EVOLUTION OF SETS AND GEOMETRIES: SHAPE DIFFERENTIALS AND TOPOLOGICAL SEMIDIFFERENTIALS

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Outline

1. Some Motivation: Local Minimizers on Non-linear Spaces
2. Metric Spaces of Subsets of a Hold-all
3. Differential and Semidifferential
   - Hadamard Differential and Semidifferential
   - Hadamard Semidifferential in Non-linear Spaces: Semi-differential Geometry!
4. Metric Group of Characteristic Functions
   - Shape Derivative and Velocity Method: Tangents to $X(\mathbb{R}^N)$
   - Topological Derivative: Semi-tangents to $X(\mathbb{R}^N)$
     - Dilatation of a Hole and Semi-tangents to $X(\mathbb{R}^3)$
   - Dilatation, Minkowski Content, and Rectifiable Sets
     - Dilatation of a Smooth Curve and Semi-tangents to $X(\mathbb{R}^3)$
     - Dilatation of an Hypersurface and Semi-tangents to $X(\mathbb{R}^N)$
     - Generalization to Submanifolds in $\mathbb{R}^N$
     - Minkowski Content and Rectifiable Sets
   - Sets of Positive Reach
     - Sets of Positive Reach and Steiner Formula
     - Dilatation of A and Normal Dilatations of Subsets E of ∂A
     - Subset of an Hypersurface and Semi-tangents to $X(\mathbb{R}^3)$
5. Some Concluding Remarks
6. References

Given U, $\varnothing \neq U \subset \mathbb{R}^N$, and a function $f : U \to \mathbb{R}$,
a necessary condition for the existence of a local minimizer $\hat{u} \in U$ of f over U is

$$\boxed{\exists \hat{u} \in U \text{ such that } d_H f(\hat{u}; v) \geq 0, \quad \forall v \in T_{\hat{u}} U,}$$

- $T_{\hat{u}} U$ is the Bouligand's tangent cone to U at $\hat{u}$ and
- $d_H f(\hat{u}; v)$ is the Hadamard semidifferential at $\hat{u}$ in the direction $v \in T_{\hat{u}} U$.

In optimization the Bouligand's tangent cone $T_{\hat{u}} U$ is the *set of admissible directions*.

Terminology:
- semidifferential: one-sided directional derivative

$$df(\hat{u}; v) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{f(\hat{u} + tv) - f(\hat{u})}{t}$$

- differential: $df(\hat{u}; v)$ exists for all v and

$$v \mapsto df(\hat{u}; v) \quad \text{linear and continuous}$$

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Two types of metric spaces (with at best a group structure) will be considered:
- Images of a fixed set by a group of diffeomorphisms: no topological changes ;
- Sets of families of functions parametrized by sets: various topological changes - such sets will have no interior in the surrounding vector space where they live.

They will generally not be Riemannian or Finsler manifolds and their tangent spaces will not be linear and need to be characterized.

Objective: to extend the notion of Hadamard semidifferential to such metric spaces.
1) shape derivatives: Zolésio introduced the notion of differential in his 1979 thesis - the tangent space is the vector space of velocities that generate the diffeomorphisms;
2) topological derivatives: to replace current expansions techniques by semidifferentials - in a first step, we shall characterize tangent spaces.

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- Identify a set with its characteristic function: (Lebesgue [26, 27](1907) measure as a relaxation of the notion of volume, Federer [16, 17], (1951, 1959) geometric measure theory, De Giorgi [8] (1954) perimeter, ...)

$$\boxed{\Omega \longleftrightarrow \chi_\Omega} \quad \chi_\Omega \overset{\text{def}}{=} \begin{Bmatrix} 1, & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega \end{Bmatrix} \qquad \begin{array}{l} [\Omega] \overset{\text{def}}{=} \{\Omega' : \Omega' = \Omega \quad m_N\text{-a.e.}\} \\ L^\infty(\mathbb{R}^N),\ L^p_{loc}(\mathbb{R}^N),\ 1 \leq p < \infty \end{array}$$

- Identify a set with its distance function: Hausdorff (1914)-Pompéju (1905) metric, sets of positive reach (Federer 1959), set-valued analysis, ...

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In 1923 J. Hadamard [25] gave a (geometric) definition of differentiability by introducing an auxiliary function $t \mapsto x(t) : \mathbb{R} \to \mathbb{R}^N$ of the auxiliary variable t such that

$$x(0) = a \quad \text{and} \quad x'(0) \stackrel{\text{def}}{=} \lim_{t\to 0} \frac{x(t)-a}{t} \text{ exists in } \mathbb{R}^N.$$

We shall use the terminology time for the *auxiliary function t* and trajectory for the *auxiliary function x*. Note that x need not be continuous or differentiable at $t \neq 0$.

The vector $x'(0)$ is the tangent to the trajectory x at the point $x(0) = a$. Scaling t by an arbitrary non-zero real number, a whole line tangent to x at a is obtained.

