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Optimal Control of Semilinear Parabolic Equations by BV-Functions

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A joint work with Florian Kruse and Karl Kunisch (Karl Franzens University of Graz)

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Introduction

We wish to control the system

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + f(y) = \displaystyle\sum_{j=1}^{m} u_j g_j & \text{in } Q = \Omega \times (0,T) \\ y = 0 & \text{on } \Sigma = \Gamma \times (0,T) \\ y(0) = y_0(x) & \text{in } \Omega \end{cases}$$

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We wish to get the state y as close as possible to y_d.

The Tikhonov Formulation

The classical formulation of the control problem is

$$\min_{u \in L^2(0,T)^m} J(u) = \frac{1}{2}\int_Q |y_u - y_d|^2 \, dx\, dt + \sum_{j=1}^{m} \frac{\nu_j}{2} \int_0^T u_j^2(t)\, dt$$

with $\nu_j > 0$.

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with $\nu_j > 0$. Hence the optimal controls are given by

$$\bar{u}_j(t) = \frac{-1}{\nu_j} \int_{\omega_j} g_j \bar{\varphi}(t) \, dx$$

where $\bar{\varphi}$ is the adjoint state, solution of the linear equation

$$\begin{cases} -\dfrac{\partial \bar{\varphi}}{\partial t} - \Delta \bar{\varphi} + f'(\bar{y})\bar{\varphi} = \bar{y} - y_d \text{ in } Q, \\ \bar{\varphi} = 0 \text{ on } \Sigma, \ \bar{\varphi}(T) = 0 \text{ in } \Omega \end{cases}$$

$\bar{y}$ being the state associated to the optimal control $\bar{u}$.

The Bang-Bang Formulation

$$\min_{\alpha_j \le u_j \le \beta_j} J(u) = \frac{1}{2} \int_Q |y_u - y_d|^2 \, dx \, dt$$

where $-\infty < \alpha_j < \beta_j < +\infty$ for $j = 1, \ldots, m$.

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This time the optimal controls satisfy

$$\bar{u}_j(t) = \begin{cases} \alpha_j & \text{if } \displaystyle\int_{\omega_j} g_j \bar{\varphi}(t) \, dx > 0 \\ \beta_j & \text{if } \displaystyle\int_{\omega_j} g_j \bar{\varphi}(t) \, dx < 0 \end{cases}$$

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An Alternative Formulation

$$\text{(P)} \quad \min_{u \in BV(0,T)^m} J(u)$$

where

$$J(u) = \frac{1}{2}\|y_u - y_d\|^2_{L^2(Q)} + \sum_{j=1}^m \Big(\alpha_j \|u_j'\|_{\mathcal{M}(0,T)} + \frac{\beta_j}{2}\Big(\int_0^T u_j(t)\,dt\Big)^2\Big)$$

and $u = (u_j)_{j=1}^m$ and y_u is the solution to the parabolic state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + f(y) = \displaystyle\sum_{j=1}^m u_j g_j & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = y_0(x) & \text{in } \Omega \end{cases}$$

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Assumptions

- $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$, a bounded domain with a Lipschitz boundary

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- $y_0 \in L^\infty(\Omega)$, $y_d \in L^{\hat{p}}(Q)$, $\hat{p} > 1 + \frac{n}{2}$ if $n > 1$, and $\hat{p} \ge 2$ if $n = 1$

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- $\alpha_j > 0$ and $\beta_j \geq 0$ for $1 \leq j \leq m$
- $\{g_j\}_{j=1}^m \subset L^\infty(\Omega) \setminus \{0\}$ have pairwise disjoint supports $\omega_j = \operatorname{supp} g_j$
- $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function of class C^2, $f(0) = 0$, and $f'(y) \geq 0$

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Analysis of the State Equation

Proposition. For every $u \in L^p(0,T)^m$, with $p > 1$, the state equation has a unique solution $y_u \in L^\infty(Q) \cap L^2(0,T;H_0^1(\Omega))$. In addition, for every $M > 0$ there exists a constant K_M such that

$$\|y_u\|_{L^\infty(Q)} + \|y_u\|_{L^2(0,T;H_0^1(\Omega))} \le K_M \ \forall u \in L^p(0,T)^m : \|u\|_{L^p(0,T)^m} \le M.$$

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Differentiability of the mapping $u \to y_u$

- Let $Y = L^\infty(Q) \cap L^2(0,T;H_0^1(\Omega))$ and $S : L^p(0,T)^m \longrightarrow Y$, $S(u) = y_u$, with $p > 1$.

