The dynamics of shear flows and null-controllability of Kolmogorov-type equations

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Keywords

- Kolmogorov equations
- Enhanced dissipation
- Hypoellipticity
- Mixing
- Hypocoercivity

Passive scalars

Consider a scalar quantity on a 2D domain $f(t,x,y)$ subject to advection and diffusion, so that

$$\partial_t f + \boldsymbol{u} \cdot \nabla f = \nu \Delta f, \qquad f(0) = f_{in}. \qquad \text{(ADE)}$$

- $\boldsymbol{u}$ is an incompressible, time-independent given (Lipschitz) flow
- $\nu \ll 1$ is a diffusion parameter
- $f_{in} \in L^2$ is a given, mean-free initial datum
- example: f is the concentration of milk in coffee

Kolmogorov equation

When $\boldsymbol{u} = (y, 0)$, and the spatial domain is $\mathbb{T} \times \mathbb{R}$ (or $\mathbb{R}^2$), the equation reduces to the Kolmogorov equation

$$\partial_t f + y \partial_x f = \nu \partial_{yy} f, \qquad f(0) = f_{in}.$$

Extremely relevant in many fields:

- basic model in Kinetic theory;
- in fluid dynamics: linearization around Couette flow;
- in control theory: do controllability results on 1D heat equation hold for Kolmogorov equations?

The control problem

Given f_{in}, $\omega \subset \mathbb{T} \times \mathbb{R}$ and $T > 0$, does there exist a control $v = v(t, x, y)$ such that the solution to

$$\partial_t f + y \partial_x f = \nu \partial_{yy} f + v 1_\omega, \qquad f(0) = f_{in},$$

is equal to 0 at time $t = T$.
Answer: YES (Beauchard, Zuazua '09).

How about for similar equations (like y^γ instead of y)?
Answer: YES-ish (for $\gamma = 2$, T cannot be small, Beauchard '14).

Many results for hypoelliptic equations in 1D with degenerate diffusion (Beauchard, Cannarsa, Guglielmi, Martinez, Raymond, Vancostonoble ...)

Key ingredient

Estimate on $\widehat{f}_k$ =Fourier transform in x of the solution: for every $k \neq 0$,

$$\|\widehat{f}_k(t)\|_{L^2_y} \leq \|\widehat{f}_k(0)\|_{L^2_y} \mathrm{e}^{-\nu k^2 t^3}$$

This estimate was deduced by Lord Kelvin in 1887.

- in the new variable $X = x - ty$, $g(t, X, y) = f(t, X + ty, y)$ satisfies $\partial_t g = \nu(\partial_y - t\partial_X)^2 g$.
- then if $\widehat{g}$ =Fourier transform in X and y: $\partial_t \widehat{g} = -\nu(\eta - kt)^2 \widehat{g}$.
- $\widehat{g}(t, k, \eta) = \widehat{f}_{in}(k, \eta) \exp\left[-\nu \int_0^t (\eta - k\tau)^2 \mathrm{d}\tau\right]$

Problem: if $u(y)$ instead of y, no Fourier transform in y!
Actually: with y^2, one gets Schrödinger...

The diffusive time-scale

(ADE) generates a well-defined evolution semigroup

$$S_\nu(t) : L^2 \to L^2.$$

A simple L^2 estimate that uses the incompressibility of $\boldsymbol{u}$ and the Poincaré inequality entails

$$\|S_\nu(t)\|_{L^2 \to L^2} \leq \mathrm{e}^{-\nu t}.$$

- There is a characteristic (diffusive) time-scale $\sim 1/\nu$
- Not much information is used about the flow $\boldsymbol{u}$ other than incompressibility

Enhanced dissipation

Can the flow $\boldsymbol{u}$ enhance diffusion?

Definition (CKRZ '08)

$\boldsymbol{u} : \mathbb{T}^2 \to \mathbb{R}^2$ is called relaxation enhancing if for every $\tau > 0$ and $\delta > 0$, there exists $\nu_0 = \nu_0(\tau, \delta)$ such that for any $\nu < \nu_0$ and any $f_0 \in L^2$ we have

$$\|S_\nu(\tau/\nu)\|_{L^2 \to L^2} < \delta$$

- By the diffusion time-scale, energy is already dissipated
- If $\|S_\nu(t)\|_{L^2 \to L^2} \lesssim \mathrm{e}^{-\varepsilon \nu^p t}$, for some $p \in (0,1)$, then $\boldsymbol{u}$ is relaxation enhancing

Spectral properties

We define the closed subspace

$$E = \overline{\mathrm{span}\left\{\varphi \in H^1 : \boldsymbol{u}\cdot\nabla\varphi = i\lambda\varphi,\ \lambda\in\mathbb{R}\right\}}^{L^2},$$

generated by H^1-eigenfunctions of $\boldsymbol{u}\cdot\nabla$.

Theorem (CKRZ '08)

$\boldsymbol{u}$ is relaxation enhancing if and only if $E = \{0\}$.

