Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

A phase field approach for optimal boundary control of damage processes in two-dimensional viscoelastic media

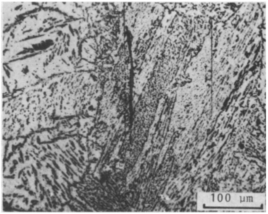
M. Hassan Farshbaf-Shaker, Christian Heinemann (WIAS)

Mohrenstrasse 39 · 10117 Berlin · Germany · Tel. +49 30 20372 0 · www.wias-berlin.de

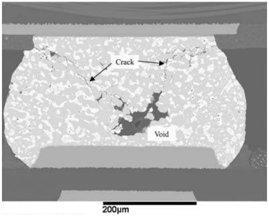
June 20-24

Overview: global-in-time existence, uniqueness and optimal control results for a phase field system describing damage processes

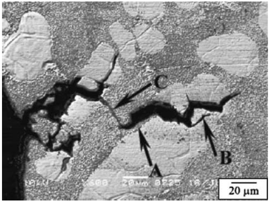
- **Introduction**
  - phase field model for damage
  - mathematical setting
- **Well-posedness** of strong solutions
  - approximation and discretization scheme with partial convex-concave splittings
  - energy estimates, enhanced regularity estimates, limit analysis
- **Optimal boundary control**
  - existence of minimizers
  - adjoint system for the regularized problem



Steel



Lead-Tin alloy



Silver-Tin alloy

- Mathematical modeling and investigation of damage phenomena (highly nonlinear) is of great interest
- Controlling the growth of damage is an important engineering task: Suppose that a workpiece is exposed to external forces during an experiment and that certain parameters related to those forces can be controlled. A control problem could be to choose optimal parameters in order to prevent further damage in the material or to redirect crack spreading to non-critical components of the structure and to avoid complete failure.
- Another problem might be the determination of external forces in order to deliberately induce a damage progression. For instance, it might be desirable to separate certain parts of the workpiece in industrial processes.

Viscous/rate-dependent model for damage processes (PFDM):[1,2]

$$\partial I_{(-\infty,0]}(\chi_t)+\chi_t-\Delta\chi-\Delta\chi_t+\partial_\chi W(\chi,\varepsilon)\ni 0 \text{ in } Q, \quad \partial_\nu\chi=\partial_\nu\chi_t=0 \text{ on } \Sigma$$

coupled with force balance equation

$$u_{tt}-\mathrm{div}\big(\partial_\varepsilon W(\chi,\varepsilon)+\mathbb{D}(\chi)\varepsilon_t\big)=\ell \quad \text{in } Q, \quad \big(\partial_\varepsilon W(\chi,\varepsilon)+\mathbb{D}(\chi)\varepsilon_t\big)\nu=b \quad \text{on } \Sigma$$

with linear elasticity

$$W(\chi,\varepsilon)=\frac{1}{2}\mathbb{C}(\chi)\varepsilon:\varepsilon+f(\chi), \qquad \varepsilon=\frac{1}{2}(\nabla u+(\nabla u)^t)$$

Interpretation: The phase field χ corresponds to density of micro-defect (-voids)

$$\chi(x)=\begin{cases}1 & \leftrightarrow \text{ no damage in } x,\\ \in(0,1) & \leftrightarrow \text{ partial damage in } x,\\ 0 & \leftrightarrow \text{ maximal damage in } x.\end{cases}$$

$\lightning\lightning$ **note** in our case we merely know $\chi(x)\le 1$ provided that $\chi(x,0)\in[0,1]$ $\lightning\lightning$

Taking into account an additional indicator function $I_{[0,1]}(\chi)$ would result in a doubly nonlinear model (double inclusion) and leads to non uniqueness of solutions.

