \mathrm{div}\big(m'(\bar\varphi)\eta\nabla\,(a\bar\varphi - K*\bar\varphi)\,\big) \\
&\ + \mathrm{div}\big(m(\bar\varphi)\,(\eta\nabla a - \nabla K * \eta)\,\big) + \mathrm{div}\big(\eta\lambda'(\bar\varphi)\nabla\bar\varphi\big) \qquad \text{in } Q \\
&\mathrm{div}(\boldsymbol{\xi}) = 0 \qquad \text{in } Q \\
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Remark. States $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}]$ need to be **strong sols** to **(nloc CHNS)**

Set

$$\mathcal{Z} := \big[C\big([0,T];H_{div}\big)\cap L^2\big(0,T;V_{div}\big)\big]\times\big[C([0,T];H)\cap L^2(0,T;V)\big]$$

Theorem

The control-to-state operator $S:\mathcal{V}\to\mathcal{Z}$ is Frechét differentiable on $\mathcal{V}$ and the Frechét derivative $S'(\overline{\boldsymbol{v}})\in\mathcal{L}(\mathcal{V},\mathcal{Z})$ is given by

$$S'(\overline{\boldsymbol{v}})\boldsymbol{k} = [\boldsymbol{\xi}^{\boldsymbol{k}},\eta^{\boldsymbol{k}}],\qquad \forall\boldsymbol{k}\in\mathcal{V},$$

where $[\boldsymbol{\xi}^{\boldsymbol{k}},\eta^{\boldsymbol{k}}]$ is the unique sol to the linearized system at $[\overline{\boldsymbol{u}},\overline{\varphi}]=S(\overline{\boldsymbol{v}})$ and corresponding to $\boldsymbol{k}\in\mathcal{V}$

Key tool for the proof: stability estimates

Lemma (Stability estimate I)

Suppose that ***(K)***, ***(V)***, ***(M)***, ***(A1)***$_1$, ***(A2)–(A7)*** *are satisfied and that* $K \in W^{2,1}$ *or* K *admissible. Assume that controls* $\boldsymbol{v}_i \in \mathcal{V}$, $i = 1, 2$, *are given and that* $[\boldsymbol{u}_i, \varphi_i] := \mathcal{S}(\boldsymbol{v}_i)$, $i = 1, 2$, *are the associated sols to* (0.1)–(0.4). *Then,* $\exists$ *a continuous function* $\mathbb{Q}_2 : [0, \infty)^2 \to [0, \infty)$, *which is nondecreasing in both its arguments and only depends on the data* F, m, K, ν, Ω, T, $\boldsymbol{u}_0$ *and* φ_0, *s.t. we have for every* $t \in (0, T]$

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{C^0([0,t];H_{div})} + \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^2(0,t;V_{div})} + \|\varphi_2 - \varphi_1\|^2_{C^0([0,t];H)}$$

$$+ \|\varphi_2 - \varphi_1\|^2_{L^2(0,t;V)} \leq \mathbb{Q}\big(\|\boldsymbol{v}_1\|_{L^2(0,T;H_{div})}, \|\boldsymbol{v}_2\|_{L^2(0,T;H_{div})}\big) \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{L^2(0,T;V'_{div})}$$

Key tool for the proof: stability estimates

A slight strengthening of **(M)** and of **(A1)** is required, namely

(M)$_1$ The mobility satisfies **(M)** and also $m \in C^2([-1,1])$

(A1)$_2$ $F \in C^4(-1,1)$ and $mF'' \in C^2([-1,1])$

Lemma (Stability estimate II)

*Suppose that **(K)**, **(V)**, **(M)**$_1$, **(A1)**$_2$, **(A2)–(A7)** are satisfied and that $K \in W^{2,1}$ or K admissible. Assume that controls $\boldsymbol{v}_i \in \mathcal{V}$, $i = 1, 2$, are given and that $[\boldsymbol{u}_i, \varphi_i] := \mathcal{S}(\boldsymbol{v}_i)$, $i = 1, 2$, are the associated sols to* (0.1)–(0.4). *Then, $\exists$ a continuous function $\mathbb{Q}_2 : [0,\infty)^2 \to [0,\infty)$, which is nondecreasing in both its arguments and only depends on the data F, m, K, ν, Ω, T, $\boldsymbol{u}_0$ and φ_0, s.t. we have for every $t \in (0,T]$*

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{C^0([0,t];H_{div})} + \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^2(0,t;V_{div})} + \|\varphi_2 - \varphi_1\|^2_{C^0([0,t];V)}$$

$$+ \|\varphi_2 - \varphi_1\|^2_{L^2(0,t;H^2(\Omega))} \le \mathbb{Q}\big(\|\boldsymbol{v}_1\|_{L^2(H_{div})}, \|\boldsymbol{v}_2\|_{L^2(H_{div})}\big) \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{L^2(0,T;V'_{div})}$$

If $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ is an optimal control for Problem **(CP)**, then

$$f'(\overline{\boldsymbol{v}})(\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