Definition (Hadamard differentiability)

A function $f : \mathbb{R}^N \to \mathbb{R}^K$ is Hadamard differentiable at $a \in \mathbb{R}^N$ if

(i) for all trajectories x the limit

$$(f \circ x)'(0) \stackrel{\text{def}}{=} \lim_{t\to 0} \frac{f(x(t)) - f(a)}{t} \text{ exists in } \mathbb{R}^K,$$

(ii) there exists a linear function $Df(a) : \mathbb{R}^N \to \mathbb{R}^K$ such that for all trajectories x at a

$$(f \circ x)'(0) = Df(a)\,(x'(0)).$$

$Df(a)$ is called the differential of f at a. The associated matrix is the Jacobian matrix.

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In finite dimension, the definition of Hadamard differentiability is equivalent to the one of Fréchet differentiability.

In his 1937 paper, Fréchet [20] observed that, in function spaces, the Hadamard differentiability is not only a notion more general than the one [23] he introduced in 1911, but that the linearity in part (ii) is not necessary to preserve the continuity of the function and the chain rule.

He introduces a "relaxed notion" without the linearity and gives the following function $f : \mathbb{R}^2 \to \mathbb{R}$ as an example.

$$f(x,y) = x\sqrt{\frac{x^2}{x^2+y^2}} \quad \text{for } (x,y) \neq (0,0) \quad \text{with } f(0,0) = 0.$$

([24, p. 239]). Indeed, it is readily checked that for any trajectory $x : \mathbb{R}^2 \to \mathbb{R}$ such that $x(0) = (0,0)$ and $x'(0)$ exists

$$(f \circ x)'(0) \overset{\text{def}}{=} \lim_{t \to 0} \frac{f(x(t)) - f(0,0)}{t} = f(x'(0)).$$

Hadamard always insisted on the linearity. This new notion was criticized by Paul Levy. Yet, his example shows that such (nondifferentiable) functions exist.

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$$(f \circ x)'(0) \stackrel{\text{def}}{=} \lim_{t \to 0} \frac{f(x(t)) - f(0,0)}{t} = f(x'(0)).$$

Hadamard always insisted on the linearity. This new notion was criticized by Paul Levy. Yet, his example shows that such (nondifferentiable) functions exist.

With the relaxed notion of Fréchet, we can deal with some families of non-differentiable functions. Unfortunately, many convex continuous functions and, in particular, the norm $\|x\|$ are not differentiable in this relaxed sense. To get around this, we need the notion of semidifferential.

For instance, consider the function $x \mapsto f(x) = \|x\| : \mathbb{R}^N \to \mathbb{R}$. Let $x : [0, +\infty) \to \mathbb{R}^N$ be a half (or semi) trajectory such that $x(0) = 0 \in \mathbb{R}^N$ and the right-hand limit $x'(0^+)$ exists (semi tangent).

The semidifferential at $x = 0$ is defined as

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A similar result for convex and semiconvex continuous functions. The (geometrical) definition can be found in J. Durdil [15, p. 457] in 1973 and the (analytical) definition in J.P. Penot [30, p. 250] in 1978, but they do not really explore the consequences.

The hypothesis of linearity of the differential is also a severe restriction to introduce a differential of a function $f : A \subset \mathbb{R}^N \to B \subset \mathbb{R}^K$ since it requires that the tangent space to A at a and the tangent space to B at $f(a)$ be linear subspaces.

This usually requires that the sets A and B be "smooth" in the sense that, at each point, the tangent space is a linear space. The introduction of semidifferentials eliminates this a priori smoothness assumption while preserving the continuity and the chain rule.

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$x'(0) \stackrel{\text{def}}{=} \lim_{t\to 0} \frac{x(t)-a}{t}$ exists

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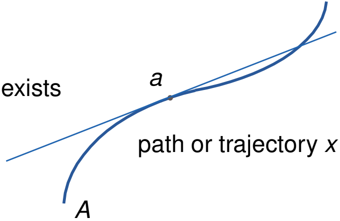

path or trajectory $x(t)$ through a

FIGURE 1: Tangent $x'(0)$) to the path $x(t)$ in A at the point $x(0) = a$.

However, the linearity of $T_a A$ puts a severe restriction on the sets A. For instance, the requirement that $T_a A$ be linear rules out a curve in $\mathbb{R}^2$ with kinks

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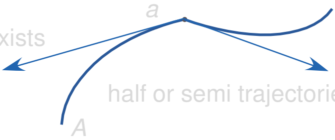

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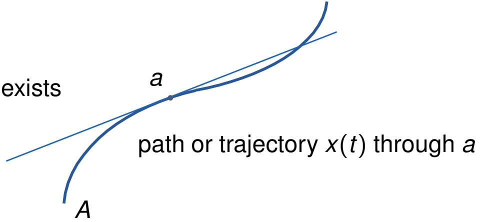

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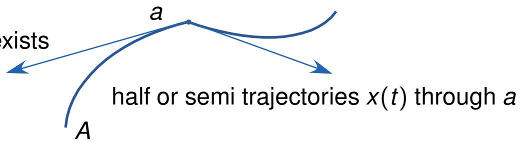

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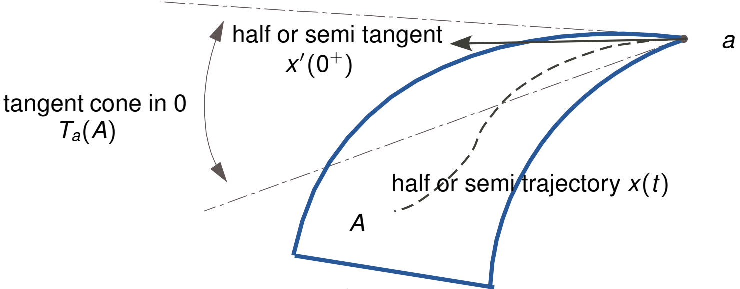

FIGURE 3: Half or semi tangent $x'(0^+)$) to the path $x(t)$ in A at the point $x(0) = a$.