- continuum damage model: continuum-mechanics based framework allowing to characterize, represent and model at the macroscopic scale the effects of distributed defects and their growth on the material behavior.
- obtained via (thermodynamic consistent) modeling approach from Frémond & Nedjar[1,2]
- sub-differential $\partial I_{(-\infty,0]}(\chi_t)$: Irreversibilty
  $\longrightarrow$ healing forbidden, i.e. $\chi_t \leq 0 \;\Leftrightarrow\; \chi(t) \leq \chi(s)$ for all $s \leq t$
- some possibilities:
  - $\longrightarrow$ set $\mathbb{C}(\chi) = \mathbb{C}(0)$ and $\mathbb{D}(\chi) = \mathbb{D}(0)$ for all $\chi < 0$
  - $\longrightarrow$ or use model with different interpretation[2]

$$\chi(x) = \begin{cases} 0 & \leftrightarrow \text{ no damage in } x, \\ \in (0, +\infty) & \leftrightarrow \text{ partial damage in } x, \\ \to +\infty & \leftrightarrow \text{ maximal damage in } x. \end{cases}$$

Reference:

[2] *B. J. Dimitrijevic, K. Hackl: A Method for Gradient Enhancement of Continuum Damage Models, Technische Mechanik, Band 28, Heft 1 (2008)*

Reference:

[1] *M. Frémond, B. Nedjar: Damage, gradient of damage and principle of virtual power. Int. J. Solids S. (1996)*

[2] *M. Frémond: Non-smooth thermomechanics, Springer (2002)*

Dissipation potential and Free energy

$$R(u_t,\chi_t) = \int_\Omega \Big(\frac{1}{2}|\chi_t|^2 + \frac{1}{2}|\nabla\chi_t|^2 + \frac{1}{2}\mathbb{D}(\chi)\varepsilon(u_t):\varepsilon(u_t) + I_{(-\infty,0]}(\chi_t)\Big)\,\mathrm{d}x$$

$$F(u,\chi) = \int_\Omega \Big(\frac{1}{2}|\nabla\chi|^2 + W(\chi,\varepsilon)\Big)\,\mathrm{d}x, \quad W(\chi,\varepsilon) = \frac{1}{2}\mathbb{C}(\chi)\varepsilon:\varepsilon + f(\chi)$$

dependence of dissipation on gradients of velocities assumed

$\rightarrow$ (higher-order) viscosities: $-\Delta\chi_t$ and $-\mathrm{div}(\mathbb{D}(\chi)\varepsilon_t)$

$$0 \in \partial_{\chi_t} R(u_t,\chi_t) + d_\chi F(u,\chi)$$

or equivalently $\xi = -\chi_t + \Delta\chi_t - d_\chi F(u,\chi)$ and $\xi \in \partial I_{(-\infty,0]}(\chi_t)$

By virtue of the complementarity formulation of $\xi \in \partial I_{(-\infty,0]}(\chi_t)$: χ is governed by a gradient flow with respect to χ in the H^1-norm whenever the driving force $-d_\chi F(u,\chi)$ is non-positive and $\chi_t = 0$ otherwise.

Rate-independent gradient damage models (selected works)

→ 2003 Giacomini: existence, convergence to brittle fracture

→ 2005 Mielke/Roubíček: nonlinear elasticity

→ 2009/13 Mielke/Thomas: general p-Laplacian

↑ **Vanishing viscosity** limit: 2013/15 Knees/Rossi/Zanini ↑

Viscous damage models (selected works)

→ 1997 Frémond/Kuttler/Shillor: 1-D local well-posedness

→ 2004/05 Bonetti/Segatti/Schimperna, Bonetti/Bonfanti: 3-D local well-posedness

→ 2010 H./Kraus: weak solutions for double inclusion law

→ 2012/15 Rocca/Rossi: with enhanced regularity, coupling with temperature

(this talk: Farshbaf-Shaker/Heinemann: 2-D global well-posedness)

Optimal control of viscous damage models

→ 2008 Hild/Münch/Ousset: On the control of crack growth in elastic media (more modeling aspects)

→ 2015 Kogut/Leugering: Optimal and approximate boundary controls of an elastic body with quasistatic evolution of damage (no necessary optimal conditions)

