Stability of solutions to nonlinear wave equations

Genni Fragnelli

Mathematics Department

University of Bari

Optimal Control for Evolutionary PDEs and Related Topics

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Model:

$$(P)\begin{cases} u'' + h(t)u' = \Delta u + f(u) & \text{in } (0,+\infty)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,+\infty)\times\Gamma, \\ u(0,x) = u_0(x),\ \ u'(0,x) = u_1(x) & x\in\Omega, \end{cases}$$

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- h "damping",
- $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$,
- $\Omega \subset \mathbb{R}^N$ bounded and smooth, $N \geq 1$, $\Gamma := \partial\Omega$.

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Question: asymptotic stability of solutions, i.e.

$$\text{“} \lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0 \text{”}.$$

Two competitors: h and f

Prototype: $f(u) = -|u|^p u,\ p > 0.$

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Remark

If $f(u) = |u|^p u,\ p > 0$ and $h \geq 0$, then solutions may blow up in finite time:
Levine, SIAM J. Math. Anal. (1974);
Levine - Park - Serrin, J. Math. Anal. Appl. (1998).

If $u'' + h(t)u' + u = 0$, set $E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|u(t)|^2$.

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If u solves (P), denote by E_u the associated energy

$$E_u(t) = \frac{1}{2}\|u'(t)\|_H^2 + \frac{1}{2}\|u(t)\|_V^2\, dx - \mathcal{F}(u),$$

where

- ⋆ $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$
- ⋆ $\mathcal{F}(u) := \int_\Omega F(u)dx,\ F(s) := \int_0^s f(s)ds$:
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For the prototype $\mathcal{F}(u) = -\displaystyle\int_\Omega \frac{|u|^{p+2}}{p+2}\, dx$.

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Lemma

If u solves (P), then $E_u \in AC(0,\infty)$ and

$$E_u'(t) = -\langle h(t)u'(t), u'(t)\rangle_H \qquad \text{a.e. in } (0,\infty).$$

ODE's or Systems of ODE's with $h \geq 0$:

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f linear: Hatvani, P.AMS (1996);
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f nonlinear: Pucci - Serrin, Acta Math. (1993),
SIAM J. Math. Anal. (1994), JDE (1994)

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$f \equiv 0$: Cox - Zuazua, Comm. PDE (1994);
López-Gómez, JDE (1997) ($h = h(x) > 0$);
Martinez, Israel JM (2000) (h increasing > 0);
Alabau-Boussouira, CRAS (2004) (h increasing > 0);
Haraux - Martinez - Vancostenoble, SIAM J. Contr. Opt. (2005) ($h \geq 0$);

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Haraux - Martinez - Vancostenoble, SIAM J. Contr. Opt. (2005) ($h \geq 0$);

f nonlinear with linear growth: Zhang, D.E.D.S. (1994) ($h \geq 0$);

f "generic": Messaoudi, Int. J. Appl. Math. (2000) (h constant > 0);

Pucci - Serrin, Comm. Pure Appl. Math. (1996) (h continuous+growth condition > 0);

Lasiecka - Toundykov, Nonlinear Anal. (2006) ($h = h(x) > 0$);

F. - Mugnai, SIAM J. Control Optim. (2008) ($h \geq 0$);

F. - Mugnai, Discrete Contin. Dyn. Syst. - S (2011) (h changes sign).

In F. - Mugnai 2008:

- h is *integrally positive* i.e. $\forall\, a > 0\ \exists\ \delta > 0$ s.t.

$$\int_t^{t+a} h(s)\, ds \geq \delta \qquad \forall\, t > 0$$

(as in Zhang, f with linear growth)

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• h is *on - off*
(as in Haraux-Martinez-Vancostenoble, $f \equiv 0$)


GEWA

On–off damping

$(t_n, t_{n+1})_n$ sequence of open disjoint intervals of $(0,\infty)$, $t_n \to \infty$, and there exist $M_n \geq m_n > 0$ such that

$$m_n \leq h(t) \leq M_n \qquad \forall\, t \in (t_n, t_{n+1}).$$

Theorem (F. - Mugnai, SIAM J. Cont. Opt. 2008)

Assume $sf(s) \leq \int_0^s f(\sigma)\, d\sigma, \forall\, s \in \mathbb{R}$. Any solution $u(u_0, u_1)$ of (1) is *asymptotically stable*, i.e. $\|u\|_V + \|u'\|_H \to 0$ as $t \to \infty$, if

$$(*) \sum_{n=1}^{\infty} m_n (t_{n+1} - t_n) \min\left((t_{n+1} - t_n)^2, \frac{1}{1 + m_n M_n} \right) = \infty.$$

- We don't know if $(*)$ is also necessary.
- $(*)$ essentially introduced by Pucci-Serrin for ODE's.

