The HJB-POD approach for infinite dimensional control problems

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Outline

1. HJ equations, DP schemes and feedback synthesis
   - HJ Equations
   - Numerical scheme for HJ equation
2. Efficient numerical methods for HJ equations
   - A quick overview
   - Accelerated iterative schemes
3. HJB-POD method for high dimensional problem
   - A-priori estimates for the HJB approximation
   - A-priori estimates for the HJB-POD approximation
4. Numerical Tests

Introduction

The Dynamic Programming Principle allows to derive a first order partial differential equation describing the value function associated to the optimal control problem (in finite or infinite dimension).

The theory of viscosity solutions allows to characterize the value function as the unique weak solution of the Bellman equation.

This characterization has been used also to construct numerical schemes for the value function and to compute optimal feedbacks.

Reference

M. Bardi, I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, 1997.

DP's advantages and disadvantages

PROS

1. The characterization of the value function is valid for all classical problems in any dimension.
2. The approximation is based on a-priori error estimates in L^∞ and is valid in any dimension
3. DP (semi-Lagrangian) schemes can work on structured and unstructured grids.
4. The computation of feedbacks is almost built in and there are nice results in low dimension.

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CONS

The "curse of dimensionality" makes the problem difficult to solve in high dimension due to

1. computational cost
2. huge memory allocations.

Outline

HJB equation for the infinite horizon problem

Controlled Dynamics and Cost Functional

$$\begin{cases} \dot{y}(t) = f(y(t), u(t)), & t \in (t_0, +\infty] \\ y(t_0) = x, \end{cases}$$

Infinite horizon cost functional

$$J_x(y, u) = \int_0^{+\infty} g(y(s), u(s)) e^{-\lambda s} \, ds$$

Value Function

$$v(x) := \inf_{u(\cdot) \in \mathcal{U}} J_x(y, u(\cdot)).$$

Value and HJB equation: infinite horizon problem

Dynamic Programming Principle

$$v(x) = \min_{u \in \mathcal{U}} \left\{ \int_{t_0}^{\tau} e^{-\lambda s} g(y_x(s), u(s))\, ds + v(y_x(\tau)) e^{-\lambda \tau} \right\}$$

By Dynamic Programming we get the stationary Bellman equation

$$\lambda v(x) + \max_{u \in U} \{ -f(x,u) \cdot \nabla v(x) - g(x,u) \} = 0, \qquad x \in \mathbb{R}^n$$

Since the value function is in general not regular we need to use weak solutions, typically Lipschitz continuous. The value function is the unique viscosity solution of the Bellman equation.

The construction of the approximation scheme can be obtained via a discrete dynamic programming approach.
State constraints can also be included.

Synthesis of feedback controls

The numerical solution of optimal control problems via HJB PDEs leads to the computation of feedback controls for generic nonlinear Lipschitz continuous vectorfields and costs.

Solving

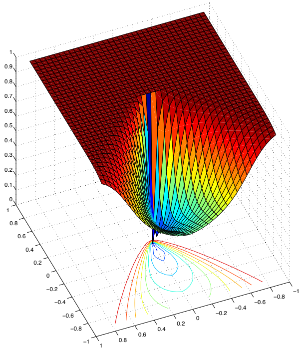
$$\lambda v(x) + \max_{u \in U}\{-f(x,u) \cdot \nabla v(x) - g(x,u)\} = 0\,, \qquad x \in \Omega$$

we get the value function on a grid and we extend it to the whole domain. Then, we can also compute a feedback map $u^* : \Omega \to U$

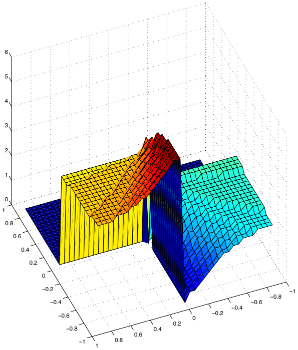
$$u^*(x) \equiv \arg\min_{u \in U}\{f(x,u) \cdot \nabla v(x) + g(x,u)\}\,, \qquad x \in \Omega$$

which is used to compute optimal trajectories by an ODE scheme.

The Zermelo navigation problem



Value Function



Feedback

Outline

How can we compute the value function?

The bottleneck of the DP approach is the computation of the value function, since this requires to solve a non linear PDE in high-dimension.
This is a challenging problem due to the huge number of nodes involved and to the singularities of the solution.