Optimal control of viscous damage phase field models

→ 2014 Wheeler/Wick/Wollner: An augmented-Lagrangian method for the phase-field approach for pressurized fractures (”first disretize then optimize” approach)

→ 2015 Farshbaf-Shaker/Heinemann: A phase field approach for optimal boundary control of damage processes in two-dimensional viscoelastic media (minimizers)

→ 2016 Farshbaf-Shaker/Heinemann: Necessary conditions of first-order for an optimal boundary control problem for viscous damage processes in 2D

Mathematical setting

- $\Omega \subseteq \mathbb{R}^2$ bounded domain with C^2-boundary Γ
- strong solution space $\boldsymbol{\mathcal{Q}} := \boldsymbol{\mathcal{U}} \times \boldsymbol{\mathcal{X}}$

$$\boldsymbol{u} \in \boldsymbol{\mathcal{U}} := H^1(0,T;\boldsymbol{H^2(\Omega)}) \cap W^{1,\infty}(0,T;\boldsymbol{H^1(\Omega)}) \cap H^2(0,T;\boldsymbol{L^2(\Omega)}),$$
$$\chi \in \boldsymbol{\mathcal{X}} := H^1(0,T;\boldsymbol{H^2(\Omega)})$$

- initial-boundary data and source term

$$\boldsymbol{u}^0 \in \boldsymbol{H^2(\Omega)}, \quad \boldsymbol{v}^0 \in \boldsymbol{H^1(\Omega)}, \quad \chi^0 \in \boldsymbol{H^2(\Omega)} \text{ with } \partial_\nu \chi^0 = 0,$$
$$\boldsymbol{b} \in H^1(0,T;\boldsymbol{L^2(\Gamma)}) \cap L^2(0,T;\boldsymbol{H^{1/2}(\Gamma)}), \quad \boldsymbol{\ell} \in L^2(0,T;\boldsymbol{L^2(\Omega)})$$

- damage-dependent tensors

$$\mathbb{C}(\cdot) = \mathsf{c}(\cdot)\mathbf{C} \quad \text{with} \quad \mathsf{c} = \mathsf{c}_1 + \mathsf{c}_2 \geq 0, \quad \mathsf{c}_1 \text{ convex}, \quad \mathsf{c}_2 \text{ concave},$$
$$\mathbb{D}(\cdot) = \mathsf{d}(\cdot)\mathbf{C} \quad \text{with} \quad \mathsf{d}(\cdot) \geq \eta > 0, \ (\text{Uniqueness for } \mathsf{d}(\cdot) \equiv \text{ const.})$$
$$\mathbf{C}_{ijlk} = \mathbf{C}_{jilk} = \mathbf{C}_{lkij}, \quad \boldsymbol{\varepsilon} : \mathbf{C}\boldsymbol{\varepsilon} \geq \eta|\boldsymbol{\varepsilon}|^2 \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{R}^{n\times n}_{\text{sym}},$$
$$\mathsf{c}_1, \mathsf{c}_2, \mathsf{d} \text{ continuously differentiable} \quad \text{and} \quad \mathsf{c}, \mathsf{c}_1', \mathsf{c}_2' \text{ bounded and Lipschitz}$$

Goal: find strong solution $(\boldsymbol{u}, \chi)$ with $\boldsymbol{u}(0) = \boldsymbol{u}^0$, $\boldsymbol{u}_t(0) = \boldsymbol{v}^0$ and $\chi(0) = \chi^0$ such that

$$\partial I_{(-\infty,0]}(\chi_t) + \chi_t - \Delta\chi - \Delta\chi_t + \partial_\chi W(\chi, \varepsilon) \ni 0 \text{ in } Q, \quad \partial_\nu \chi = \partial_\nu \chi_t = 0 \text{ on } \Sigma$$

coupled with force balance equation

$$\boldsymbol{u}_{tt} - \mathrm{div}\big(\partial_\varepsilon W(\chi, \varepsilon) + \mathbb{D}(\chi)\varepsilon_t\big) = \boldsymbol{\ell} \quad \text{in } Q, \quad \big(\partial_\varepsilon W(\chi, \varepsilon) + \mathbb{D}(\chi)\varepsilon_t\big)\boldsymbol{\nu} = \boldsymbol{b} \quad \text{on } \Sigma$$

