Global strong solutions of the full Navier-Stokes and Q-tensor system for nematic liquid crystal flows in two dimensions

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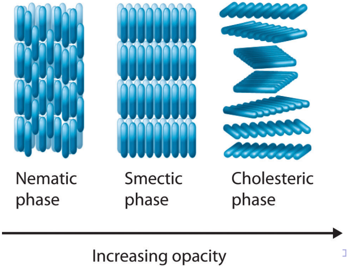
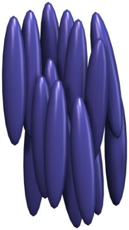
Liquid crystals

- Fourth state of matter besides gas, liquid and solid
- Intermediate state between crystalline and isotropic
- The molecules do not possess a positional order (even partial) but display an orientational order
- Different liquid crystal phases, depending on the *amount* of order in the material
- nematic (*thread*), smectic (*soap*), cholesteric (*bile - solid*)



Nematic liquid crystals

- Simplest liquid crystal phase, *close* to the liquid one
- The molecules float around as in a liquid phase, but have the tendency to align along a preferred direction due to their orientation



The state variables $\boldsymbol{u}$ and Q

- Beris & Edward model (1994)
- $\boldsymbol{u}$ velocity field of the flow
- Q symmetric traceless $d \times d$ tensor describing the local orientation and degree of ordering of the molecules ($d = 2, 3$ spatial dimension)

- coupled PDE system describing the interaction between the fluid velocity and the alignment of liquid crystal molecules

- incompressible Navier-Stokes equations for $\boldsymbol{u}$, with highly nonlinear anisotropic force terms
- nonlinear convection-diffusion parabolic equation for Q

The Navier-Stokes and Q-tensor system

$$
\begin{cases}
\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla P = \lambda \nabla \cdot (\tau + \sigma) \\
\\
\operatorname{div} \boldsymbol{u} = 0 \\
\\
\partial_t Q + \boldsymbol{u} \cdot \nabla Q - S(\nabla \boldsymbol{u}, Q) = \Gamma H(Q)
\end{cases}
$$

$\boldsymbol{u}(x,t) : \mathbb{T}^d \times (0,\infty) \to \mathbb{R}^d \qquad Q(x,t) : \mathbb{T}^d \times (0,\infty) \to S_0^{(d)}$

$\mathbb{T}^d$ periodic box with period $a_i = 1$ in the i*th* direction

$S_0^{(d)} = \{Q \in \mathbb{R}^{d \times d} \,|\, Q_{ij} = Q_{ji},\ \operatorname{tr}(Q) = 0,\ i,j = 1,\ldots,d\}$

τ, σ symmetric and skew-symmetric parts of the stress tensor

P pressure

$\nu > 0$ viscosity

λ competition between kinetic and elastic potential energy

Γ Deborah number

The tensor H and the matrix S

H variational derivative of the free energy $\mathcal{F}(Q)$ w.r.t. Q

S describes the rotating and stretching effects on Q

$$H(Q) = L\Delta Q - aQ + b\left(Q^2 - \frac{1}{d}\mathrm{tr}(Q^2)\mathbb{I}\right) - cQ\,\mathrm{tr}(Q^2)$$

$$S(\nabla \boldsymbol{u}, Q) = (\xi D + \Omega)\left(Q + \frac{1}{d}\mathbb{I}\right) + \left(Q + \frac{1}{d}\mathbb{I}\right)(\xi D - \Omega)$$
$$-2\xi\left(Q + \frac{1}{d}\mathbb{I}\right)\mathrm{tr}(Q\nabla \boldsymbol{u})$$

$\mathbb{I}$ identity matrix

D, Ω symmetric and skew-symmetric part of the strain tensor

$$D = \frac{\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}}{2} \qquad \Omega = \frac{\nabla \boldsymbol{u} - \nabla^T \boldsymbol{u}}{2}$$

The stress tensors τ and σ

$$\tau = \xi\left(Q + \frac{1}{d}\mathbb{I}\right)H(Q) - \xi H(Q)\left(Q + \frac{1}{d}\mathbb{I}\right)$$
$$+2\xi\left(Q + \frac{1}{d}\mathbb{I}\right)\mathrm{tr}(QH(Q)) - L\nabla Q \odot \nabla Q$$

where

$$(\nabla Q \odot \nabla Q)_{ij} = \sum_{k,l=1}^{d} \nabla_i Q_{kl} \nabla_j Q_{kl}$$

$$\sigma = QH(Q) - H(Q)Q$$

Further features

$\xi \in \mathbb{R}$ depends on the molecular shapes of the liquid crystal and measures the ratio between the tumbling and the aligning effect that a shear flow exerts on the liquid crystal directors