DEFINITION (ADMISSIBLE SEMI-TRAJECTORY)

Given $A \subset \mathbb{R}^N$, an *admissible semi-trajectory* (or *half trajectory*) in A at a point $a \in \overline{A}$ is a function $x : [0, \tau] \to A$, $\tau > 0$, such that the *semi-tangent* (or *half tangent*) at a

$$x'(0^+) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{x(t) - a}{t}$$

exists. When the limit $x'(0^+)$ exists, it follows that $x(t) \to a$ as $t \searrow 0$.

DEFINITION

The Bouligand tangent cone[a] to a set A at a point $a \in \overline{A}$ is the set

$$T_a A \overset{\text{def}}{=} \left\{ v \in \mathbb{R}^N : \exists \{x_n\} \subset A \text{ and } \{t_n \searrow 0\} \text{ such that } \lim_{n\to\infty} \frac{x_n - a}{t_n} = v \right\}$$

(cf. G. Bouligand [6] in 1931).

[a] Note that $T_a A$ is a closed cone and $T_a \overline{A} = T_a A$.

When the boundary ∂A of A is smooth, $T_a A$ is a linear subspace of $\mathbb{R}^N$.

Admissible trajectories give another characterization of the Bouligand's cone.

THEOREM

$T_a A = \{x'(0^+) : x \text{ is an admissible semi-trajectory in } A \text{ at } a\}$.

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Similarly, we can introduce the tangent cone $T_b B$ at any point $b \in B$.

DEFINITION (GEOMETRIC DEFINITION - RELAXATION OF THE LINEARITIES)

We say that $f : A \subset \mathbb{R}^N \to B \subset \mathbb{R}^K$ is Hadamard semidifferentiable at $a \in A$ if

(i) for all admissible semitrajectories x in A at a, the limit

$$(f \circ x)'(0^+) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{f(x(t)) - f(a)}{t} \text{ exists,}$$

(ii) and there exists a (positively homogeneous) function $d_A f(a) : T_a A \to T_{f(a)} B$ such that for all admissible semitrajectories x at a

$$(f \circ x)'(0^+) = d_A f(a) \left(x'(0^+) \right).$$

$d_A f(a)$ is called the tangential semidifferential of f at $a \in A$. It can be shown that $d_A f(a)$ is continuous on $T_a A$.

The two important properties are preserved:

(i) continuity at a

(ii) and the chain rule is applicable to the composition of such functions.

DEFINITION (HADAMARD DIFFERENTIABILITY)

Given $A \subset \mathbb{R}^N$ and $B \subset \mathbb{R}^K$, $f : A \to B$ is Hadamard differentiable at $a \in A$ if

(i) f is *semidifferentiable* at a,

(ii) $T_a A$ and $T_{f(a)} B$ are linear subspaces and $v \mapsto d_A f(a; v) : T_a A \to T_{f(a)} B$ is linear.

The semidifferential $d_A f(a)$ will called the *differential* of f and denoted $D_A f(a)$.

- The previous definitions extend to subsets A and B of topological vector spaces, but we have to be careful and retain the abstract notions that are really meaningful.

- For shapes and geometries, the subset A will be a complete metric space with or without a group structure in a surrounding Banach or Fréchet space.

- For instance, Courant metrics of Micheletti on spaces of diffeomorphisms and metric spaces associated with characteristic and oriented distance functions.

- The beginnings of differentiation in infinite-dimensional spaces go back to Vito Volterra [33] in 1887. There was an important activity on extensions to topological vector spaces in the sixties (cf., for instance, de Foglio [7] in 1959). Since then a considerable literature on differentials in topological vector spaces has appeared (see the well written and documented 207-page paper of M. Z. Nashed [28] for a rather complete account until 1971, and V. I. Averbuh and O. G. Smoljanov [3, 4, 5] in 1987 and 1988). Yet, most of this material is not immediately pertinent for our purposes.

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Go back to the metric Abelian group of characteristic functions on $\mathbb{R}^N$

$$X(\mathbb{R}^N) = \left\{\chi_\Omega : \Omega \subset \mathbb{R}^N \text{ Lebesgue measurable}\right\} \subset L^\infty(\mathbb{R}^N).$$

It is a closed subset without interior of the Banach space $L^\infty(\mathbb{R}^N)$ and the Fréchet spaces $L^p_{\text{loc}}(\mathbb{R}^N)$, $1 \le p < \infty$. The analog would be the sphere in $\mathbb{R}^3$.