On–off damping

Theorem (Abstract Lemma)

- **Given** $a, b, M, m > 0$ **s.t.**

$$m \leq h(t) \leq M \qquad \forall\, t \in [a, b].$$

- $sf(s) \leq \int_0^s f(\sigma)\, d\sigma \;\forall\, s \in \mathbb{R}$

$$\Downarrow$$

$\forall\, (u_0, u_1) \in V \times H$ **any solution** u **of problem (P) satisfies**

$$E_u(b) \leq \frac{1}{1 + \frac{(b-a)^3}{30} \frac{1}{\frac{4}{\lambda_1} + \frac{3(b-a)^2}{32m} + \frac{M(b-a)^2}{16\lambda_1}}} E_u(a).$$

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- h is *integrally positive* i.e. $\forall\, a > 0\ \exists\ \delta > 0$ s.t.

$$\int_t^{t+a} h(s)\, ds \geq \delta \qquad \forall\, t > 0$$

(as in Zhang, f with linear growth)

- h is *on - off*
(as in Haraux-Martinez-Vancostenoble, $f \equiv 0$)

In F. - Mugnai 2011:

- h is *positive - negative*.

Sign–changing damping:
$h \leq 0$ in $I_{2n+1} := (t_{2n+1}, t_{2n+2})$ and $h > 0$ in $I_{2n} := (t_{2n}, t_{2n+1})$, $t_{2n} < t_{2n+1}, t_n \to +\infty$.

Theorem (F. - Mugnai, DCDS - S 2011)

- $\forall\, n \in \mathbb{N}\ \exists\, m_{2n},\, M_{2n},\, M_{2n+1} > 0$ such that

$$m_{2n} \le h(t) \le M_{2n} \qquad \forall\, t \in I_{2n},$$

$$-M_{2n+1} \le h(t) \le 0 \qquad \forall\, t \in I_{2n+1}.$$

- $sf(s) \le \int_0^s f(\sigma)\, d\sigma \le 0 \quad \forall\, s \in \mathbb{R}$,
- $$\sum_{n=0}^{\infty} \Bigg[2M_{2n+1}|I_{2n+1}| - \log\left(1 + \frac{|I_{2n}|^3}{\frac{120}{\lambda_1 m_{2n}} + \frac{45|I_{2n}|^2}{16 m_{2n}} + \frac{15 M_{2n}|I_{2n}|^2}{8\lambda_1}}\right)\Bigg] = -\infty,$$
- u solves (P).

$$\Downarrow$$

u is asymptotically stable.

Problem: Asymptotic stability for models with delay.
Example:

$$\begin{cases} u'' = \Delta u, & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0, & \text{on } (0,\infty)\times\Gamma_D, \\ \dfrac{\partial u}{\partial \nu}(t,x) = -\mu_1 u'(t,x) - \mu_2 u'(t-\tau,x), & \text{on } (0,\infty)\times\Gamma_N, \\ u(0,x) = u_0(x) \text{ and } u'(0,x) = u_1(x), & \text{in } \Omega, \\ u'(t-\tau,x) = f_0(t-\tau,x), & \text{in } (0,\tau)\times\Gamma_N, \end{cases}$$

- $\Gamma(=\partial\Omega) = \Gamma_D \cup \Gamma_N$, $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$ and $\Gamma_D \neq \emptyset$
- $\nu(x)$ is the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative
- $\tau, \mu_1, \mu_2 > 0$
- (u_0, u_1, f_0) belong to suitable spaces

Stability Results:

$\star$ $\mu_2 = 0 \Rightarrow$ asymptotic stability holds

- Chen, SIAM J. Control Optim. (1979), (1981);
- Lagnese, J. Differential Equations, 1983 or SIAM J. Control Optim. (1988);
- Lasieska - Triggiani, J. Differential Equations (1987);
- Komornik - Zuazua, J. Math. Pures Appl. (1990);

$\star$ $\mu_1 = 0 \Rightarrow$ system is unstable

- Datko - Lagnese - Polis, SIAM J. Control Optim. (1986);

$\star$ $\mu_2 < \mu_1 \Rightarrow$ asymptotic stability holds

$\star$ $\mu_2 > \mu_1 \Rightarrow$ system is unstable

$\star$ $\mu_2 = \mu_1 \Rightarrow$ instabilities may occur

- Nicaise - Pignotti, J. Differential Equations (2006);
- Xu - Yung - Li, ESAIM Control Optim. Calc. Var. (2006).