Overview of the existence proof

Step 1: solve approximate problem

- $\longrightarrow$ Moreau-Yosida regularization
- $\longrightarrow$ discretization scheme + partial convex-concave splittings
- $\longrightarrow$ existence by recursive minimization

Step 2: a priori estimates

- $\longrightarrow$ energy-dissipation estimate
- $\longrightarrow$ enhanced regularity estimates

Step 3: limit passage

- $\longrightarrow$ apply compactness results
- $\longrightarrow$ limsup-estimate and Minty's trick

Step 1 (a): Regularization of the indicator function $\boldsymbol{I := I_{(-\infty,0]}}$

- functional setting: define $I: \; L^2(Q) \to \mathbb{R} \cup \{+\infty\}$ by

$$I(v) := \begin{cases} 0 & \textit{if } v \leq 0 \textit{ a.e. in } Q, \\ +\infty & \textit{else} \end{cases}$$

- define the Moreau-Yosida approximation by

$$I_\beta(v) := \inf_{w \in L^2(Q)} \left\{ \frac{\|v-w\|_{L^2}^2}{2\beta} + I(w) \right\} = \frac{\|\max\{v,0\}\|_{L^2}^2}{2\beta}$$

$\rightarrow$ general result:[3] I_β is *convex* and *Gâteaux-differentiable* and

$$\forall v \in L^2(Q): \qquad \limsup_{\beta \searrow 0} I_\beta(v) \leq I(v),$$

$$\forall v_\beta \rightharpoonup v \text{ in } L^2(Q): \qquad \liminf_{\beta \searrow 0} I_\beta(v_\beta) \geq I(v)$$

- differential inclusion is **approximated** by a **pde**
$\boldsymbol{\partial I_\beta(\chi_t) + \chi_t - \Delta\chi - \Delta\chi_t + \partial_\chi W(\chi, \ldots) = 0}$ (note that "$=$" instead of "$\ni$")

Reference:
[3] *T. Roubíček: Nonlinear Partial Differential Equations with Applications, Birkhäuser (2013)*

Step 1 (b): Setting up discretization scheme

- **properties**
  - $\rightarrow$ discrete in time, continuous in space
  - $\rightarrow$ equidistant partition of the time interval $[0,T]$ with time step size τ
  - $\rightarrow$ partial splitting of $W = W^1 + W^2$ into $W^1(\cdot,\varepsilon)$ convex and $W^2(\cdot,\varepsilon)$ concave
- **regularized differential inclusion**

$$\partial I_\beta(\chi_t) + \chi_t - \Delta\chi - \Delta\chi_t + \partial_\chi W(\chi,\varepsilon) = 0, \quad W(\chi,\varepsilon) = \frac{1}{2}\mathbb{C}(\chi)\varepsilon : \varepsilon + f(\chi)$$

$$\Downarrow$$

- **discretization of the regularized inclusion**

$$\partial I_\beta\Big(\frac{\chi^k - \chi^{k-1}}{\tau}\Big) + \frac{\chi^k - \chi^{k-1}}{\tau} - \Delta\chi^k - \Delta\frac{\chi^k - \chi^{k-1}}{\tau}$$
$$+ \partial_\chi W^1(\chi^k, \varepsilon^{k-1}) + \partial_\chi W^2(\chi^{k-1}, \varepsilon^{k-1}) = 0 \qquad \text{in } \Omega$$

with boundary condition

$$\partial_\nu \chi^k = 0 \qquad \text{on } \Gamma$$

Step 1 (b): Setting up discretization scheme

- **force balance equation**

$$\boldsymbol{u}_{tt} - \operatorname{div}\big(\partial_{\boldsymbol{\varepsilon}} W(\chi, \boldsymbol{\varepsilon}) + \mathbb{D}(\chi)\boldsymbol{\varepsilon}_t\big) = \boldsymbol{\ell} \qquad \textit{in } Q$$