This goal has motivated new efforts in several directions:
Domain Decomposition (DD)
Fast Marching Methods (FMM)
Accelerated iterative schemes

Semi-Lagrangian discretization of HJB

Dynamic Programming Principle

$$v(x) = \min_{u \in \mathcal{U}} \left\{ \int_{t_0}^{\tau} e^{-\lambda s} g(y_x(s), u(s))\, ds + v(y_x(\tau)) e^{-\lambda \tau} \right\}$$

Time-Discrete Approximation via Value Iteration

$$V_i^{k+1} = \min_{u \in U} \{ e^{-\lambda \Delta t} V^k\left(x_i + \Delta t\, f\,(x_i, u)\right) + \Delta t\, g\,(x_i, u) \}$$

- Fix a grid in Ω with $\Omega \subset \mathbb{R}^n$ bounded,
- Steps Δx. Nodes: $\{x_1, \ldots, x_N\}$,
- Discrete solution: $V_i \approx v(x_i)$.
- Stability for large time steps Δt

Semi-Lagrangian discretization of HJB

The most standard way to solve this system is the Value Iteration (VI).

Fully discrete SL-FEM/ Value Iteration (VI) scheme

$$\begin{aligned} V^{k+1} &= T(V^k), \quad \text{for } i = 1, \dots, N \\ \left(T(V^k)\right)_i &\equiv \min_{u \in U}\{e^{-\lambda \Delta t} I[V^k](x_i + \Delta t\, f(x_i, u)) + g(x_i, u)\} \end{aligned}$$

Some advantages:

- Simple to implement (for $I = I_1$,the P_1 interpolation operator)
- (VI) Converges under very general assumptions.

WARNING: Rather expensive in terms of CPU time, since $\beta = e^{-\lambda \Delta t}$ the Lipschitz constant of T goes to 1 when $\Delta t \to 0$.

Semi-Lagrangian discretization of HJB

Fully-Discrete Approximation (Value Iteration)

$$V_i^{k+1} = \min_{u \in U} \{ e^{-\lambda \Delta t} I \left[V^k \right] (x_i + \Delta t\, f(x_i, u)) + \Delta t\, L(x_i, u) \}$$

This algorithm converges for any initial guess V^0.

Error Estimate: [F. 1987]

$$\max_{i \in N_G} \| v(x_i) - V_i \| \leq C \Delta t^{1/2} + \frac{L_f}{\lambda(\lambda - L_f)} \frac{\Delta x}{\Delta t}.$$

N_G = number of nodes, L_f = Lipschitz constant of the dynamics f.

Policy iteration

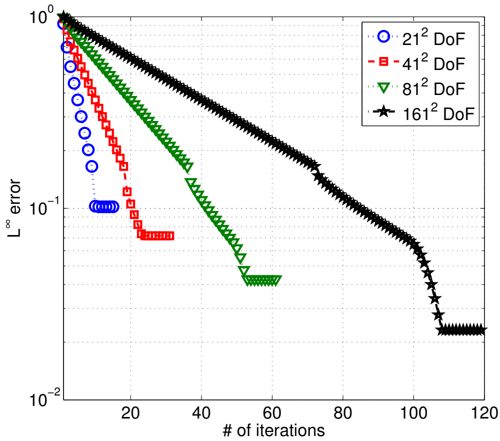
An alternative form to solve this problem is the iteration in the policy space [Bellman 1955, Howard 1960].

Fully discrete SL-FEM/ Policy Iteration (PI) scheme

1. Fix $u_i^0 \in U$, for $i = 1, \ldots, K$.
2. Solve $(V^k)_i = \beta I_1[V^k](x_i + \Delta t\, f(x_i, u_i^k)) + \Delta t\, g(x_i, u_i^k)$.
3. Update $u_i^{k+1} = \underset{u \in U}{\operatorname{argmin}} \{ I_1[V^k](x_i + \Delta t\, f(x_i, u)) + \Delta t\, g(x_i, u) \}$.
4. Repeat until matching convergence criteria (can be set on V or u).Typically we use $||V^{k+1} - V^k||_\infty < \epsilon$ as stopping rule.