For the *velocity method*, consider the following continuous trajectory in $X(\mathbb{R}^N)$

$$t \mapsto \chi_{T_t(V)(\Omega)} : [0,1] \to X(\mathbb{R}^N), \quad \frac{dT_t(V)}{dt} = V(t) \circ T_t(V), \; T_0(V) = I.$$

The semi-tangent at χ_Ω is obtained by considering the limit of the differential quotient

$$\frac{1}{t}\left(\chi_{T_t(V)(\Omega)} - \chi_\Omega\right) \in L^\infty(\mathbb{R}^N).$$

which does not exist in $L^\infty(\mathbb{R}^N)$ or in $L^p_{\text{loc}}(\mathbb{R}^N)$, $1 \le p < \infty$.

To get a derivative consider the distribution associated with $T_t(V)(\Omega)$

$$\phi \mapsto \int_{\mathbb{R}^N} \chi_{T_t(V)(\Omega)}\, \phi \, dx = \int_{T_t(V)(\Omega)} \phi \, dx = \int_\Omega \phi \circ T_t \, \det DT_t \, dx : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}$$

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If $V \in C^{0,1}(\overline{\mathbb{R}^N}, \mathbb{R}^N)$, then

$$\frac{d}{dt}\bigg|_{t=0^+} \int_\Omega \phi \circ T_t \, \det DT_t \, dx = \int_\Omega \operatorname{div}\,(V(0)\,\phi)\; dx = \int_{\mathbb{R}^N} \chi_\Omega \operatorname{div}\,(V(0)\,\phi)\; dx.$$

The bilinear function

$$(\phi, V) \mapsto \int_{\mathbb{R}^N} \chi_\Omega \operatorname{div}\,(V(0)\,\phi)\; dx : H_0^1(\mathbb{R}^N) \times C^{0,1}(\overline{\mathbb{R}^N}, \mathbb{R}^N) \to \mathbb{R}$$

is continuous. This generates the continuous linear mapping

$$V \mapsto \nabla\chi_\Omega \cdot V : C^{0,1}(\overline{\mathbb{R}^N}, \mathbb{R}^N) \to H^{-1}(\mathbb{R}^N), \quad (\nabla\chi_\Omega \cdot V)\,\phi \overset{\text{def}}{=} \int_{\mathbb{R}^N} \chi_\Omega \operatorname{div}\,(V(0)\,\phi)\; dx,$$

where $\nabla\chi_\Omega$ is the distributional gradient of χ_Ω. The support of $\nabla\chi_\Omega \cdot V$ is in $\Gamma = \partial\Omega$.

So, the tangent space to $X(\mathbb{R}^N)$ (considered as a subset of the space of distributions) at χ_Ω contains the linear subspace (a set of full tangents to $X(\mathbb{R}^N)$ at χ_Ω)

$$\left\{ \nabla\chi_\Omega \cdot V : V \in C^{0,1}(\overline{\mathbb{R}^N}, \mathbb{R}^N) \right\} \subset H^{-1}(\mathbb{R}^N) \subset \mathcal{D}(D)'$$

of functions in $H^{-1}(\mathbb{R}^N)$.

When Ω is an open set with Lipschitz boundary, it is a measure

$$\frac{d}{dt}\bigg|_{t=0^+} \int_{\mathbb{R}^N} \chi_{T_t(V)(\Omega)}\, \phi \, dx = \int_\Gamma V(0) \cdot n_\Gamma \, \phi \, dx.$$

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The *shape derivative* perturbs the set via a diffeomorphism of the space; the *topological derivative* introduced in 1999 by Sokołowski and Zȯchowski [31, 32] perturbs the set Ω through a *geometric perturbation* by introducing *small holes* around a point $a \in \Omega$, that is, a *dilatation* of the point a.

Notation. Introduce the distance function d_E of x to E and the *r-dilatation*, $r > 0$, of a subset E of $\mathbb{R}^N$

$$d_E(x) \overset{\text{def}}{=} \inf_{e \in E} |x - e|, \quad E_r \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^N : d_E(x) \leq r \right\}. \tag{4.1}$$

In the special case $E = \{a\}$, assume that $\Omega \subset \mathbb{R}^3$ is open and $a \in \Omega$:

$$r > 0, \quad E_r \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^N : d_E(x) \leq r \right\} = \overline{B_r(a)}$$

auxiliairy variable: $t \overset{\text{def}}{=} m_3(B_r(a)) = \alpha_3 r^3, \quad \alpha_3 = 4\pi/3 =$ volume of unit ball in $\mathbb{R}^3$

Consider the continuous trajectory in the group $X(\mathbb{R}^N)$

$$t \mapsto \chi_{\Omega_t} : [0, \tau] \to X(\mathbb{R}^N), \quad \Omega_t \overset{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash \overline{B}_{\sqrt[3]{t/\alpha_3}}(a)$$

The differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ does not converge in $L^p_{\text{loc}}(\mathbb{R}^N)$, but the quotient of their associated distributions do.