$L > 0$ elastic constant

$a, b, c \in \mathbb{R}$ material and temperature dependent coefficients

we assume $c > 0 \Rightarrow$ free energy $\mathcal{F}(Q)$ is bounded from below

$$\mathcal{F}(Q) = \int_{\mathbb{T}^d} \left(\frac{L}{2} |\nabla Q|^2 + f_b(Q) \right)$$

$f_b(Q) = \frac{a}{2} \mathrm{tr}(Q^2) - \frac{b}{3} \mathrm{tr}(Q^3) + \frac{c}{4} \mathrm{tr}^2(Q^2)$ bulk functional

$\frac{L}{2} |\nabla Q|^2$ elastic functional

Initial and boundary conditions

- periodic boundary conditions

$$\boldsymbol{u}(x + e_i, t) = \boldsymbol{u}(x, t) \quad \text{for } (x, t) \in \mathbb{T}^d \times \mathbb{R}^+$$

$$Q(x + e_i, t) = Q(x, t) \quad \text{for } (x, t) \in \mathbb{T}^d \times \mathbb{R}^+$$

where $\{e_i\}_{i=1}^d$ is the canonical orthonormal basis of $\mathbb{R}^d$

- spatially 1-periodic initial data

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x), \ \ x \in \mathbb{T}^d, \quad \text{with } \nabla \cdot \boldsymbol{u}_0 = 0$$

$$Q(x, 0) = Q_0(x), \ \ x \in \mathbb{T}^d, \quad \text{with } Q_0 \in S_0^{(d)}$$

Remark

The system preserves for all time both the symmetry and tracelessness of any solution Q associated to an initial datum with the same properties.

Literature: $\xi = 0$

- Paicu & Zarnescu (2012)
  - $\exists$ of global weak solutions in 2D and 3D
  - higher order global regularity in 2D
  - weak-strong uniqueness in 2D
- Dai, Feireisl, Rocca, Schimperna & Shonbek (2014)
  - asymptotic behavior in 3D in particular cases
- Abels, Dolzmann & Liu (2015)
- Guillen-Gonzales & Rodriguez-Bellido (2014), (2015)
  - $\exists$ of global weak solutions in 2D and 3D
  - $\exists!$ of local strong solutions in 2D and 3D
  - regularity results in 2D and 3D

Literature: $\xi \neq 0$

- Paicu & Zarnescu (2011)

  • $\exists$ of global weak solutions in 2D and 3D for small $|\xi|$

- De Anna & Zarnescu (2015)

  • uniqueness of weak solutions in 2D

- Abels, Dolzmann & Liu (2014)

  • local in time well-posedness and $\exists$ of global weak solutions in 2D and 3D with nonhomogeneous Dirichlet/Neumann boundary conditions

Literature: (B-E) system $\Rightarrow$ (E-L) system

- Ericksen (1976) and Leslie (1978) developed the hydrodynamics theory of liquid crystals based on the evolution of the *velocity field* $\boldsymbol{u}$ and the *director field* $\boldsymbol{d}$
- Wang, Zhang & Zhang (2015)

  • rigorous derivation of the classical Ericksen-Leslie system from the Beris-Edwards system
- Lin & Liu (1995)
- Grasselli & Wu (2011), (2013)
- Wang, Zhang & Zhang (2013)
- Wu, Xu & Liu (2013)
- Cavaterra & Rocca (2013)
- Huang, Lin & Wang (2014)
- Cavaterra, Rocca & Wu (2014)

- (E-L) system with Leslie coefficients
- Existence of suitably defined weak solutions in 3D without any restriction on the size of the fluid viscosity and of the initial data
- Main difficulty: the lack of control of $\|\boldsymbol{d}\|_{L^\infty}$ (no maximum principle for $\boldsymbol{d}$)