with boundary condition

$$\Big(\partial_{\boldsymbol{\varepsilon}} W(\chi, \boldsymbol{\varepsilon}) + \mathbb{D}(\chi)\boldsymbol{\varepsilon}_t\Big)\nu = \boldsymbol{b} \qquad \text{on } \Sigma$$

$$\Downarrow$$

- **time-discretization of the force balance equation**

$$\frac{\boldsymbol{u}^k - 2\boldsymbol{u}^{k-1} + \boldsymbol{u}^{k-2}}{\tau^2} - \operatorname{div}\Big(\partial_{\boldsymbol{\varepsilon}} W(\chi^k, \boldsymbol{\varepsilon}^k) + \mathbb{D}(\chi^k)\frac{\boldsymbol{\varepsilon}^k - \boldsymbol{\varepsilon}^{k-1}}{\tau}\Big) = \boldsymbol{\ell}^k \qquad \text{in } \Omega$$

with boundary condition

$$\Big(\partial_{\boldsymbol{\varepsilon}} W(\chi^k, \boldsymbol{\varepsilon}^k) + \mathbb{D}(\chi^k)\frac{\boldsymbol{\varepsilon}^k - \boldsymbol{\varepsilon}^{k-1}}{\tau}\Big)\nu = \boldsymbol{b}^k \qquad \text{on } \Gamma$$

Step 3 (b): Limit passage

- **remains to show:** $\partial I_\beta(\partial_t\chi_\beta) \rightharpoonup \xi \in \partial I_{(-\infty,0]}(\partial_t\chi)$

  note that we know $\partial_t\chi_\beta \rightharpoonup \partial_t\chi$ weakly in $L^2(0,T;\boldsymbol{H^2(\Omega)})$;
- **approach:**

  $\partial I_\beta(\partial_t\chi_\beta)$ is bounded in $\boldsymbol{L^2(Q)}$

  $\Rightarrow \quad \partial I_\beta(\partial_t\chi_\beta) \rightharpoonup S$ weakly in $\boldsymbol{L^2(Q)}$ for some cluster point $S \in \boldsymbol{L^2(Q)}$

  task: $S \in \partial I_{(-\infty,0]}(\partial_t\chi)$
- exploit convexity of I_β

$$I_\beta(\partial_t\chi_\beta) + \langle \partial I_\beta(\partial_t\chi_\beta), \varphi - \partial_t\chi_\beta \rangle_{L^2} \le I_\beta(\varphi), \qquad \forall \varphi \in \boldsymbol{L^2(Q)},$$

- to pass to the limit we need the properties (*Minty's trick*)

$\rightarrow$ $\liminf$ -estimate: $\liminf\limits_{\beta\searrow 0} I_\beta(\partial_t\chi_\beta) \ge I(\partial_t\chi)$ $\checkmark$

$\rightarrow$ $\limsup$ -estimates: $\limsup\limits_{\beta\searrow 0} \langle \partial I_\beta(\partial_t\chi_\beta), \partial_t\chi_\beta \rangle_{L^2} \le \langle S, \partial_t\chi \rangle_{L^2}$ (have to show)

$$\limsup_{\beta\searrow 0} I_\beta(\varphi) \le I(\varphi), \qquad \forall \varphi \in \boldsymbol{L}^2(Q) \quad \checkmark$$

(CP) *Minimize the cost-functional*

$$\mathcal{J}(\chi, b) := \frac{\lambda_T}{2}\|\chi(T) - \chi_T\|^2_{L^2(\Omega)} + \frac{\lambda_\Sigma}{2}\|b\|^2_{L^2(\Sigma;\mathbb{R}^n)} \tag{1}$$

subject to our viscous/rate-dependent model for damage processes **(PFDM)** (2)

and subject to the control constraint (other types will also be allowed)

$$\mathcal{B}_{adm} := \left\{b \in \mathcal{B} \mid b_{min} \le b \le b_{max} \text{ a.e. in } \Sigma \text{ and } \|b\|_{\mathcal{B}} \le R\right\}. \tag{3}$$

- Banach space $\mathcal{B} := L^2(0,T;H^{1/2}(\Gamma;\mathbb{R}^n)) \cap H^1(0,T;L^2(\Gamma;\mathbb{R}^n))$
- The box constraints $b_{min},\ b_{max} \in L^\infty(\Sigma)$ satisfy $b_{min} \le b_{max}$ a.e. in Σ.
- $\chi_T \in L^2(\Omega)$ is a given target damage profile function
- λ_T, λ_Σ and R are some prescribed positive constants.