Shear flows

Take $\boldsymbol{u} = (u(y), 0)$, we assume that $u \in C^{n_0+2}(\mathbb{T})$ has a *finite* number of critical points, where $n_0 \in \mathbb{N}$ denotes the maximal order of vanishing of u' at the critical points.

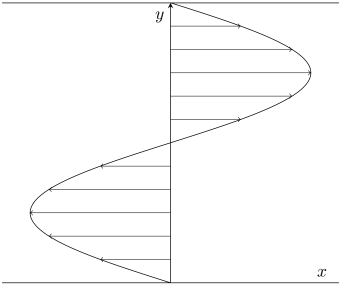

Figure: $u(y) = \sin y$

Shear flows

(ADE) becomes

$$\partial_t f + u\partial_x f = \nu \begin{cases} \Delta f \\ \partial_{yy} f \end{cases} \qquad f(0) = f_0.$$

- $E = \{\varphi \in L^2 : \varphi(x,y) = \varphi(y) \; a.e.\}$
- $\partial_t \widehat{f}_k - iku\widehat{f}_k = \nu \partial_{yy} \widehat{f}_k$ where $k =$Fourier variable in x, so that

$$\partial_t \widehat{f}_0 = \nu \partial_{yy} \widehat{f}_0$$

- enhanced dissipation on $E^\perp$, since $P_0 : L^2 \to E$ commutes with $S_\nu(t)$

Main result

Theorem (Bedrossian, CZ '15)

There exist positive constants $\varepsilon \ll 1$, $C \geq 1$ and $\kappa_0 \ll 1$ (depending only on u) such that the following holds: for every $\nu > 0$ and every integer $k \neq 0$ such that $\nu|k|^{-1} \leq \kappa_0$, there hold the L^2 decay estimates

$$\|S_\nu(t)P_k\|_{L^2 \to L^2} \leq C\mathrm{e}^{-\varepsilon\lambda_{\nu,k}t}, \qquad \forall t \geq 0,$$

where P_k denotes the projection to the k-th Fourier mode in x and

$$\lambda_{\nu,k} = \frac{\nu^{\frac{n_0+1}{n_0+3}}|k|^{\frac{2}{n_0+3}}}{(1 + \log|k| + \log\nu^{-1})^2}$$

is the decay rate.

Remarks

$$\lambda_{\nu,k} = \frac{\nu^{\frac{n_0+1}{n_0+3}} |k|^{\frac{2}{n_0+3}}}{(1+\log|k| + \log \nu^{-1})^2}$$

- quantification of the faster time-scale $O(1/\nu^p)$, hence enhanced dissipation
- rate is believed to be sharp, up to log-corrections
- in the channel case, modify n_0 with $n_c = \max\{n_0, 1\}$
- in real variables, if $\nu \leq \kappa_0$, we obtain

$$\|f(t) - \langle f(t)\rangle_x\|_{L^2} \lesssim \mathrm{e}^{-\varepsilon \frac{\nu^{\frac{n_0+1}{n_0+3}}}{(1+\log \nu^{-1})^2} t}$$

Hypoellipticity

The equation

$$\partial_t f + u\partial_x f = \partial_{yy} f$$

does not have an obvious regularization in x.

- Hypoellipticity: setting $\nu = 1$ and ∂_{yy} dissipation, data which are only initially L^2 become instantly Gevrey-p regular for all $p > \frac{n_0+3}{2}$, where Gevrey-p, $\mathcal{G}^p$ with $p \geq 1$, is defined via the space
$$\mathcal{G}^p = \bigcup_{\lambda>0} \left\{ f \in L^2 : \|\mathrm{e}^{\lambda|\nabla|^{1/p}} f\|_{L^2} < \infty \right\}.$$
- Prototypical example in kinetic theory: $\partial_t f + v \cdot \nabla_x f = \Delta_v f + \dots$
- Precise rate in k is used in null-controllability of Kolmogorov-type equations

Inviscid mixing

The result is consistent with the mixing properties of the inviscid equation

$$\partial_t f + u\partial_x f = 0.$$

Theorem (Bedrossian, CZ '15)

For all $k \neq 0$ we have

$$\|\widehat{f}_k(t)\|_{H_y^{-1}} \leq C\langle kt\rangle^{-1/(n_0+1)}\|\widehat{f}_k(0)\|_{H_y^1},$$

where $\langle kt\rangle = \sqrt{1+(kt)^2}$.

The H^{-1} norm is often used to quantify mixing (Lin, Thiffeault, Doering '11) as it provides an averaged measure of the characteristic length-scale of the solution.

Faster time-scales

- $\widehat{f}_k(y)$ is concentrated in y-frequencies η which satisfy $|\eta| \gtrsim \langle kt \rangle^{\frac{1}{n_0+1}}$
- For small ν, the inviscid mixing is dominant: $\nu\partial_{yy}$ acts like $-\nu\langle kt \rangle^{\frac{2}{n_0+1}} \widehat{f}_k$ damping.