Navier-Stokes and Q-tensor system + (PBC) +(IC)

$$
(NSQ)\begin{cases}
\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla P = \lambda\nabla\cdot(\tau+\sigma), & \mathbb{T}^d\times\mathbb{R}^+ \\
\mathrm{div}\,\boldsymbol{u} = 0, & \mathbb{T}^d\times\mathbb{R}^+ \\
\partial_t Q + \boldsymbol{u}\cdot\nabla Q - S(\nabla\boldsymbol{u}, Q) = \Gamma H(Q), & \mathbb{T}^d\times\mathbb{R}^+ \\
\boldsymbol{u}(x+e_i,t) = \boldsymbol{u}(x,t), & (x,t)\in\mathbb{T}^d\times\mathbb{R}^+ \\
Q(x+e_i,t) = Q(x,t), & (x,t)\in\mathbb{T}^d\times\mathbb{R}^+ \\
\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x) \ \text{ with } \ \nabla\cdot\boldsymbol{u}_0(x) = 0, & x\in\mathbb{T}^d \\
Q(x,0) = Q_0(x)\in S_0^{(d)}, & x\in\mathbb{T}^d
\end{cases}
$$

Our main results on problem (NSQ)

- $\exists\,!$ of a global strong solution $(\boldsymbol{u}, Q)$ with periodic boundary conditions in 2D for arbitrary ξ
- convergence of the unique global strong solution to a single steady state

- For $\xi \neq 0$ the Q-equation in (NSQ) does not satisfy a maximum principle
- The highly nonlinear terms are very difficult to handle

Functional spaces

$$L^p(\mathbb{T}^d) = L^p(\mathbb{T}^d, M), \qquad M = \mathbb{R},\ \mathbb{R}^2,\ \mathbb{R}^3$$
$$W^{m,p}(\mathbb{T}^d) = W^{m,p}(\mathbb{T}^d, M), \qquad M = \mathbb{R},\ \mathbb{R}^2,\ \mathbb{R}^3$$
$$H^m(\mathbb{T}^d) = W^{m,2}(\mathbb{T}^d)$$
$$\mathbf{H} = \left\{ \boldsymbol{u} \in L^2(\mathbb{T}^d),\ \nabla \cdot \boldsymbol{u} = 0,\ \int_{\mathbb{T}^d} \boldsymbol{u}\, dx = \mathbf{0} \right\}$$
$$\mathbf{V} = \left\{ \boldsymbol{u} \in H^1(\mathbb{T}^d),\ \nabla \cdot \boldsymbol{u} = 0,\ \int_{\mathbb{T}^d} \boldsymbol{u}\, dx = \mathbf{0} \right\}$$

$$L^2(\mathbb{T}^d, S_0^{(d)}) = \{Q : \mathbb{T}^d \to S_0^{(d)}, \int_{\mathbb{T}^d} |Q|^2 < \infty\}$$
$$H^1(\mathbb{T}^d, S_0^{(d)}) = \{Q : \mathbb{T}^d \to S_0^{(d)}, \int_{\mathbb{T}^d} |\nabla Q|^2 + |Q|^2 < \infty\}$$
$$H^2(\mathbb{T}^d, S_0^{(d)}) = \{Q : \mathbb{T}^d \to S_0^{(d)}, \int_{\mathbb{T}^d} |\Delta Q|^2 + |\nabla Q|^2 + |Q|^2 < \infty\}$$
$$|Q| = \sqrt{tr(Q^2)} = \sqrt{Q_{ij}Q_{ij}}$$
$$|\nabla Q|^2 = \nabla_k Q_{ij} \nabla_k Q_{ij} \qquad |\Delta Q|^2 = \Delta Q_{ij} \Delta Q_{ij}$$

Preliminary results

$$\mathcal{E}(t) = \lambda \mathcal{F}(Q(t)) + \frac{1}{2}\int_{\mathbb{T}^d} |\boldsymbol{u}|^2, \quad \text{total energy}$$

$\mathcal{E}(t)$ = sum of free energy and kinetic energy for $\boldsymbol{u}$

$c > 0 \ \Rightarrow \ \mathcal{E}(t)$ is bounded from below

Lemma

Let $(\boldsymbol{u}, Q)$ be a smooth solution to problem (NSQ). Then, for $d = 2, 3$, we have

$$\frac{d}{dt}\mathcal{E}(t) = -\nu \int_{\mathbb{T}^d} |\nabla \boldsymbol{u}|^2 - \lambda\Gamma \int_{\mathbb{T}^d} |H(Q)|^2 \leq 0, \quad \forall\, t > 0$$