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Consider the continuous trajectory in the group $X(\mathbb{R}^N)$

$$t \mapsto \chi_{\Omega_t} : [0, \tau] \to \mathrm{X}(\mathbb{R}^N), \quad \Omega_t \overset{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash \overline{B}_{\sqrt[3]{t/\alpha_3}}(a)$$

The differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ does not converge in $L^p_{\text{loc}}(\mathbb{R}^N)$, but the quotient of their associated distributions do.

Given $\phi \in \mathcal{D}(\mathbb{R}^3)$, the *weak limit* is

$$\frac{1}{t}\left[\int \chi_{\Omega_t}\phi\, dx - \int \chi_\Omega \phi\, dx\right] = \frac{1}{t}\left[\int_{\Omega_t} \phi\, dx - \int_\Omega \phi\, dx\right]$$

$$= -\frac{1}{\mathrm{m}_3(\overline{B}_{\sqrt[3]{t/\alpha_3}}(a))}\int_{B_{\sqrt[3]{t/\alpha_3}}(a)} \chi_\Omega\, \phi\, dx = -\frac{1}{\mathrm{m}_3(\overline{B}_r(a))}\int_{\overline{B}_r(a)} \chi_\Omega\, \phi\, dx \to -\phi(a)$$

$\phi \mapsto -\phi(a) : \mathcal{D}(\mathbb{R}^3) \to \mathbb{R}$ is a distribution (a measure).

Moreover, for a scalar $\rho > 0$,

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dx - \int_\Omega \phi\, dx\right] \to -\rho\, \phi(a).$$

and we get a *half tangent* to $X(\mathbb{R}^N)$ at $\chi_\Omega \in X(\mathbb{R}^N)$, but not a full tangent.

For a point $b \in \operatorname{int} \complement\Omega$, we would use an open ball and the distribution

$$\phi \mapsto \int_{\mathbb{R}^N} \chi_{\Omega \cup B_{(t/\alpha_N)^{1/N}(b)}}\, \phi\, dx : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R},$$

to obtain the semi-tangent

$$\phi \mapsto \rho\, \phi(b) : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}, \quad \rho \geq 0.$$

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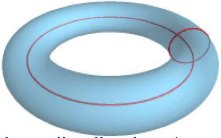
Let E be a compact non-intersecting C^2-curve in $\mathbb{R}^3$ such that $H_1(E)$ is finite. Consider the r-dilatation of the curve:

auxiliary variable : $t = \pi\, r^2 = \alpha_2\, r^2$
$\alpha_2 =$ volume of the unit ball in $\mathbb{R}^2$
$E_r \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N : d_E(x) \leq r\}$



torus

$$\phi \mapsto \int_E \phi\, dH_1 : \mathcal{D}(\mathbb{R}^3) \to \mathbb{R} \text{ is a distribution (measure).}$$

Consider the trajectory

$$t \mapsto \Omega_t : [0, \tau] \to X(\mathbb{R}^N), \quad \Omega_t \stackrel{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash E_{\sqrt[2]{t/\alpha_2}}$$

Given $\phi \in \mathcal{D}(\mathbb{R}^3)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\frac{1}{t}\left[\int_{\Omega_t} \phi\, dx - \int_\Omega \phi\, dx\right] = -\frac{1}{t}\int_{E_{\sqrt[2]{t/\alpha_2}}} \chi_\Omega\, \phi\, dx = -\frac{1}{\alpha_2\, r^2}\int_{E_r} \chi_\Omega\, \phi\, dx \to -\int_E \phi\, dH_1.$$

This distribution is a *semi-tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dx - \int_\Omega \phi\, dx\right] \to -\rho \int_E \phi\, dH_1.$$

Let $A \subset \mathbb{R}^N$ be a compact set of class $C^{1,1}$. Its boundary $E = \partial A$ is a $C^{1,1}$-hypersurface of dimension $N-1$ such that $H_{N-1}(\partial A) < \infty$.

Shell or sandwich of thickness $t = 2r$ around the hypersurface $E = \partial A$

$$r > 0, \quad E_r \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^N : d_{\partial A}(x) \le r\right\} = \left\{x \in \mathbb{R}^N : |b_A(x)| \le r\right\}$$

auxiliary variable: $t = 2r = \alpha_1\, r$, $\quad \alpha_1 = 2 =$ volume of the unit ball in $\mathbb{R}^1$.

$$\phi \mapsto \int_E \phi\, dH_{N-1} : \mathcal{D}(\mathbb{R}^3) \to \mathbb{R} \text{ is a distribution (measure).}$$

Consider the perturbed set and the continuous trajectories will be

$$t \mapsto \Omega_t \stackrel{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash E_{t/2}, \quad t \mapsto \chi_{\Omega \backslash E_{t/2}}; [0, \tau] \to X(\mathbb{R}^N).$$