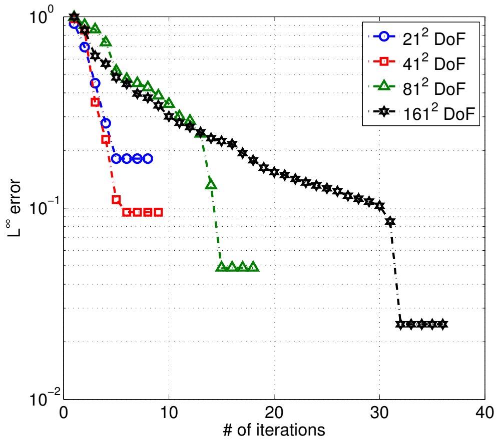
Numerical "Fact" #1: PI is faster

$||V^k - V^*||_\infty$ evolution for VI (2D MTP)



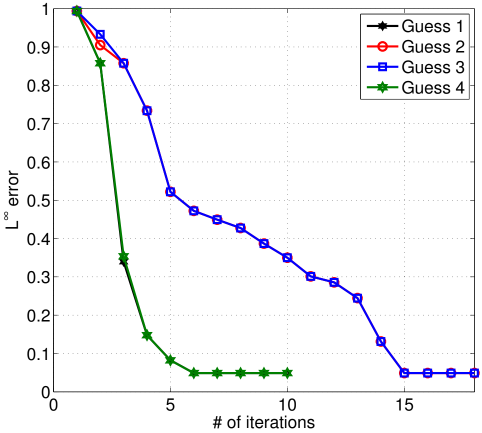
Numerical “Fact” #1: PI is faster

$||V^k - V^*||_\infty$ evolution for PI (2D MTP)



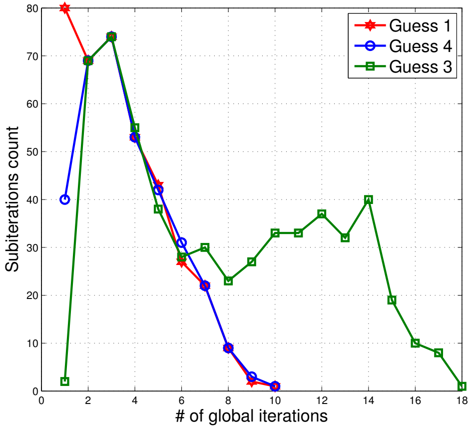
Numerical “Fact” #2: PI is sensitive to u_i^0.

$||V^k - V^*||_\infty$ evolution for PI with different u_i^0



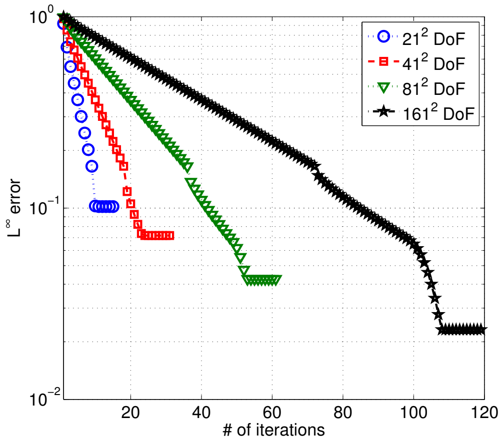
Numerical “Fact” #2: PI is sensitive to u_i^0.

PI inner solver iteration subcount (2D MTP)



Num. “Fact” #3: VI is fast in coarse meshes.

$||V^k - V^*||_\infty$ evolution for VI (2D MTP)



The accelerated algorithm (Alla, F., Kalise, 2015)

Theoretical results have established a link between VI and PI in control problems and games [Puterman and Brumelle (1979), Bokanowski, Maroso and Zidani (2009)].

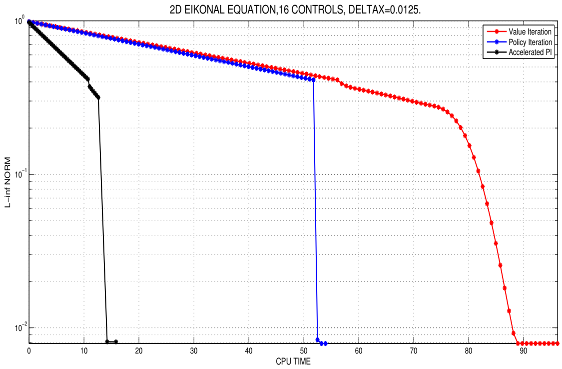
Numerically, PI strongly depends on a good initialization.
Similar approaches have been studied by González and Sagastizábal (1990), Chow (1991), Seeck (1997), and Grüne (2004).

The Accelerated PI (API) algorithm

1. Consider two mesh parameters k_1 and k_2, $k_1 > k_2$. Set $V^0_{k_1}$.
2. Perform a coarse VI in the k_1- mesh with result $V^*_{k_1}$.
3. $V^0_{k_2} = I_1[V^*_{k_1}]$ and $u^0_{k_2} = \underset{u\in U}{\text{argmin}}\{I_1[V^0_{k_2}](x_i + hf(x_i, u))\}$.
4. start PI solver over the k_2-mesh.