- energy dissipation of the liquid crystal flow
- $\mathcal{E}$ is a Lyapunov functional for (NSQ)
- $\mathcal{E}(t) + \int\int_{(0,T)\times\mathbb{T}^d} \left(\nu|\nabla \boldsymbol{u}|^2 + \lambda\Gamma|H(Q)|^2\right) = \mathcal{E}(0)$

Preliminary results

Lemma

Assume $(\boldsymbol{u}_0, Q_0) \in \mathbf{H} \times H^1(\mathbb{T}^d, S_0^{(d)})$.
Let $(\boldsymbol{u}, Q)$ *be a smooth solution to the problem (NSQ).*
Then, for $d = 2, 3$, *we have*

$$\|\boldsymbol{u}(t)\|_{L^2} + \|Q(t)\|_{H^1} \leq C, \qquad \forall\, t > 0$$

where $C > 0$ *depends on* $\|\boldsymbol{u}_0\|_{L^2}$, $\|Q_0\|_{H^1}$, L, λ, a, b, c.
Moreover, it holds

$$\int_0^T \int_{\mathbb{T}^d} |\nabla \boldsymbol{u}|^2 + |\Delta Q|^2 < C_T, \qquad \forall\, T > 0$$

where $C_T > 0$ *may further depend on* ν, Γ *and* T.

Preliminary results

Existence of global weak solutions for $d = 2, 3$
(Abels, Dolzmann & Liu (2014))

Lemma

Let $d = 2, 3$ *and* $\xi \in \mathbb{R}$.
Assume $(\boldsymbol{u}_0, Q_0) \in \mathbf{H} \times H^1(\mathbb{T}^d, S_0^{(d)})$.

Then problem (NSQ) admits at least one global-in-time weak solution $(\boldsymbol{u}, Q)$ *such that*

$$\boldsymbol{u} \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$$

$$Q \in L^\infty(0, T; H^1(\mathbb{T}^d, S_0^{(d)})) \cap L^2(0, T; H^2(\mathbb{T}^d, S_0^{(d)}))$$

Moreover the following energy inequality holds:

$$\mathcal{E}(t) + \int_0^t \int_{\mathbb{T}^d} \nu |\nabla \boldsymbol{u}|^2 + \lambda\Gamma |H(Q)|^2 \leq \mathcal{E}(0), \quad a.e.\ t \in (0, T)$$

The case $d = 2$

- $\mathrm{tr}(Q^3) = 0$
- $f_B(Q) = \dfrac{a}{2}\mathrm{tr}(Q^2) + \dfrac{c}{4}\mathrm{tr}^2(Q^2)$
- $H(Q) = L\Delta Q - aQ - cQ\,\mathrm{tr}(Q^2)$

Definition

Assume $d = 2$ and $(\boldsymbol{u}_0, Q_0) \in \mathbf{V} \times H^2(\mathbb{T}^2, S_0^{(2)})$.

$(\boldsymbol{u}, Q)$ is called global strong solution to problem (NSQ) if

- $\boldsymbol{u} \in C([0, +\infty); \mathbf{V}) \cap L^2_{loc}(0, +\infty; H^2(\mathbb{T}^2, \mathbb{R}^2))$
- $Q \in C([0, +\infty); H^2(\mathbb{T}^2, S_0^{(2)}) \cap L^2_{loc}(0, +\infty; H^3(\mathbb{T}^2, S_0^{(2)}))$
- the equations for $\boldsymbol{u}$ and Q in (NSQ) are satisfied a.e.

Main result: $\exists\,!$ of global strong solutions

Theorem

Assume $d = 2$ and $\xi \in \mathbb{R}$.

Then, for any $(\boldsymbol{u}_0, Q_0) \in \mathbf{V} \times H^2(\mathbb{T}^2, S_0^{(2)})$, problem (NSQ) admits a unique global strong solution $(\boldsymbol{u}, Q)$ such that

$$\|\boldsymbol{u}(t)\|_{H^1} + \|Q(t)\|_{H^2} \leq C, \quad \forall\, t \geq 0$$

where C is a positive constant depending on $\|\boldsymbol{u}_0\|_{H^1}$, $\|Q_0\|_{H^2}$, ν, Γ, L, λ, a, c and ξ, but it is independent of t.