Reference:

[10] *M.H. Farshbaf-Shaker, C. Heinemann: A phase field approach for optimal boundary control of damage processes in two-dimensional viscoelastic media, M3AS (2015)*

[11] *M.H. Farshbaf-Shaker, C. Heinemann: Necessary conditions of first-order for an optimal boundary control problem for viscous damage processes in 2D*

Existence of minimizers to (CP)

The optimal control problem **(CP)** admits a solution $(\chi, b) \in \mathcal{X} \times \mathcal{B}_{adm}$

Overview of the existence proof

→ Moreau-Yosida regularization

→ Existence of minimizers for the regularized subproblems

→ Limit passage (convergence along subsequences $\rightarrow$ no ”global” result)

Necessary conditions of first-order

- Problem **(CP)** is a non-smooth optimization problem
- Classical constraint qualifications in Banach spaces are violated, and existence of Lagrange multipliers are not guaranteed.
- But for the regularized subproblems, we derive necessary conditions of first-order
- We are able to prove the existence of very weak solutions to the corresponding regularized adjoint problem

"Reduced control problem"

$\mathcal{S} : \mathcal{B} \to \mathcal{Q} \subset \dot{\mathcal{Q}} := \dot{\mathcal{U}} \times \dot{\mathcal{X}}$ defined by $\mathcal{S}(b) = (\mathcal{S}_1(b), \mathcal{S}_2(b)) := (u(b), \chi(b))$ solving our regularized PDE system with the following definition

$$\dot{\mathcal{U}} := H^1(0,T; H^1(\Omega;\mathbb{R}^n) \cap W^{1,\infty}(0,T; L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T; H^1(\Omega;\mathbb{R}^n)^*),$$
$$\dot{\mathcal{X}} := H^1(0,T; H^1(\Omega)).$$

With the "reduced functional" $j : \mathcal{B} \to \mathbb{R}$ defined by $j(b) := \mathcal{J}(\mathcal{S}_2(b), b)$, where $\mathcal{J} : \dot{\mathcal{X}} \times \mathcal{B} \to \mathbb{R}$ is the cost functional, our optimal control problem **(CP)** can now be restated as

(CP') find a minimizer of j over $\mathcal{B}_{adm}$.

Necessary condition of first-order

$$0 \leq \langle Dj(b), \widehat{b} - b \rangle_{\mathcal{B}} = \int_\Sigma (p + \lambda_\Sigma b) \cdot (\widehat{b} - b) \,\mathrm{d}x\,\mathrm{d}t \quad \text{for every } \widehat{b} \in \mathcal{B}_{adm}.$$

Adjoint system for the regularized problem I $\quad (\xi(\chi_t) := \partial I_\beta(\chi_t))$

Let $(u,\chi) \in \mathcal{Q}$ be a solution of the state system **(PFDM)** to $b \in \mathcal{B}$, i.e. $(u,\chi) := \mathcal{S}(b)$. Then there exists a pair of function $(p,q) \in \overline{\mathcal{Q}} :=:= \overline{\mathcal{U}} \times \overline{\mathcal{X}}$ (very weak solution) with

$$\overline{\mathcal{U}} := L^2(0,T;H^1(\Omega;\mathbb{R}^n)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)),$$
$$\overline{\mathcal{X}} := L^2(0,T;H^1(\Omega)).$$