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Proposition. The mapping $S : L^p(Q)^m \longrightarrow Y$ is of class C^2. For all elements u, v and w of $L^p(0,T)^m$, the functions $z_v = S'(u)v$ and $z_{vw} = S''(u)(v,w)$ are the solutions of the problems

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial t} - \Delta z + f'(y_u)z = \displaystyle\sum_{j=1}^{m} v_j g_j \text{ in } Q \\ \qquad\qquad z = 0 \text{ on } \Sigma,\ z(0) = 0 \text{ in } \Omega \end{array}\right.$$

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Study of the Cost Functional

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$$\bullet G(u) = \sum_{j=1}^{m} \alpha_j g(u_j')$$

where $g : \mathcal{M}(0,T) \longrightarrow \mathbb{R}$ is given by $g(\mu) = \|\mu\|_{\mathcal{M}(0,T)}$

Differentiability of F

Proposition. The functional $F : L^p(0,T)^m \longrightarrow \mathbb{R}$, with $p > 1$, is of class C^2. Denoting $z_v = S'(u)v$ and $z_w = S'(u)w$, we have

$$F'(u)v \quad = \sum_{j=1}^{m} \int_0^T \Big(\int_{\omega_j} \varphi_u(x,t) g_j(x)\, dx + \beta_j \int_0^T u_j(t)\, dt \Big) v_j(t)\, dt$$

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F''(u)(v,w) &= \int_Q \big(1 - \varphi_u f''(y_u)\big) z_v z_w \, dx\, dt \\
&\quad + \sum_{j=1}^{m} \beta_j \int_0^T v_j\, dt \int_0^T w_j\, dt
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Existence of Solution of (P)

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Theorem. Let us assume that one of the following assumptions hold.

1. $\beta_j > 0$ for every $1 \leq j \leq m$.
2. There exist $q \in [1, 2)$ and $C > 0$ such that $f'(y) \leq C(1 + |y|^q)$ $\forall y \in \mathbb{R}$.

Then, problem (P) has at least one solution. Moreover, if f is affine, the solution is unique.

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Local Solutions

Definition. We shall call $\bar{u}$ a local solution of (P) if $\exists\varepsilon > 0$ so that

$$J(\bar{u}) \leq J(u) \ \ \forall u : \|u - \bar{u}\|_{BV(0,T)^m} \leq \varepsilon.$$

We say that $\bar{u}$ is an $L^p(0,T)^m$-local solution $(1 \leq p \leq \infty)$ if the above inequality holds in an $L^p(0,T)^m$-ball around $\bar{u}$. Finally, $\bar{u}$ is called a strong local solution if for some $\varepsilon > 0$

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• Since $BV(0,T) \subset L^p(0,T)$ continuously $\forall p \in [1, +\infty]$, we deduce that if $\bar{u}$ is an $L^p(0,T)^m$-local solution of (P), then it is a local solution.

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• Any strong local solution is an $L^p(0,T)^m$-local solution $\forall p \in (1,+\infty]$.

First Order Optimality Conditions

- Given $\bar{u} \in BV(0,T)^m$ with associated state and adjoint state $\bar{y}$ and $\bar{\varphi}$, respectively, we define

$$\bar{\Phi}_j(t) = \int_0^t \int_{\omega_j} \bar{\varphi}(x,s) g_j(x)\,dx\,ds + \beta_j t \int_0^T \bar{u}_j(s)\,ds, \quad 1 \le j \le m.$$

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Theorem If $\bar{u}$ is a local solution of (P), then $\bar{\Phi}_j \in C^1[0,T] \cap C_0(0,T)$ for $1 \le j \le m$ and they satisfy

$$\|\bar{\Phi}_j\|_{C_0(0,T)} \begin{cases} = \alpha_j & \text{if } \bar{u}_j' \neq 0, \\ \le \alpha_j & \text{if } \bar{u}_j' = 0, \end{cases}$$

$$\int_0^T \bar{\Phi}_j\, d\bar{u}_j' = \|\bar{\Phi}_j\|_{C_0(0,T)} \|\bar{u}_j'\|_{\mathcal{M}(0,T)}$$

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On the Structure of $\bar{u}'$

Corollary. Under the assumptions of previous theorem, for each $j \in \{1, \ldots, m\}$ such that $\bar{u}_j$ is not a constant function on $[0, T]$, we have

$$\begin{cases} \mathrm{supp}(\bar{u}_j'^{+}) \subset \{t \in [0, T] : \bar{\Phi}_j(t) = +\alpha_j\}, \\ \mathrm{supp}(\bar{u}_j'^{-}) \subset \{t \in [0, T] : \bar{\Phi}_j(t) = -\alpha_j\}, \end{cases}$$

where $\bar{u}_j' = \bar{u}_j'^{+} - \bar{u}_j'^{-}$ is the Jordan decomposition of the measure $\bar{u}_j'$.