Stability results for distributed damping with delay

$$(P)\begin{cases} u'' + h_1(t)u' + h_2(t)u'(t-\tau, x) = \Delta u + f(u) & \text{in } Q, \\ u(t,x) = 0 & \text{in } (0,\infty)\times\Gamma, \\ u(0,x) = u_0(x),\ u'(0,x) = u_1(x) & x \in \Omega, \end{cases}$$

- $Q := (0,+\infty)\times\Omega, \Gamma := \partial\Omega$, u_0, u_1, Ω are as before.
- f as in F. - Mugnai, 2008 or F. - Mugnai, 2011.
- $\tau \leq T_{2n}$, $\forall n \in \mathbb{N}$, where $T_{2n} := t_{2n+1} - t_{2n}$
$(I_{2n} := (t_{2n}, t_{2n+1}))$.
- h_i, $i = 1, 2$, "damping" s.t.

$$h_1(t)h_2(t) = 0, \forall\, t > 0$$

(In particular $h_2(t) = 0, \forall\, t \in I_{2n}$ and $h_1(t) = 0$, $\forall\, t \in I_{2n+1}$).

Question: asymptotic stability of solutions, i.e.

$$\text{“} \lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0 \text{”}.$$

$f \equiv 0$: Nicaise - Pignotti, Adv. Differential Equations (2012).
$h_2 \equiv 0$: F. - Mugnai, (2008).

Theorem (F. - Pignotti, Dynamics of PDE 2016)

$\exists\ m_{2n}, M_{2n}, M_{2n+1}$, s.t. $\forall\ u \in H := L^2(\Omega)$:
i) $h_1 \in C^1(\bar{I}_{2n})$, $0 < m_{2n} \leq h_1(t) \leq M_{2n}$ and $h_2(t) = 0$ $\forall\ t \in I_{2n}$, $\forall\ n \in \mathbb{N}$,
ii) $h_2 \in C^1(\bar{I}_{2n+1})$, $|h_2(t)| \leq M_{2n+1}$ and $h_1(t) = 0$ $\forall\ t \in I_{2n+1}$, $\forall\ n \in \mathbb{N}$.

$$\Downarrow$$

$$\sum_{n=0}^{\infty} M_{2n+1} T_{2n+1} < +\infty \text{ and } \sum_{n=0}^{\infty} \ln \tilde{c}_n = -\infty$$

are sufficient for asymptotic stability to hold. Here

$$\tilde{c}_n = \frac{1}{1 + \frac{T_{2n}^3}{30} \frac{1}{\frac{4}{\lambda_1 m_{2n}} + \frac{3T_{2n}^2}{32 m_{2n}} + \frac{M_{2n} T_{2n}^2}{16\lambda_1}}} + M_{2n+1} T_{2n+1}$$

$\tilde{c}_n$ follows by the Abstract Lemma applied in I_{2n} (F. - Mugnai, SIAM J. Cont. Opt. 2008)

Remark

Under the previous assumptions $E(t) \to 0$, as $t \to +\infty$, where

$$E(t) = E_u(t) + \frac{\xi}{2} \int_{t-\tau}^{t} |h_2(s+\tau)| \int_{\Omega} (u')^2(s,x)\,dx\,ds.$$

In general E is not decreasing as it happens for the standard energy E_u. For a decay estimate we should assume (as in Nicaise -Pignotti, J. Dynam. Differential Equations 2014)

$$\inf_{n\in\mathbb{N}} \frac{m_{2n}}{M_{2n+1}} > 0$$

and

$$\xi \in \left(0, \inf_{n\in\mathbb{N}} \frac{m_{2n}}{M_{2n+1}}\right)$$

More general problems

$$\begin{cases} u'' + h_1(t)g(u') + h_2(t)u'(t-\tau) = \Delta u + f(u) & \text{in } Q, \\ u(t,x) = 0 & \text{in } (0,\infty)\times\Gamma, \\ u(0,x) = u_0(x),\ u'(0,x) = u_1(x) & x \in \Omega, \end{cases}$$

$g : \mathbb{R} \longrightarrow \mathbb{R}$ C^1, $g(0) = 0$ and $0 < m \leq g'(v) \leq M$ $\forall\, v \in \mathbb{R}$;
$u_0 \in H^1_0(\Omega)$ and $u_1 \in H := L^2(\Omega)$:

as long as i) is replaced by
$i')$: $m_{2n}\|v\|^2_H \leq \int_\Omega h_1(t)g(v)v dx \leq M_{2n}\|v\|^2_H$ $\forall v \in H$,
$\forall\, t \in I_{2n}$, $\forall\, n \in \mathbb{N}$.