Test 1: different performances

L^1-norm $||V^k - V^*||_1$ evolution.



Outline

Control of Partial Differential Equations via DP

The discretization of a PDE leads to a large system of ODEs.
The approximation of the correspondent HJB equations becomes unfeasible because of the curse of dimensionality.

Model Reduction via Proper Ortoghonal Decomposition

POD decomposition allows to reduce the number of variables to approximate partial differential equations.
GOAL: to approximate PDE optimal control problems in a rather small dimension via POD using numerical schemes for HJB equations.

References

Kunisch and Volkwein (2001, ...), Kunisch, Volkwein and Xie (2004), Alla-F. (2013, 2014), Alla-Hinze (2015).

Proper Orthogonal Decomposition and SVD

Given snapshots : $(y(t_0, u), \ldots, y(t_m, u)) \in \mathbb{R}^n$

We look for an orthonormal basis $\{\psi_i\}_{i=1}^{\ell}$ in $\mathbb{R}^m$ with $\ell \leq \min\{n, m\}$ s.t.

$$J(\psi_1, \ldots, \psi_\ell) = \sum_{j=1}^{m} \alpha_j \left\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle \psi_i \right\|^2 = \sum_{i=\ell+1}^{d} \sigma_i^2$$

reaches a minimum where $\{\alpha_j\}_{j=1}^{n} \in \mathbb{R}^+$.

$$\min J(\psi_1, \ldots, \psi_\ell) \quad s.t. \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

Singular Value Decomposition: $Y = \Psi \Sigma V^T$.

For $\ell \in \{1, \ldots, d = rank(Y)\}$, $\{\psi_i\}_{i=1}^{\ell}$ is called the POD basis of rank ℓ.
Ansatz:

$$y(x, t) \approx \sum_{i=1}^{\ell} y_i^{\ell}(t) \psi_i(x)$$

Reduced Order Modeling Control Problem

$$\min J^\ell(y^\ell, u)$$

y^ℓ satisfies the reduced dynamics

$$\begin{cases} \dot{y}^\ell(t) = F^\ell(y^\ell(t), u(t)) & t > 0, \\ y^\ell(t_0) = y_0^\ell \in \mathbb{R}^\ell. \end{cases}$$

The cost functional is:

$$J^\ell_{y_0^\ell}(y^\ell, u) = \int_0^\infty g^\ell(y^\ell(t), u(t)) e^{-\lambda t}\, dt$$

Reduced Value Function

$$v(y_0^\ell) = \inf_{u \in \mathcal{U}_{ad}} J^\ell(y^\ell_{y_0^\ell}, u)$$

Reduced Domain (in the POD space) for HJB

We are going to solve the HJB equation in a box: $[a_1, b_1] \times \ldots \times [a_\ell, b_\ell]$

The box must be chosen such that the dynamic contains all the possible controlled trajectories.

We discretize the control space $\mathcal{U}$ by a discrete set $\{u_1, \ldots, u_M\}$.

$$y(t) = \sum_{i=1}^{\ell} < y(t, u_j), \psi_i > \psi_i = \sum_{i=1}^{\ell} y_i^\ell(t, u_j)\psi_i$$

and we choose the box to guarantee

$$y_i^\ell(t, u_j) \in [a_i, b_i],\ j = 1, \ldots, M.$$

Back to the infinite horizon problem

Let us consider the controlled dynamics

$$\dot{y}(t) = f\big(y(t), u(t)\big) \in \mathbb{R}^n \text{ for } t > 0, \quad y(0) = y_\circ \in \mathbb{R}^n \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$
The infinite horizon cost functional is

$$J(y, u) = \int_0^\infty g\big(y(t), u(t)\big) e^{-\lambda t} \, \mathrm{d}t \tag{2}$$

where $\lambda > 0$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.
The set of admissible controls has the form

$$\mathbb{U}_{\mathrm{ad}} = \big\{ u \in \mathbb{U} \,\big|\, u(t) \in U_{\mathrm{ad}} \text{ for almost all } t \geq 0 \big\},$$

where $\mathbb{U} = L^2(0, \infty; \mathbb{R}^m)$ and $U_{\mathrm{ad}} \subset \mathbb{R}^m$ is a compact, convex subset.