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Additional Comments

• If the set of points $\{t \in [0,T] : \bar{\Phi}_j(t) \in \{-\alpha_j, +\alpha_j\}\}$ is finite, then $\bar{u}'_j$ is a combination of Dirac measures centered at those points.

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• There is a threshold $M_j < +\infty$ such that if $\alpha_j > M_j$, then $\bar{u}'_j = 0$, i.e., $\bar{u}_j$ is constant in $[0, T]$. Moreover, there exists a vector $\bar{\xi} \in \mathbb{R}^m$ such that for any $\alpha \in \mathbb{R}^m$ with $\alpha_j > M_j$ for all $1 \leq j \leq m$, the constant function $\bar{\xi}$ is a solution of (P).

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Necessary Second Order Conditions

- The critical cone

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Theorerm. If $\bar{u}$ is a local minimum of (P), then $F''(\bar{u})v^2 \geq 0$ for all $v \in C_{\bar{u}}$.

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Sufficient Second Order Conditions

- The extended critical cone: for every $\tau > 0$ we set

$$C_{\bar u}^{\tau} = \Big\{v \in BV(0,T)^m : \; F'(\bar u)v + G'(\bar u; v) \leq \tau\Big(\|z_v\|_{L^2(Q)} + \sum_{j=1}^m \beta_j\Big|\int_0^T v_j(t)\,dt\Big|\Big)\Big\},$$

where $z_v = S'(\bar u)v$.

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where $z_v = S'(\bar u)v$.

Theorem. Let $\bar u \in BV(0,T)^m$ satisfy the first order optimality conditions and

$$F''(\bar u)v^2 \ge \kappa \|z_v\|^2_{L^2(Q)} \quad \forall v \in C_{\bar u}^{\tau}$$

Then, there exist positive constants $\varepsilon > 0$ and $\delta > 0$ such that

$$J(\bar u) + \frac{\delta}{2}\|y_u - \bar y\|^2_{L^2(Q)} \le J(u) \text{ for all } u \in BV(0,T)^m : \|y_u - \bar y\|_{L^\infty(Q)} < \varepsilon$$

Numerical Approximation

• In th sequel we assume that Ω is a convex set and $y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$.

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• In th sequel we assume that Ω is a convex set and $y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$.

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• We assume that $\exists \rho_T > 0$ such that $\tau \leq \rho_T \tau_k$ for $1 \leq k \leq N_\tau$.

• We set $\sigma = (h, \tau)$.

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Discretization of the Controls

$$U_\tau = \{u_\tau \in BV(0,T) : u_\tau = \sum_{k=1}^{N_\tau} u_k \chi_k, \text{ with } \{u_k\}_{k=1}^{N_\tau} \subset \mathbb{R}\},$$

where χ_k denotes the characteristic function of the interval $I_k = (t_{k-1}, t_k]$.

Discretization of the Controls

$$U_\tau = \{u_\tau \in BV(0,T) : u_\tau = \sum_{k=1}^{N_\tau} u_k \chi_k, \text{ with } \{u_k\}_{k=1}^{N_\tau} \subset \mathbb{R}\},$$

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Approximation of $BV(0,T)$ by U_τ

$$\Lambda_\tau : BV(0,T) \longrightarrow U_\tau, \quad \Lambda_\tau u = \sum_{k=1}^{N_\tau} \left(\frac{1}{\tau_k} \int_{I_k} u(t)\, dt \right) \chi_k.$$

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Discretization of the States

$$Y_h = \{y_h \in C_0(\Omega) : y_h = \sum_{j=1}^{N_h} y_j e_j \text{ with } \{y_j\}_{j=1}^{N_h} \subset \mathbb{R}\}$$

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$$y_\sigma = \sum_{k=1}^{N_\tau} y_{k,h}\chi_k = \sum_{k=1}^{N_\tau}\sum_{j=1}^{N_h} y_{kj}\chi_k e_j$$

with $\{y_{k,h}\}_{k=1}^{N_\tau} \subset Y_h$ and $\{y_{kj}\}_{\substack{1\le k\le N_\tau\\ 1\le j\le N_h}} \subset \mathbb{R}$.