More general problems

$$\begin{cases} u'' + h_1(t)g(u') + h_2(t)u'(t-\tau) = \Delta u + f(u) & \text{in } Q, \\ u(t,x) = 0 & \text{in } (0,\infty)\times\Gamma, \\ u(0,x) = u_0(x),\ u'(0,x) = u_1(x) & x\in\Omega, \end{cases}$$

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$\forall\, t \in I_{2n}$, $\forall\, n \in \mathbb{N}$.

$$\begin{cases} u'' + h_1(t)g(u') + h_2(t)u'(t-\tau) = \Delta^{2m} u + f(u) & \text{in } Q, \\ Cu(t,x) = 0 \in \mathbb{R}^{2m} & \text{in } (0,\infty)\times\Gamma, \\ u(0,x) = u_0(x) \in D(\Delta^m), u'(0,x) = u_1(x) & x\in\Omega, \end{cases}$$

$m \in \mathbb{N}$, g as before and C is a boundary operator.

Example:

$m = 1$ and $Cu = (u, \partial u/\partial\nu)$ with $D(\Delta) = H^2_0(\Omega)$ or
$Cu = (u, \Delta u)$ with $D(\Delta) = H^1_0(\Omega) \cap H^2(\Omega)$.

Stability results also for localized damping with delay

$$(P)\begin{cases} u''+h_1(t)\chi_\omega u'+h_2(t)\chi_{\tilde\omega}u'(t-\tau)=\Delta u+f(u) & \text{in } Q,\\ u(t,x)=0 & \text{in } (0,\infty)\times\Gamma,\\ u(0,x)=u_0(x),\ u'(0,x)=u_1(x) & x\in\Omega,\end{cases}$$

- $Q := (0,+\infty)\times\Omega, \Gamma := \partial\Omega.$
- u_0, u_1, Ω are as before.
- f as in F. - Mugnai, 2008 or F. - Mugnai, 2011.
- $\tau \le T_{2n}$, $\forall n\in\mathbb{N}$, where $T_{2n} := t_{2n+1}-t_{2n}$ $(I_{2n} := (t_{2n}, t_{2n+1}))$.
- h_i, $i=1,2$, as before.
- $\tilde\omega \subset \omega \subset \Omega$.

Question: asymptotic stability of solutions, i.e.

$$\text{“}\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0\text{”}.$$

Hypothesis

⋆ $\exists\ m_{2n}, M_{2n}, M_{2n+1}$ s.t. $\forall\ u \in L^2(\Omega)$ i) and ii) hold.
⋆ $\exists\ \{\bar{T}_n\}$ s.t. $\forall n \in \mathbb{N}\ T_{2n} > \bar{T}_n$ and $\forall\ T \in (\bar{T}_n, T_{2n})$
$\exists\ d_n(T) > 0$ s.t., for every solution of (P) in I_{2n},

$$(OI) \quad E_u(t_{2n} + T) \le d_n \int_{t_{2n}}^{t_{2n}+T} \int_{\omega} h_1(t)(u')^2(t, x)dxdt.$$

Remark

(OI) holds if f is G. Lipschitz or $(2+\delta)F(s) \ge sf(s)$, for some $\delta > 0$. Here $F(s) := \int_0^s f(t)dt$ (Zuazua, Comm. Partial Differential Equations, 1990)

Theorem (F. - Pignotti, Dynamics of PDE 2016)

Assume the previous Hypothesis. If

$$\sum_{n=0}^{\infty} M_{2n+1}T_{2n+1} < +\infty \text{ and } \sum_{n=0}^{\infty} \ln \hat{d}_n = -\infty$$

$$\Downarrow$$

u is asymptotically stable. Here

$$\hat{d}_n = \frac{d_n}{d_n + 1}.$$

Proof: $E_u(t_{2n+1}) \leq \hat{d}_n E_u(t_{2n}), \quad \forall\, n \in \mathbb{N}.$

Remark

If $f \equiv 0$: this result improves the one of Nicaise - Pignotti, J. Dynam. Differential Equations 2014, by removing

$$\inf_{n\in\mathbb{N}} \frac{m_{2n}}{M_{2n+1}} > 0,$$

crucial for their result.

Some extensions:

- $h_1(t)\chi_\omega u'$ replaced by $h_1(t)\chi_\omega g(u')$:

i') is replaced by

$ii')$: $m_{2n}\|v\|^2_{L^2(\omega)} \leq \int_\omega h_1(t)g(v)v dx \leq M_{2n}\|v\|^2_{L^2(\omega)}$
$\forall\ v \in H, \forall\ t \in I_{2n}, \forall\ n \in \mathbb{N}$.

Remark

(OI) holds if f is G. Lipschitz or $(2+\delta)F(s) \geq sf(s)$, for some $\delta > 0$ and $\exists\ c > 0$ s.t. $g(s)s \geq c|s|^2$, $\forall s \in \mathbb{R}$.
(Zuazua, Comm. Partial Differential Equations, 1990)

- distributed positive - negative damping with delay
- localized positive - negative damping without delay
- localized positive - negative damping with delay