Variational formulation of the dynamics

Let $\mathrm{M} \in \mathbb{R}^{n\times n}$ denote a symmetric, positive definite (mass) matrix with smallest and largest positive eigenvalues λ_{min} and λ_{max}, respectively. We introduce the following weighted inner product in $\mathbb{R}^n$:

$$\langle y, \tilde{y}\rangle_{\mathrm{M}} = y^\top \mathrm{M}\tilde{y} \quad \text{for } y, \tilde{y} \in \mathbb{R}^n,$$

By $\|\cdot\|_{\mathrm{M}} = \langle \cdot\,,\cdot\rangle_{\mathrm{M}}^{1/2}$ we define the associated induced norm. Recall that we have

$$\lambda_{min}\,\|y\|_2^2 \le \|y\|_{\mathrm{M}}^2 \le \lambda_{max}\,\|y\|_2^2 \quad \text{for all } y \in \mathbb{R}^n.$$

Then, y is a trajectory if

$$\begin{aligned} \langle \dot{y}(t) - f(y(t), u(t)), \varphi\rangle_{\mathrm{M}} &= 0 \quad \text{for all } \varphi \in \mathbb{R}^n \text{ and for almost all } t > 0, \\ \langle y(0) - y_\circ, \varphi\rangle_{\mathrm{M}} &= 0 \quad \text{for all } \varphi \in \mathbb{R}^n \end{aligned} \tag{3}$$

The infinite horizon problem

Assume that the dynamics has a unique solution $y = y(u; y_\circ) \in \mathbb{Y} = H^1(0, \infty; \mathbb{R}^n)$ for every admissible control $u \in \mathbb{U}_{\mathrm{ad}}$ and for every initial condition $y_\circ \in \mathbb{R}^n$, denoted by $y(u; y_\circ)$.

Reduced cost functional

$$\widehat{J}(u; y_\circ) = J(y(u; y_\circ), u) \quad \text{for } u \in \mathbb{U}_{\mathrm{ad}} \text{ and } y_\circ \in \mathbb{R}^n,$$

Then, our optimal control problem for $y_\circ \in \mathbb{R}^n$ is

$$\min_{u \in \mathbb{U}_{\mathrm{ad}}} \widehat{J}(u; y_\circ). \tag{$\widehat{\mathbf{P}}$}$$

ASSUMPTIONS

(A1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and

$$\|f(y,u) - f(\tilde{y},u)\|_2 \leq L_f \|y - \tilde{y}\|_2 \text{ for all } y, \tilde{y} \in \mathbb{R}^n \text{ and } u \in U_{\text{ad}}$$

Moreover, $\|f(y,u)\|_\infty \leq M_f$ for all $(y,u) \in \overline{\Omega} \times U_{\text{ad}}$.
(A2) $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and globally Lipschitz continuous.
Moreover, $\|g(y,u)\|_\infty \leq M_g$ for all $(y,u) \in \overline{\Omega} \times U_{\text{ad}}$.

If (A1)–(A2) hold and $\lambda > L_f$, v_h is globally Lipschitz-continuous

$$\left|v_h(y) - v_h(\tilde{y})\right| \leq \frac{L_g}{\lambda - L_f} \|y - \tilde{y}\|_2 \quad \forall y, \tilde{y} \in \overline{\Omega} \text{ and } h \in [0, 1/\lambda)$$

A-priori estimate for HJB [F87]

Theorem

Let (A1)–(A2) hold and $\lambda > \max\{L_g, L_f\}$. Let v and v_h be the continuous and semi-discrete solutions respectively. Moreover, assume semiconcavity, i.e.

$$\begin{aligned} \|f(y+\tilde{y},u) - 2f(y,u) + f(y-\tilde{y},u)\|_2 &\le C_f \,\|\tilde{y}\|_2^2, \\ \left|g(y+\tilde{y},u) - 2g(y,u) + g(y-\tilde{y},u)\right| &\le C_g \,\|\tilde{y}\|_2^2 \end{aligned} \tag{4}$$

for all $(y,\tilde{y},u) \in \mathbb{R}^n \times \mathbb{R}^n \times U_{\mathrm{ad}}$. Then,

$$\sup_{y\in\mathbb{R}^n} \left|v(y) - v_h(y)\right| \le Ch \quad \text{for any } h \in [0, 1/\lambda).$$

An estimate for the HJB-POD approximation

We introduce two different POD approximations for the HJB equation. A first estimate is based on the fully discrete HJB equation, where we project all vertices $\{y_i\}_{i=1}^{n_{\mathbb{S}}}$ into $\mathbb{R}^\ell$ by setting

$$y_i^\ell = \Psi^\top \mathrm{M} y_i \quad \text{for } i = 1, \ldots, n_{\mathbb{S}}.$$