Sketch of the proof

- semi-Galerkin approximation scheme $\Rightarrow$ problem $(NSQ)_N$
- existence of approximate solutions $(\boldsymbol{u}^N, Q^N)$ to $(NSQ)_N$
- lower order estimates for $(\boldsymbol{u}^N, Q^N)$, independent of N and t
- higher order estimates for $(\boldsymbol{u}^N, Q^N)$, independent of N and t
- passage to the limit for $N \to \infty$
- existence of solutions for (NSQ)
- uniqueness of solution for (NSQ)

Semi-Galerkin approximation scheme / 1

- $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$ eigenvectors of Stokes operator $\mathcal{S}$ with zero mean

$$\mathcal{S}\boldsymbol{v}_n = \kappa_n \boldsymbol{v}_n, \quad \nabla \cdot \boldsymbol{v}_n = 0, \quad \int_{\mathbb{T}^2} \boldsymbol{v}_n(x)\, dx = 0, \ \text{ in } \mathbb{T}^2$$

$$\boldsymbol{v}_n(x + \boldsymbol{e}_i) = \boldsymbol{v}_n(x), \quad x \in \mathbb{T}^2$$

$0 < \kappa_1 \leq \kappa_2 \leq \ldots \nearrow +\infty$ corresponding eigenvalues

- $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$ orthogonal basis of **H** and **V**
- $V_N = \mathrm{span}\{\boldsymbol{v}_n\}_{n=1}^{N}$
- $\Pi_N : \mathbf{H} \to V_N$ orthogonal projection operators

- $\boldsymbol{u}^N = \sum_{i=1}^{N} h_i(t)\boldsymbol{v}_i(x)$ approximation of velocity, solution to

$$\int_{\mathbb{T}^2} (\boldsymbol{u}^N)_t \cdot v_k - \int_{\mathbb{T}^2} \left(\boldsymbol{u}^N \otimes \boldsymbol{u}^N\right) : \nabla \boldsymbol{v}_k + \nu \int_{\mathbb{T}^2} \nabla \boldsymbol{u}^N : \nabla \boldsymbol{v}_k$$

$$= - \int_{\mathbb{T}^2} (\sigma^N + \tau^N) : \nabla \boldsymbol{v}_k, \quad \text{in } (0, T), \quad \forall\, k = 1, \ldots, N$$

- Q^N is determined in terms of u^N and solves uniquely

$$Q^N_t + \boldsymbol{u}^N \cdot \nabla Q^N - S^N(\nabla \boldsymbol{u}^N, Q^N) = \Gamma H^N(Q^N), \quad \mathbb{T}^2 \times \mathbb{R}^+$$

- $\tau^N, \sigma^N, S^N, H^N$ approximation of τ, σ, S, H, where
  $\boldsymbol{u} \Rightarrow \boldsymbol{u}^N, \quad Q \Rightarrow Q^N$
  $$D \Rightarrow D^N = \frac{\nabla \boldsymbol{u}^N + \nabla^T \boldsymbol{u}^N}{2}, \quad \Omega \Rightarrow \Omega^N = \frac{\nabla \boldsymbol{u}^N - \nabla^T \boldsymbol{u}^N}{2}$$

The approximation problem $(NSQ)_N$

$$
(NSQ)_N \begin{cases} \displaystyle\int_{\mathbb{T}^2} (\boldsymbol{u}^N)_t \cdot v_k - \int_{\mathbb{T}^2} \left(\boldsymbol{u}^N \otimes \boldsymbol{u}^N\right) : \nabla \boldsymbol{v}_k + \nu \int_{\mathbb{T}^2} \nabla \boldsymbol{u}^N : \nabla \boldsymbol{v}_k \\ \displaystyle = - \int_{\mathbb{T}^2} (\sigma^N + \tau^N) : \nabla \boldsymbol{v}_k, \quad \text{in } (0, T), \quad \forall\, k = 1, ..., N \\ \\ Q_t^N + \boldsymbol{u}^N \cdot \nabla Q^N - S^N(\nabla \boldsymbol{u}^N, Q^N) = \Gamma H^N(Q^N), \;\; \mathbb{T}^2 \times \mathbb{R}^+ \\ \\ \boldsymbol{u}^N(x + e_i, t) = \boldsymbol{u}^N(x, t), \quad \mathbb{T}^2 \times \mathbb{R}^+ \\ Q^N(x + e_i, t) = Q^N(x, t), \quad \mathbb{T}^2 \times \mathbb{R}^+ \\ \boldsymbol{u}^N|_{t=0} = \Pi_N \boldsymbol{u}_0, \quad Q^N|_{t=0} = Q_0, \quad \mathbb{T}^2 \end{cases}
$$

Existence of approximate solutions $(\boldsymbol{u}^N, Q^N)$

Proposition

Assume $\boldsymbol{u}_0 \in \mathbf{V}$, $Q_0 \in H^2(\mathbb{T}^2, S_0^{(2)})$.