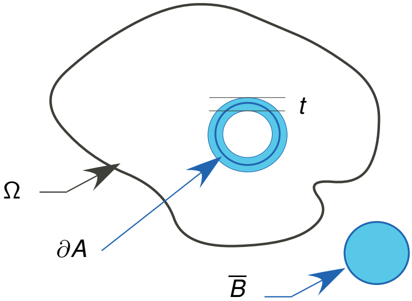
Given $\phi \in \mathcal{D}(\mathbb{R}^N)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\frac{1}{t}\left[\int_{\Omega_t} \phi\, dx - \int_\Omega \phi\, dx\right] = -\frac{1}{t}\int_{E_{t/2}} \chi_\Omega\, \phi\, dx = -\frac{1}{\alpha_1 r}\int_{E_r} \chi_\Omega\, \phi\, dx \to -\int_E \phi\, dH_{N-1}.$$

This distribution is a *semi-tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dx - \int_\Omega \phi\, dx\right] \to -\rho\int_E \phi\, dH_{N-1}.$$

When $N = 2$, we create a new connected component.



We can extend those construction to submanifolds of dimension d, $1 \le d < N-1$, in $\mathbb{R}^N$. Consider the *r*-dilated set E_r around E

$$\phi \mapsto \int_E \phi \, dH_d : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R} \text{ is a distribution (measure).}$$

Let $t = \alpha_{N-d} r^{N-d}$ be the auxiliary variable. The continuous trajectory will be

$$t \mapsto \chi_{\Omega_t} : [0, \tau] \to X(\mathbb{R}^N), \quad \Omega_t \overset{\text{def}}{=} \Omega \backslash E_r = \Omega \backslash E_{\sqrt[N-d]{t/\alpha_{N-d}}}.$$

Given $\phi \in \mathcal{D}(\mathbb{R}^N)$, the weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is

$$\begin{aligned} \frac{1}{t}\left[\int_{\Omega_t} \phi \, dm_N - \int_\Omega \phi \, dm_N\right] &= -\frac{1}{t}\int_{E_{\sqrt[N-d]{t/\alpha_{N-d}}}} \chi_\Omega \, \phi \, dm_N \\ &= -\frac{1}{\alpha_{N-d}\, r^{N-d}} \int_{E_r} \chi_\Omega \, \phi \, dm_N \to -\int_E \phi \, dH_d. \end{aligned}$$

This distribution is a *semi-tangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi \, dm_N - \int_\Omega \phi \, dm_N\right] \to -\rho \int_E \phi \, dH_d.$$

In the previous examples, the limit of the differential quotient for the dilatation E_r is related to the *d-dimensional Minkowski content* of a subset E of $\mathbb{R}^N$.

DEFINITION

Given d, $0 \leq d \leq N$, the *d-dimensional upper and lower Minkowski contents* of a subset E of $\mathbb{R}^N$ are defined through an r-dilatation of the set E as follows

$$M^{*d}(E) \overset{\text{def}}{=} \limsup_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d}\, r^{N-d}}, \quad M_*^d(E) \overset{\text{def}}{=} \liminf_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d}\, r^{N-d}}, \tag{4.2}$$

where m_N is the Lebesgue measure in $\mathbb{R}^N$ and α_{N-d} is the volume of the unit ball in $\mathbb{R}^{N-d}$. When the two limits exist and are equal, we say that E admits a d-dimensional *Minkowski content* and the common value will be denoted $M^d(E)$.

Since the dilatation E_r does not distinguish between E and its closure, it can be assumed that E is closed in $\mathbb{R}^N$.

Intuitively, $M^d(E)$ is a measure of the d-dimensional "area" or "volume" of an object E in $\mathbb{R}^N$. It plays a role similar to the *d-dimensional Hausdorff or Radon measure* in $\mathbb{R}^N$, but it is generally not a measure.

We are interested in closed subsets E of $\mathbb{R}^N$ such that

$$M^d(E) \overset{\text{def}}{=} \lim_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d}\, r^{N-d}} \text{ exists} \tag{4.3}$$

and M^d be a *measure* such that the following distribution makes sense

$$\phi \mapsto \int_E \phi\, dM^d = \lim_{r \searrow 0} \frac{1}{\alpha_{N-d}\, r^{N-d}} \int_{E_r} \phi\, dm_N : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}\,. \tag{4.4}$$

In applications to d-dimensional objects, $0 \leq d \leq N$, the choice of the auxiliary variable, is the volume $t = \alpha_{N-d}\, r^{N-d}$ of the ball of radius $r = (t/\alpha_{N-d})^{1/(N-d)}$ in $\mathbb{R}^{N-d}$. Given $\Omega \subset \mathbb{R}^N$ open, a continuous trajectory $t \mapsto \chi_{\Omega_t}$ is obtained in $X(\mathbb{R}^N)$ such that

$$\chi_{\Omega_t} \to \chi_{\Omega \setminus E} \text{ in } L^p_{\text{loc}}(\mathbb{R}^N), \quad 1 \leq p < \infty.$$

If $m_N(E) = 0$, then $\chi_{\Omega_t} \to \chi_\Omega$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, $1 \leq p < \infty$. The expected semi-tangents would be the distribution (measure)

$$\phi \mapsto -\int_E \phi\, dM_d : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}\,. \tag{4.5}$$

We are interested in closed subsets E of $\mathbb{R}^N$ such that

$$M^d(E) \stackrel{\text{def}}{=} \lim_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d}\, r^{N-d}} \text{ exists} \tag{4.3}$$

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$$\phi \mapsto -\int_E \phi \, dM_d : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}\,. \tag{4.5}$$

Thanks to the pioneering and seminal work of Federer [18] and the extension of his work by Ambrosio et al [1], the previous construction can be readily extended to the dilation of some families of rectifiable sets for which the d-dimensional Minkowski content is equal to the d-dimensional Hausdorff measure H_d:

$$M^d(E) \overset{\text{def}}{=} \lim_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d}\, r^{N-d}} = H_d(E). \tag{4.6}$$

This property is related to the property that E is d-rectifiable in $\mathbb{R}^N$.