Here we assume $y_i^\ell \neq y_j^\ell$ for $i, j \in \{1, \ldots, n_{\mathbb{S}}\}$ with $i \neq j$.
Then, a POD discretization of HJB in the reduced space is given by

$$v_{hk}^\ell(y_i^\ell) = \min_{u \in U_{\mathrm{ad}}} \left\{(1 - \lambda h) v_{hk}^\ell\left(y_i^\ell + h f^\ell(y_i^\ell, u)\right) + h g^\ell(y_i^\ell, u)\right\} \qquad (5)$$

for $1 \leq i \leq n_{\mathbb{S}}$.

An estimate for the HJB-POD approximation

Let us define the mapping $\widetilde{v}^{\ell}_{hk} : \overline{\Omega} \to \mathbb{R}$ by

$$\widetilde{v}^{\ell}_{hk}(y) = v^{\ell}_{hk}(\Psi^{\top}\mathrm{M}y) \quad \text{for all } y \in \overline{\Omega} \text{ with } \Psi^{\top}\mathrm{M}y \in \overline{\Omega}.$$

Thus, (5) can be written as

$$\widetilde{v}^{\ell}_{hk}(y_i) = \min_{u \in U_{\text{ad}}} \left\{(1 - \lambda h)\widetilde{v}^{\ell}_{hk}\big(y_i + hf(\mathcal{P}^{\ell} y_i, u)\big) + hg(\mathcal{P}^{\ell} y_i, u)\right\} \qquad (6)$$

for $1 \leq i \leq n_{\mathbb{S}}$.

An estimate for the HJB-POD approximation

To simplify, let us assume the
invariance condition for the reduced order dynamics

$$y_i + hf(\mathcal{P}^\ell y_i, u) \in \overline{\Omega} \quad \text{for } i = 1, \ldots, n_{\mathbb{S}}, \ \forall u \in U_{\mathrm{ad}} \tag{A3}$$

Theorem

*Assume that (A1)–(A2), (A3) and $\lambda > L_f$ hold.
Then, there exist two constants $\widehat{C}_0, \widehat{C}_1$ such that*

$$\sup_{y \in \overline{\Omega}} \left| v_h(y) - \widetilde{v}^\ell_{hk}(y) \right| \leq \widehat{C}_0 \, \frac{k}{h} + \widehat{C}_1 \left(\sum_{i=1}^{n_{\mathbb{S}}} \| y_i - \mathcal{P}^\ell y_i \|_2^2 \right)^{1/2}$$

for any $h \in [0, 1/\lambda)$

An estimate for the HJB-POD approximation

By the previous results we conclude

Theorem

Assume that (A1)–(A2), (A3) and (A4) hold. Let f, g satisfy the semiconcavity conditions.
If $\lambda > \max\{L_f, L_g\}$, then there exists constants $C_0, C_1, C_2 \geq 0$ such that

$$\sup_{y \in \overline{\Omega}} \left| v(y) - \widetilde{v}^{\ell}_{hk}(y) \right| \leq C_0 h + C_1 \frac{k}{h} + C_2 \left(\sum_{i=1}^{n_{\mathbb{S}}} \|y_i - \mathcal{P}^\ell y_i\|_2^2 \right)^{1/2}$$

for any $h \in [0, 1/\lambda)$.

Outline

Test 3: Advection-Diffusion Equation

Advection-Diffusion Equation

$$\begin{cases} y_t - \varepsilon y_{xx} + \beta y_x = \chi_{\Omega_c}(x)u(t) & \text{in} \quad \Omega \times (0,\infty], \\ y(\cdot,0) = y_0, & \text{in} \quad \Omega, \\ y(\cdot,t) = 0 & \text{in} \quad \partial\Omega \times (0,T), \end{cases} \tag{7}$$