For any $N \in \mathbb{N}$, *there exists* $T_N > 0$ *depending on* N, $\|\boldsymbol{u}_0\|_{H^1}$ *and* $\|Q_0\|_{H^2}$ *s.t. problem (NSQ)$_N$ admits a solution* (u^N, Q^N) *satisfying*

$$\boldsymbol{u}^N \in L^\infty(0, T_N; \mathbf{V}) \cap L^2(0, T_N; H^2(\mathbb{T}^2, \mathbb{R}^2)) \cap H^1(0, T_N; \mathbf{H})$$

$$Q^N \in L^\infty(0, T_N; H^2(\mathbb{T}^2, S_0^{(2)})) \cap L^2(0, T_N; H^3(\mathbb{T}^2, S_0^{(2)}))$$

$$\cap H^1(0, T_N; H^1(\mathbb{T}^2, S_0^{(2)}))$$

Existence of approximate solutions: sketch of the proof

- given $\tilde{\boldsymbol{u}} \in C([0,T];V_N)$, then there exists a unique $Q = Q[\tilde{\boldsymbol{u}}]$ solution to the equation for Q_N in $(\text{NSQ})_N$ with $\boldsymbol{u}^N$ replaced by $\tilde{\boldsymbol{u}}$
- inserting $Q[\tilde{\boldsymbol{u}}]$ in the equation for $\boldsymbol{u}_N$ in $(\text{NSQ})_N$, the solution $\boldsymbol{u}$ to the resulting ODE system defines a mapping $\mathcal{T} : \tilde{\boldsymbol{u}} \mapsto \mathcal{T}[\tilde{\boldsymbol{u}}] = \boldsymbol{u}$
- $\mathcal{T}$ admits a fixed point by means of fixed point Schauder's theorem on $(0, T_N)$, for some $T_N > 0$ depending on $\|\boldsymbol{u}_0\|_{H^1}$, $\|Q_0\|_{H^2}$ and N

Uniform-in-time lower-order estimates

- $\|\boldsymbol{u}^N(t)\|_{L^2} + \|Q^N(t)\|_{H^1} \leq C, \qquad \forall\, t \in [0, T_N)$
- $\displaystyle\int_0^t \int_{\mathbb{T}^d} \left(|\nabla \boldsymbol{u}^N(\tau)|^2 + |\Delta Q^N(\tau)|^2\right) < C(1+t), \quad \forall\, t \in [0, T_N)$

  where the constant $C > 0$ depends on $\|\boldsymbol{u}_0\|_{L^2}$, $\|Q_0\|_{H^1}$, L, λ, ν, Γ, a, c, but it is independent of N and t

Uniform-in-time higher-order estimates

- $\|\boldsymbol{u}^N(t)\|_{H^1} + \|Q^N(t)\|_{H^2} \le C, \qquad \forall\, t \in [0, T_N)$
- $\displaystyle\int_0^t \int_{\mathbb{T}^2} \Big|\Delta \boldsymbol{u}^N(\tau)|^2 + |\nabla\Delta Q^N(\tau)|^2\Big) < C(1+t), \ \forall\, t \in [0, T_N)$

  where the constant $C > 0$ depends on $\|\boldsymbol{u}_0\|_{H^1}$, $\|Q_0\|_{H^2}$, L, λ, ν, Γ, a, c, but it is independent of N and t

Extension of $(\boldsymbol{u}^N, Q^N)$ to $[0, T]$, for any $T > 0$

- uniform-in-time lower-order and higher-order estimates $\Rightarrow$ $(\boldsymbol{u}^N, Q^N)$ does not blow up in finite time
- every approximate solution (u^N, Q^N) can be extended to the time interval $[0, T]$, for any $T > 0$
- since the uniform estimates are also independent of N $\Rightarrow$

$$\boldsymbol{u}^N \in L^\infty(0, T; \mathbf{V}) \cap L^2(0, T; H^2(\mathbb{T}^2, \mathbb{R}^2)) \cap H^1(0, T; \mathbf{H})$$

$$Q^N \in L^\infty(0, T; H^2(\mathbb{T}^2, S_0^{(2)})) \cap L^2(0, T; H^3(\mathbb{T}^2, S_0^{(2)})) \cap H^1(0, T; H^1(\mathbb{T}^2, S_0^{(2)}))$$