DEFINITION (FEDERER [16, PP. 251–252])

Let E be a subset of a metric space X. $E \subset X$ is d-rectifiable if it is the image of a compact subset K of $\mathbb{R}^d$ by a Lipschitz continuous function $f : \mathbb{R}^d \to X$.

THEOREM ([16, P. 275])

If $E \subset \mathbb{R}^N$ is compact and d-rectifiable, then $M^d(E) = H_d(E)$.

DEFINITION (RECTIFIABLE SETS [2, DFN. 2.57, P. 80])

Let $E \subset \mathbb{R}^N$ be an H_d-measurable.

(i) E is *countably d-rectifiable* if there exist countably many Lipschitzian functions $f_i : \mathbb{R}^d \to \mathbb{R}^N$ such that

$$E \subset \cup_{i=0}^{\infty} f_i(\mathbb{R}^d). \tag{4.7}$$

(ii) E is said to be *countably H_d-rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^d \to \mathbb{R}^N$ such that $E \backslash \cup_i f_i(\mathbb{R}^d)$ is H_d-negligible, that is,

$$H_d\left(E \backslash \cup_{i=0}^{\infty} f_i(\mathbb{R}^d)\right) = 0. \tag{4.8}$$

(iii) E is said to be *H_d-rectifiable* if E is *countably H_d-rectifiable* and $H_d(E) < \infty$. □

Ambrosio et al [2] have noticed that an $(N-1)$-rectifiable set is always contained in the support of a probability measure satisfying a suitable $(N-1)$-dimensional lower bound: there exists $\gamma > 0$ and a probability measure η in $\mathbb{R}^N$ satisfying:

$$\eta(B_r(x)) \geq \gamma\, r^d, \quad \forall r \in (0,1),\ \forall x \in E \quad \text{(density assumption)}. \tag{4.9}$$

Then they extend Theorem 9 to countably H_d-rectifiable compact sets.

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In all the examples of the previous section, the closed set E has positive reach. For such sets Federer [17, Thm. 5.6, p. 455] has extended the *Steiner formula* and, as a consequence, they have a normal *Minkowski content* for some d, $0 \leq d \leq N$.

THEOREM

Given a closed subset $A \subset \mathbb{R}^N$ of positive reach, that is, reach $(A) > 0$, then there exist unique Radon measures $\psi_0, \psi_1, , \ldots, \psi_N$ over $\mathbb{R}^N$ such that, for $0 \leq r <$ reach (A),

$$\mathrm{m}_N\left(\left\{x \in \mathbb{R}^N : d_A(x) \leq r \text{ and } p_A(x) \in E\right\}\right) = \sum_{m=0}^{N} \alpha_{N-m}\, r^{N-m}\, \psi_m(E), \tag{4.10}$$

whenever E is a Borel set of $\mathbb{R}^N$, and, consequently,

$$\int_{E_r^A} f \circ p_A \, d\mathrm{m}_N = \sum_{m=0}^{N} \alpha_{N-m}\, r^{N-m} \int_E f\, d\psi_m, \tag{4.11}$$

$$E_r^A \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^N : d_A(x) \leq r \text{ and } p_A(x) \in E\right\}, \tag{4.12}$$

whenever f is a bounded real valued Baire function on $\mathbb{R}^N$ with bounded support.

Consider a closed subset A of $\mathbb{R}^N$ of positive reach such that $\partial A = A$ and $m_N(A) = 0$ ($\Rightarrow \psi_N(A) = 0$). This includes smooth submanifolds of $\mathbb{R}^N$.

- For $E = A$, $E_r^A = A_r$ and there exists $0 \le d \le N-1$ such that

$$\exists d,\ 0 \le d \le N-1, \quad M^d(A) \stackrel{\text{def}}{=} \lim_{r \searrow 0} \frac{m_N(A_r)}{\alpha_{N-d}\, r^{N-d}} = \psi_d(A). \tag{4.13}$$

- For E a closed subset of $A = \partial A$, consider its *normal r-dilatation* with respect to A:

$$E_r^A \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^N : d_A(x) \le r \text{ and } p_A(x) \in E \right\}, \quad 0 \le r < \text{reach}\,(A), \quad E \subset \partial A \tag{4.14}$$

$$\exists d,\ 0 \le d \le N-1, \quad \lim_{r \searrow 0} \frac{m_N(E_r^A)}{\alpha_{N-d}\, r^{N-d}} = \psi_d(E). \tag{4.15}$$

Consider a closed subset A of $\mathbb{R}^N$ of positive reach such that $\partial A = A$ and $m_N(A) = 0$ ($\Rightarrow \psi_N(A) = 0$). This includes smooth submanifolds of $\mathbb{R}^N$.