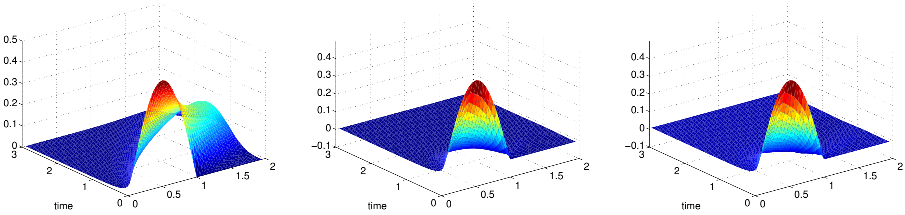
Cost Functional

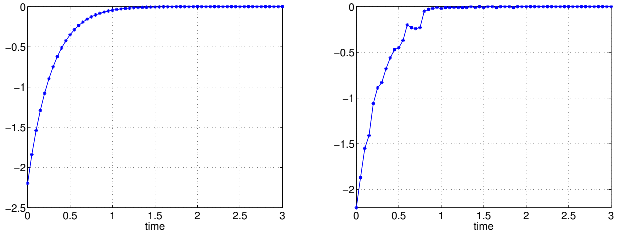
$$J(y,u,t) = \int_0^\infty \left(\|y(x,\tau)\|^2 + \gamma ||u(\tau)||^2 \right) e^{-\lambda\tau} \, d\tau.$$

Parameters

$\varepsilon = 0.1, \beta = 1, \gamma = 0.01, y_0(x) = 0.5\sin(\pi x),$
$\Omega = [0,2], \Omega_c = (0.5,1), U = [-2.2,0];$
Snapshots: $\Delta x = 0.01, \Delta t = 0.1.$
VF: $\Delta x = \{0.1, 0.05\}, \Delta t = 0.1\Delta x$; *Trajectories*: $\Delta t = 0.01$;

Test 3: Advection-Diffusion Equation





Controls

Figure: Test 3: Uncontrolled (left), LQR optimal (middle), HJB-POD (right).

Test 3: Advection-Diffusion Equation

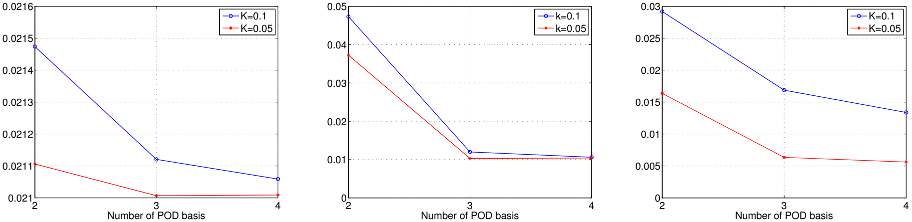


Figure: Evaluation of the cost functional (left), L^2-error for $y(u^\ell)$ and $y^\ell(u^\ell)$ (middle) and L^2-error between LQR solution and $y(u^\ell)$.

Test 4: Semi-linear Equation

Semi-Linear Equation

$$\begin{cases} y_t - \varepsilon y_{xx} + \beta y_x + \mu(y - y^3) = y_0(x)u(t) & \text{in} \quad \Omega \times (0, \infty], \\ y(\cdot, 0) = y_0, & \text{in} \quad \Omega, \\ y(\cdot, t) = 0 & \text{in} \quad \partial\Omega \times (0, T), \end{cases}$$

Cost Functional

$$J(y, u, t) = \int_0^\infty \left(\|y(x,\tau)\|^2 + \gamma ||u(\tau)||^2 \right) e^{-\lambda\tau} \, d\tau.$$

Parameters

$\varepsilon = 0.1 = \beta, \mu = 1, \gamma = 0.01, y_0(x) = \sin(\pi x), \Omega = [0,1], U = [-1,1];$
Snapshots: $\Delta x = 0.01, \Delta t = 0.1.$
VF: $\Delta x = \{0.1, 0.05\}, \Delta t = 0.1\Delta x$; *Trajectories*: $\Delta t = 0.01;$

Test 4: Semi-linear Equation

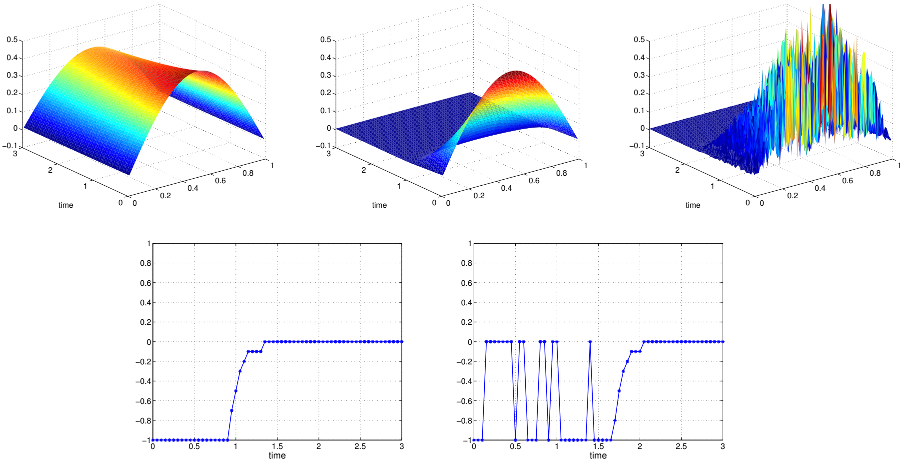


Figure: Uncontrolled (left), Optimal (middle) and optimal HJB control (right).