Limit for $N \to \infty$ and existence of solutions for (NSQ)

- uniform-in-time estimates
- standard weak compactness results
- Aubin-Lions compactness Lemma
- $(\boldsymbol{u}^N, Q^N) \to (\boldsymbol{u}, Q)$ up to a subsequence
- $(\boldsymbol{u}, Q)$ is a global strong solution to problem (NSQ)

Uniqueness of a solution for (NSQ)

Let $(\boldsymbol{u}_i, Q_i)$ be two global strong solutions to (NSQ), corresponding to the initial data $(\boldsymbol{u}_{i0}, Q_{i0})$, $i = 1, 2$.

Then we can prove, for any $\xi \in \mathbb{R}$,

$$
\begin{aligned}
&\|(\boldsymbol{u}_1 - \boldsymbol{u}_2)(t)\|_{L^2}^2 + \|(Q_1 - Q_2)(t)\|_{H^1}^2 \\
&+ \int_0^t (\|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)(\tau)\|_{L^2}^2 + \|\Delta(Q_1 - Q_2)(\tau)\|_{L^2}^2) d\tau \\
&\leq C e^{\int_0^t h(\tau)d\tau} \left(\|\boldsymbol{u}_{01} - \boldsymbol{u}_{02}\|_{L^2}^2 + \|Q_{01} - Q_{02}\|_{H^1}^2 \right), \quad \forall\, t \in (0, T)
\end{aligned}
$$

where $h \in L^1(0, T)$.

Therefore, the global strong solution to (NSQ) is unique.

Continuous dependence result

- Previous estimate shows that the map $(\boldsymbol{u}_0, Q_0) \to (\boldsymbol{u}(t), Q(t))$, where $(\boldsymbol{u}, Q)$ is the global strong solution to (NSQ), is continuous from $\mathbf{V} \times H^2 \to \mathbf{H} \times H^1$
- The continuous dependence from $\mathbf{V} \times H^2 \to \mathbf{V} \times H^2$ is an open problem

Remark

In the paper De Anna & Zarnescu the authors prove uniqueness of weak solutions by means of a continuous dependence estimate showing a similar gap in the functional spaces

Main result: convergence to equilibrium

Theorem

Assume $d = 2$ and $\xi \in \mathbb{R}$.

For any $(\boldsymbol{u}_0, Q_0) \in \mathbf{V} \times H^2(\mathbb{T}^2, S_0^{(2)})$, the unique global strong solution to problem (NSQ) converges to a single steady state $(\mathbf{0}, Q_\infty)$ as time tends to infinity.

More precisely, it holds

$$\lim_{t \to +\infty} \left(\|\boldsymbol{u}(t)\|_{H^1} + \|Q(t) - Q_\infty\|_{H^2} \right) = 0$$

where $Q_\infty \in S_0^{(2)}$ and solves the elliptic problem in $\mathbb{T}^2$

$$\begin{cases} L\Delta Q_\infty - aQ_\infty - c\,\mathrm{tr}(Q_\infty^2)Q_\infty = 0, & \mathbb{T}^2 \\ Q_\infty(x + e_i) = Q_\infty(x), & x \in \mathbb{T}^2 \end{cases}$$

Proof: main steps

- decay property of the solution
- characterization of the ω-limit set
- convergence to equilibrium

Lemma

Assume $(\boldsymbol{u}_0, Q_0) \in \mathbf{V} \times H^2\left(\mathbb{T}^2, S_0^{(2)}\right)$ *and* $\xi \in \mathbb{R}$.

Then the unique global strong solution $(\boldsymbol{u}, Q)$ *to (NSQ) has the decay property*

$$\lim_{t \to +\infty} \left(\|\boldsymbol{u}(t)\|_{H^1} + \|H(Q(t))\|_{L^2} \right) = 0$$

so that

$$\boldsymbol{u}(t) \to \mathbf{0} \quad \textit{in } H^1$$

Characterization of the ω-limit set

Lemma

Assume $(\boldsymbol{u}_0, Q_0) \in \mathbf{V} \times H^2(\mathbb{T}^2, S_0^{(2)})$ *and* $\xi \in \mathbb{R}$.