such that

$$p_{tt} - \operatorname{div}(\mathbb{C}(\chi)\boldsymbol{\varepsilon}(p) + \mathbb{C}'(\chi)\boldsymbol{\varepsilon}(u)q - \mathbb{D}\boldsymbol{\varepsilon}(p_t)) = 0 \qquad \text{in } Q,$$

$$-q_t - (\xi'(\chi_t)q)_t + \Delta q_t - \Delta q + \frac{1}{2}\mathbb{C}''(\chi)\boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(u)q + \mathbb{C}'(\chi)\boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(p) + f''(\chi)q = 0 \qquad \text{in } Q,$$

with the boundary conditions

$$(\mathbb{C}(\chi)\boldsymbol{\varepsilon}(p) + \mathbb{C}'(\chi)\boldsymbol{\varepsilon}(u)q - \mathbb{D}\boldsymbol{\varepsilon}(p_t)) \cdot \nu = 0 \qquad \text{on } \Sigma,$$

$$\nabla p \cdot \nu = 0 \qquad \text{on } \Sigma$$

and with the final-time conditions

$$p(T) = p_t(T) = 0 \qquad \text{in } \Omega,$$

$$-\Delta q(T) + q(T) + \xi'(\chi_t(T))q(T) = \lambda_T(\chi(T) - \chi_T) \qquad \text{in } \Omega,$$

$$\nabla q(T) \cdot \nu = 0 \qquad \text{on } \Gamma.$$

Overview of the existence proof

Step 1: α-regularization

$\longrightarrow$ Work with regularized state variable χ_α instead of χ: To this end, let $\{\chi_\alpha\} \subseteq C^\infty(\overline{Q})$ be a smooth approximation sequence such that

$$\chi_\alpha \to \chi \text{ in } H^1(0,T;H^2(\Omega)) \text{ as } \alpha \downarrow 0.$$

$\longrightarrow$ This enables us to time-differentiate the difficult nonlinear term $\xi'(\chi_t)q$

$\rightarrow \quad (\xi'(\chi_t)q)_t = \xi''(\chi_t)\chi_{tt}q + \xi'(\chi_t)q_t$

$\longrightarrow$ suitable time-discretization scheme which decouples the system

Step 2: a priori estimates

$\longrightarrow$ a priori estimate for the time discretization and for the α-regularization

$\longrightarrow$ testing with time-integrated versions of the adjoint variables with tricky calculations

- *Irreversibility condition.* Necessary optimality condition for the original non-smooth problem: it remains open if stationarity conditions can be obtained via a limit passage of the regularized version.
- *Different cost functionals.* We have considered an L^2-tracking type cost functional in **(CP)**. This restricts possible applications because (smooth approximations of) cracks only give rise to a small contribution with respect to the L^2-norm. More realistic choices would be the usage of higher-order or even L^∞-cost functionals.
- *Damage-dependent viscosities.* It would be desirable to let not only the stiffness tensor $\mathbb{C}(\cdot)$ but also the viscosity tensor $\mathbb{D}$ in the force balance equation to depend on the damage phase-field χ. Optimality conditions for the optimal control problem still needs to be shown.

Thank you for your attention!

Some references

[1] M.H. Farshbaf-Shaker, C. Heinemann: *Necessary conditions of first-order for an optimal boundary control problem for viscous damage processes in 2D*, (submittted)

[2] M.H. Farshbaf-Shaker, C. Heinemann: *A phase field approach for optimal boundary control of damage processes in two-dimensional viscoelastic media*, M3AS (2015)

[3] P. I. Kogut and G. Leugering: *Optimal and approximate boundary controls of an elastic body with quasistatic evolution of damage*, Math. Methods Appl. Sci., 2015

[4] M. F. Wheeler, T. Wick, and W. Wollner: *An augmented-Lagrangian method for the phase-field approach for pressurized fractures*, Comput. Methods Appl. Mech. Engrg., 2014

[5] Knees, Rossi, Zanini: *A vanishing viscosity approach to a rate-independent damage model*, M3AS (2013)

[6] E. Rocca, R. Rossi: *A degenerating PDE system for phase transitions and damage*, M3AS (2014)