Test 5: Wave Equation

Damped Wave Equation

$$\begin{cases} y_{tt} - y_{xx} - \beta y_{xxt} = \chi_{\Omega_c}(x)u(t) & \text{in} \quad \Omega \times (0,\infty], \\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = y_1 & \text{in} \quad \Omega, \\ y(\cdot,t) = 0 & \text{in} \quad \partial\Omega \times (0,T), \end{cases}$$

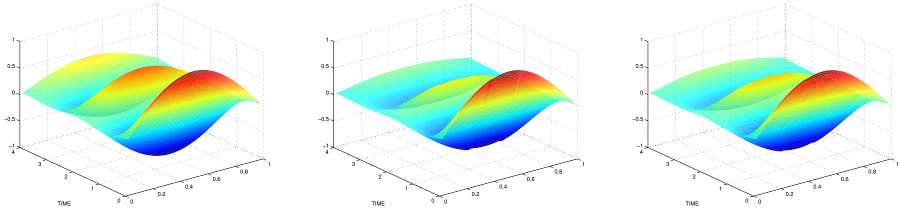
Cost Functional

$$J(y,u,t) = \int_0^\infty \left(\|y(x,\tau)\|^2 + \gamma ||u(\tau)||^2 \right) e^{-\lambda\tau} \, d\tau.$$

Parameters

$\beta = 0.05, \gamma = 0.01, y_0(x) = \sin(\pi x), y_1(x) = 0,$
$\Omega = [0,1], \Omega_c = (0.4, 0.6), U = [-1.2, 0.6];$
Snapshots: $\Delta x = 0.01, \Delta t = 0.1.$
VF: $\Delta t = 0.01, \Delta x = 0.1$; *Trajectories*: $\Delta t = 0.01$;

Test 5: Wave Equation, Optimal Solution



	$k = 0.1$	$k = 0.05$
$\ell = 3$	0.1224	0.0953
$\ell = 4$	0.1009	0.0886
$\ell = 5$	0.0648	0.0468

Table: H_0^1-error between LQR control and HJB-POD approx. at time $t = 4$.

Test 5: Wave Equation, Stabilization of the feedback

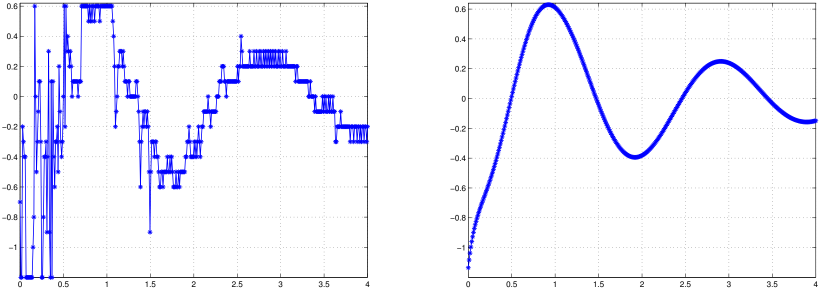


Figure: Feedback Control with Chattering (left) LQR control (right)),

Wave Equation: Stabilization of the feedback

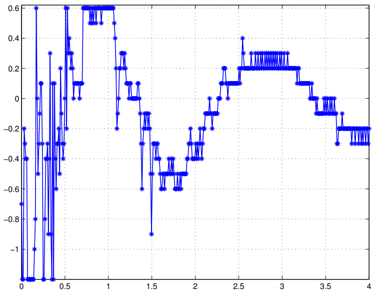


Figure: Feedback Control with Chattering (left), Stabilized feedback control with limited chattering (right)

CONCLUSIONS

- Accelerated Policy Iteration speeds up the numerical approximation of the value function
- Model reduction via POD is crucial in order to make the problem feasible
- Feedback Control for PDEs is obtained via the POD-HJB approach
- The chattering of the feedback control for finite and infinite dimensional problem can be reduced
- A-priori error estimation are available

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