Then the ω*-limit set* $\omega(\boldsymbol{u}_0, Q_0)$

$$\omega(\boldsymbol{u}_0, Q_0) = \big\{(\boldsymbol{u}_\infty, Q_\infty) \mid \exists \{t_n\} \nearrow \infty \ : \ \boldsymbol{u}(t_n) \to \boldsymbol{u}_\infty \text{ in } L^2, \\ Q(t_n) \to Q_\infty \text{ in } H^1 \text{ as } n \to \infty\big\}$$

is a non-empty bounded subset in $\mathbf{V} \times H^2(\mathbb{T}^2, S_0^{(2)})$*, satisfying*

$$\omega(\boldsymbol{u}_0, Q_0) \subset \big\{(\mathbf{0}, Q_*) : Q_* \in \mathfrak{S}\big\}, \ \text{ where}$$

$$\mathfrak{S} = \big\{Q_* : L\Delta Q_* - aQ_* - c\,\mathrm{tr}(Q_*^2)Q_* = 0, \ \ Q_* \in S_0^{(2)} \\ \text{and } Q_*(x + e_i) = Q_*(x) \text{ in } \mathbb{T}^2\big\}$$

Convergence to equilibrium

- Generalization of Simon's results in the scalar case to the matrix case
- Gradient inequality of Łojasiewicz-Simon type for the matrix valued functions
- Due to the special structure of the Q-tensor in 2D, the problem reduces to the vector case

Lemma

Let $Q_* \in H^1(\mathbb{T}^2, S_0^{(2)})$ be a critical point of free energy $\mathcal{F}(Q)$.

Then there exist $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on Q_* s.t., for any $Q \in H^1(\mathbb{T}^2, S_0^{(2)})$ satisfying $\|Q - Q_*\|_{H^1} < \beta$, we have

$$\left\| L\Delta Q - aQ - c\,\mathrm{tr}(Q^2)Q \right\|_{(H^1)'} \geq |\mathcal{F}(Q) - \mathcal{F}(Q_*)|^{1-\theta}$$

Here, $(H^1(\mathbb{T}^2, S_0^{(2)}))'$ is the dual space of $H^1(\mathbb{T}^2, S_0^{(2)})$.

Main result: convergence rate to the asymptotic limit

Theorem

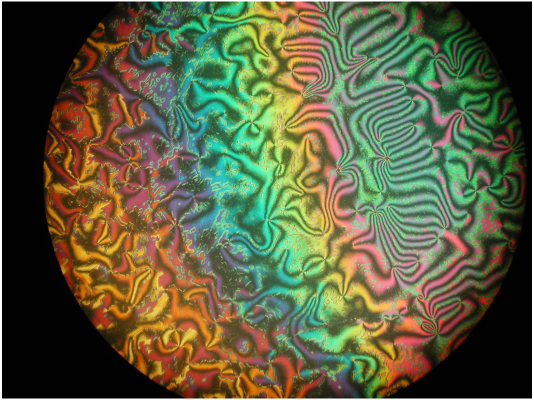
Assume $d = 2$, $\xi \in \mathbb{R}$ and $(\boldsymbol{u}_0, Q_0) \in \mathbf{V} \times H^2(\mathbb{T}^2, S_0^{(2)})$.
Let us consider the global strong solution $(\boldsymbol{u}, Q)$ to problem (NSQ).
Then the following estimate holds

$$\|\boldsymbol{u}(t)\|_{H^1} + \|Q(t) - Q_\infty\|_{H^2} \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \geq 0$$

where $C > 0$ is a constant depending on ν, Γ, L, λ, a, c, ξ, $\|\boldsymbol{u}_0\|_{H^1}$, $\|Q_0\|_{H^2}$, $\|Q_\infty\|_{H^2}$.
Here the constant $\theta \in (0, \frac{1}{2})$ depends on Q_∞.

Open issues

- generalization to the case of no-slip boundary condition for $\boldsymbol{u}$ and nonhomogeneous Dirichlet boundary data for Q
- regularization in finite time of weak solutions in the case of periodic boundary conditions or more general boundary conditions



THANKS FOR YOUR ATTENTION!