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But

$$f'(\boldsymbol{v}) = J'_y\big(S(\boldsymbol{v}), \boldsymbol{v}\big) S'(\boldsymbol{v}) + J'_{\boldsymbol{v}}\big(S(\boldsymbol{v}), \boldsymbol{v}\big)$$

and hence the Frechét differentiability result for $S : \mathcal{V} \to \mathcal{Z}$ yields

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Corollary

*Let $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ be an optimal control for Problem **(CP)** with associated state $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}] := S(\overline{\boldsymbol{v}})$. Then*

$$\beta_1 \int_0^T \int_\Omega (\overline{\boldsymbol{u}} - \boldsymbol{u}_Q) \cdot \boldsymbol{\xi}^{\boldsymbol{h}} + \beta_2 \int_0^T \int_\Omega (\overline{\varphi} - \varphi_Q)\eta^{\boldsymbol{h}} + \beta_3 \int_\Omega (\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega) \cdot \boldsymbol{\xi}^{\boldsymbol{h}}(T)$$

$$+ \beta_4 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\eta^{\boldsymbol{h}}(T) + \gamma \int_0^T \int_\Omega \overline{\boldsymbol{v}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

where $[\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$ is the unique sol to the linearized system corresponding to $\boldsymbol{h} = \boldsymbol{v} - \overline{\boldsymbol{v}}$

Aim: eliminate $\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}$ from the previous inequality. Hence, introduce the **adjoint system**

$$\widetilde{\boldsymbol{p}}_t = -\, 2\,\mathrm{div}\big(\nu(\overline{\varphi})\, D\widetilde{\boldsymbol{p}}\big) - (\overline{\boldsymbol{u}}\cdot\nabla)\,\widetilde{\boldsymbol{p}} + (\widetilde{\boldsymbol{p}}\cdot\nabla^T)\,\overline{\boldsymbol{u}} + \widetilde{q}\,\nabla\overline{\varphi} - \beta_1(\overline{\boldsymbol{u}} - \boldsymbol{u}_Q)$$

$$\begin{aligned}\widetilde{q}_t = &-\mathrm{div}\big(\beta(\cdot,\overline{\varphi})\nabla\widetilde{q}\big) + m'(\overline{\varphi})\nabla(a\overline{\varphi} - K*\overline{\varphi})\cdot\nabla\widetilde{q} + m(\overline{\varphi})\nabla a\cdot\nabla\widetilde{q}\\ &-\nabla K\dot{*}(m(\overline{\varphi})\nabla\widetilde{q}) + \lambda'(\overline{\varphi})\nabla\overline{\varphi}\cdot\nabla\widetilde{q} - (\nabla K*\overline{\varphi})\cdot\widetilde{\boldsymbol{p}} - \nabla K*(\overline{\varphi}\,\widetilde{\boldsymbol{p}})\\ &+ 2\nu'(\overline{\varphi})\,D\overline{\boldsymbol{u}} : D\widetilde{\boldsymbol{p}} + \overline{\varphi}\,\widetilde{\boldsymbol{p}}\cdot\nabla a - \overline{\boldsymbol{u}}\cdot\nabla\widetilde{q} - \beta_2(\overline{\varphi} - \varphi_Q)\end{aligned}$$

$$\mathrm{div}(\widetilde{\boldsymbol{p}}) = 0$$

$$\widetilde{\boldsymbol{p}} = 0, \qquad \frac{\partial\widetilde{q}}{\partial\boldsymbol{n}} = 0 \quad \text{on } \Sigma$$

$$\widetilde{\boldsymbol{p}}(T) = \beta_3(\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega), \qquad \widetilde{q}(T) = \beta_4(\overline{\varphi}(T) - \varphi_\Omega)$$

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Proposition

The adjoint system has a unique weak sol $\widetilde{\boldsymbol{p}}, \widetilde{q}$ satisfying

$$\widetilde{\boldsymbol{p}} \in C\big([0,T];H_{div}\big)\cap L^2\big(0,T;V_{div}\big), \quad \widetilde{q}\in C([0,T];H)\cap L^2(0,T;V)$$

Theorem

Let $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ be an optimal control for Problem **(CP)** *with associated state $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}] = S(\overline{\boldsymbol{v}})$ and adjoint state $[\widetilde{\boldsymbol{p}}, \widetilde{q}]$. Then*

$$\gamma \int_0^T \int_\Omega \overline{\boldsymbol{v}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) + \int_0^T \int_\Omega \widetilde{\boldsymbol{p}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

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- The system **(nloc CHNS)**, written for $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, the adjoint system and the above variational inequality form together the first order necessary optimality conditions

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- The system **(nloc CHNS)**, written for $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, the adjoint system and the above variational inequality form together the first order necessary optimality conditions
- Since $\mathcal{V}_{ad}$ is a nonempty, closed and convex subset of $L^2(Q)^2$, then the above variational inequality with $\gamma > 0$ is equivalent to

$$\overline{\boldsymbol{v}} = P_{\mathcal{V}_{ad}}\Big(\Big\{ -\frac{\widetilde{\boldsymbol{p}}}{\gamma} \Big\}\Big)$$

where $P_{\mathcal{V}_{ad}}$ is the orthogonal projector in $L^2(Q)^2$ onto $\mathcal{V}_{ad}$