- For $E = A$, $E_r^A = A_r$ and there exists $0 \le d \le N-1$ such that

$$\exists d,\ 0 \le d \le N-1, \quad M^d(A) \stackrel{\text{def}}{=} \lim_{r \searrow 0} \frac{m_N(A_r)}{\alpha_{N-d}\, r^{N-d}} = \psi_d(A). \tag{4.13}$$

- For E a closed subset of $A = \partial A$, consider its *normal r-dilatation* with respect to A:

$$E_r^A \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^N : d_A(x) \le r \text{ and } p_A(x) \in E \right\}, \quad 0 \le r < \text{reach}(A), \quad E \subset \partial A \tag{4.14}$$

$$\exists d,\ 0 \le d \le N-1, \quad \lim_{r \searrow 0} \frac{m_N(E_r^A)}{\alpha_{N-d}\, r^{N-d}} = \psi_d(E). \tag{4.15}$$

Consider a closed subset A of $\mathbb{R}^N$ of positive reach such that $\partial A = A$ and $\mathrm{m}_N(A) = 0$ $(\Rightarrow \psi_N(A) = 0)$. This includes smooth submanifolds of $\mathbb{R}^N$.
- For $E = A$, $E_r^A = A_r$ and there exists $0 \le d \le N-1$ such that

$$\exists d,\ 0 \le d \le N-1, \quad M^d(A) \overset{\text{def}}{=} \lim_{r \searrow 0} \frac{\mathrm{m}_N(A_r)}{\alpha_{N-d}\, r^{N-d}} = \psi_d(A). \tag{4.13}$$

- For E a closed subset of $A = \partial A$, consider its *normal r-dilatation* with respect to A:

$$E_r^A \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^N : d_A(x) \le r \text{ and } p_A(x) \in E \right\}, \quad 0 \le r < \operatorname{reach}(A), \quad E \subset \partial A \tag{4.14}$$

$$\exists d,\ 0 \le d \le N-1, \quad \lim_{r \searrow 0} \frac{\mathrm{m}_N(E_r^A)}{\alpha_{N-d}\, r^{N-d}} = \psi_d(E). \tag{4.15}$$

Shell or sandwich of thickness $t = 2r$ around the piece of surface or patch E

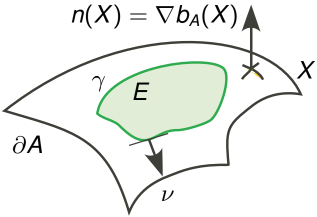
$$r > 0, \quad E_r^A \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^N : |b_A(x)| \leq r, \quad p_{\partial A}(x) \in E \right\}$$

$$t = 2r = \alpha_1\, r, \quad \alpha_1 = 2 = \text{ volume of the unit ball in } \mathbb{R}^1.$$

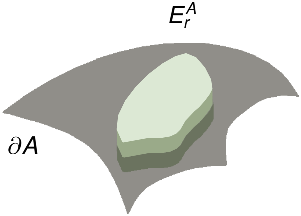
$$\phi \mapsto \int_E \phi \, dH_2 : \mathcal{D}(\mathbb{R}^3) \to \mathbb{R} \text{ is a distribution (measure).}$$

Given $U_\varepsilon(\partial A) \subset \Omega$, the perturbed set will be

$$t \mapsto \Omega_t \overset{\text{def}}{=} \Omega \backslash E_r^A = \Omega \backslash E_{t/2}^A$$



(a) E with boundary γ



(b) Sandwich E_r^A

Some Concluding Remarks

- It is possible to construct a wide range of metrics and complete metrics spaces including families of sets with prescribed boundary smoothness (Erlangen Workshop [12]). The choice is application dependent.
- In most cases the group and the metric structures are sufficiently compatible to introduce some form of tangent space (not necessarily a Hilbert or Banach space, that is non Riemannian or Finsler) and semidifferentials or differentials of functions.
- Similar techniques can be developed for families of subsets of a submanifold of $\mathbb{R}^{\mathrm{N}}$ of co-dimension greater or equal to one (Proc. of the Erlangen Workshop [12]).
- The general approach for the space $\mathrm{X}(\mathbb{R}^N)$ of characteristic functions in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ can be extended to metric spaces associated with the oriented distance function, but the convergence of the perturbed sets $\Omega\backslash E_t$ via an r-dilatation of a subset E of dimension d will not converge to Ω ($\chi_{\Omega\backslash U_t(E)} \to \chi_\Omega$) but to $\Omega\backslash E$ ($b_{\Omega\backslash U_t(E)} \to b_{\Omega\backslash E}$). The choice is obviously very much problem dependent!!
- \- Connections with Topological Optimization (see A.V. Cherkaev, R. Kohn (Eds.), Topics in the Mathematical Modelling of Composite Materials ...).

- Thank you for your attention -

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