Hence

$$\partial_t \widehat{f}_k - iku\widehat{f}_k = -\nu\langle kt \rangle^{\frac{2}{n_0+1}} \widehat{f}_k$$

implying

$$\|\widehat{f}_k(t)\|_{L^2}^2 \lesssim \exp\left[-\nu k^{\frac{2}{n_0+1}} \int_0^t \tau^{\frac{2}{n_0+1}} \mathrm{d}\tau\right]$$

and

$$-\nu k^{\frac{2}{n_0+1}} \int_0^t \tau^{\frac{2}{n_0+1}} \mathrm{d}\tau = -\nu k^{\frac{2}{n_0+1}} t^{\frac{n_0+3}{n_0+1}} = -\left[\nu^{\frac{n_0+1}{n_0+3}} k^{\frac{2}{n_0+3}} t\right]^{\frac{n_0+3}{n_0+1}}$$

Hypocoercivity

First try (as in Villani '09):

$$\Phi = \|f\|^2 + \alpha\|\partial_y f\|^2 + 2\beta \int_{\mathbb{T}^2} u'\partial_x f \partial_y f + \gamma\|u'\partial_x f\|^2,$$

since $u'\partial_x = [\partial_y, u\partial_x]$. In fact:

$$\Phi_k = \|\widehat{f}_k\|^2 + \|\sqrt{\alpha}\partial_y \widehat{f}_k\|^2 - 2k \int_{\mathbb{T}} i\beta u' \widehat{f}_k, \partial_y \widehat{f}_k \mathrm{d}y + |k|^2\|\sqrt{\gamma}u'\widehat{f}_k\|^2$$

- α, β, γ have to be ν and k dependent (Beck, Wayne '13)
- α, β, γ have to be y-dependent: each critical point has to be treated differently depending on its order
- in the end

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_k + \varepsilon\nu^{\frac{n_0+1}{n_0+3}} k^{\frac{2}{n_0+3}}\Phi_k \leq 0$$

Critical points

- $\{\phi_j\}_{j=0}^{n_0}$ is a partition of unity, such that $\phi_j = 1$ around all critical points of order j
- the weights of the energy functional are defined accordingly

$$\alpha(y) = \sum_{j=0}^{n_0} \varepsilon_{\alpha,j}\alpha_j\phi_j(y), \quad \beta(y) = \sum_{j=0}^{n_0} \varepsilon_{\beta,j}\beta_j\phi_j(y),$$
$$\gamma(y) = \sum_{j=0}^{n_0} \varepsilon_{\gamma,j}\gamma_j\phi_j(y),$$

- $\varepsilon_{\alpha,j}, \varepsilon_{\beta,j}, \varepsilon_{\gamma,j}$ are chosen small independently of ν and k
- $\alpha_j = \dfrac{\nu^{\frac{2}{j+3}}}{|k|^{\frac{2}{j+3}}}, \qquad \beta_j = \dfrac{\nu^{\frac{1-j}{j+3}}}{|k|^{\frac{4}{j+3}}}, \qquad \gamma_j = \dfrac{\nu^{-\frac{2j}{j+3}}}{|k|^{\frac{6}{j+3}}}.$

Localized spectral gap

Theorem (Bedrossian, CZ '15)

For all $\sigma > 0$ and $g : \mathbb{T} \to \mathbb{C}$ (denoting $g_j = g\sqrt{\phi_j}$),

$$\sigma^{\frac{j}{j+1}} \|g_j\|^2 \lesssim \sigma \|\partial_y g_j\|^2 + \|u' g_j\|^2.$$

The way this is used is $g = \widehat{f}_k$ and

$$|k|^{\frac{2}{j+1}} \beta_j^{\frac{1}{j+1}} \varepsilon_{\beta,j}^{\frac{1}{j+1}} \nu^{\frac{j}{j+1}} \|\widehat{f}_{k,j}\|^2 \lesssim \nu \|\partial_y \widehat{f}_{k,j}\|^2 + |k|^2 \beta_j \varepsilon_{\beta,j} \|u' \widehat{f}_{k,j}\|^2,$$

for all $j = 0, \ldots n_0$. Hence, if $\nu|k|^{-1} \leq \kappa_0 \ll 1$,

$$\sum_{j=0}^{n_0} \left[\nu \|\partial_y \widehat{f}_{k,j}\|^2 + |k|^2 \varepsilon_{\beta,j} \beta_j \|u' \widehat{f}_{k,j}\|^2 \right] \lesssim \nu \|\partial_y \widehat{f}_k\|^2 + |k|^2 \|\sqrt{\beta} u' \widehat{f}_k\|^2.$$

Summary

- Enhanced dissipation: faster decay rate induced by the flow
- Hypoellipticity: regularization in x induced by the flow
- Mixing: transfer of information from small to large frequencies

The proof emphasize that in certain singular limits, they are mathematically *equivalent* to hypocoercivity.

- Null-controllability: avoid use of Fourier transform method (in y). Is sharp rate enough to prove null-controllability?

THANK YOU