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Discrete State Equation

$$
\begin{cases}
\left(\dfrac{y_{k,h} - y_{k-1,h}}{\tau_k}, z_h\right) + a(y_{k,h}, z_h) + (f(y_{k,h}), z_h) \\
\qquad\qquad = \dfrac{1}{\tau_k} \displaystyle\sum_{j=1}^{m} (g_j, z_h) \int_{I_k} u_j(t)\, dt, \quad \forall z_h \in Y_h,\ 1 \leq k \leq N_\tau \\
y_{0,h} = y_{0h},
\end{cases}
$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$, a is the bilinear form associated to the operator $-\Delta$, i.e., and y_{0h} is the projection $P_h y_0$ of y_0 on Y_h:

$$(P_h y_0, z_h) = (y_0, z_h) \ \ \forall z_h \in Y_h.$$

Discrete Optimal Control Problem

$$(\mathrm{P}_\sigma) \quad \min_{u \in BV(0,T)^m} J_\sigma(u)$$

$$J_\sigma(u) = \frac{1}{2}\|y_\sigma - y_d\|^2_{L^2(Q_h)} + \sum_{j=1}^m \Big(\alpha_j \|u'_j\|_{\mathcal{M}(0,T)} + \frac{\beta_j}{2}\Big(\int_0^T u_j(t)\,dt\Big)^2\Big)$$

where y_σ is the discrete state associated to $u = (u_j)_{j=1}^m$.

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REMARK 1 *Given* $\{u_j\}_{j=1}^m \subset BV(0,T)$, *we observe that*

$$\int_{I_k} u_j(t)\,dt = \int_{I_k} \Lambda_\tau u_j(t)\,dt \quad \textit{for all } 1 \le j \le m \textit{ and } 1 \le k \le N_\tau.$$

Hence the discrete states associated to $\{u_j\}_{j=1}^m$ *and* $\{\Lambda_\tau u_j\}_{j=1}^m$ *coincide.*

Existence of Discrete Solutions

•*Assumption:*

(A) The mapping $z_h \in Y_h \longrightarrow ((g_j, z_h))_{j=1}^m \in \mathbb{R}^m$ is surjective.

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REMARK 2 *In the case that $\beta_j > 0$ for all $1 \le j \le m$, condition (A) is not needed to establish the existence of a solution of (P_σ). However, it is still necessary for the uniqueness in the case that f is affine.*

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Convergence Analysis

Theorem. Let us assume that either f is affine or $\beta_j > 0$ for every $1 \leq j \leq m$, and let $\{\bar{u}_\tau\}_\tau \subset BV(0,T)^m$ be a family of global solutions of problems (P_σ). Then this family is bounded in $BV(0,T)^m$. In addition, any weak* limit $\bar{u}$ of a subsequence when $\sigma \to 0$ is a global solution of (P).

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$$\|\bar{u}'_\tau\|_{\mathcal{M}(0,T)^m} \to \|\bar{u}'\|_{\mathcal{M}(0,T)^m} \text{ and } \|\bar{u} - \bar{u}_\tau\|_{L^p(0,T)^m} \to 0 \ \forall p \in [1,+\infty)$$
$$\|\bar{y} - \bar{y}_\sigma\|_{L^2(Q)} \to 0 \text{ and } J_\sigma(\bar{u}_\tau) \to J(\bar{u})$$

where $\bar{y}$ and $\bar{y}_\sigma$ are the continuous and discrete states associated to $\bar{u}$ and $\bar{u}_\tau$, respectively.

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Approximation of Local Minima

Theorem. Let $\bar{u}$ be a strict $L^p(0,T)^m$-local minimum of (P) with $p \in [1,+\infty)$. Then there exist an $L^p(0,T)^m$-ball $B_\rho(\bar{u})$ such that J_σ has a global minimum $\bar{u}_\tau$ in $\bar{B}_\rho(\bar{u}) \cap BV(0,T)^m$ for every σ. The family $\{\bar{u}_\tau\}_\tau$ converges to $\bar{u}$ as pointed out in the previous theorem. Consequently, there exists σ_0 such that $\bar{u}_\tau$ is a local solution of (P_σ) for every $|\sigma| \leq |\sigma_0|$.

Error Estimates

Theorem. Let $\bar{u}$ be a local solution of (P) satisfying the sufficient second order conditions, and let $\{\bar{u}_\tau\}_\tau$ be a family of global minima of J_σ on $B_\rho(\bar{u}) \cap BV(0,T)^m$ converging to $\bar{u}$ in $L^p(0,T)^m$, for $p > 1$. Then, there exists $C > 0$ independent of σ such that

$$\|\bar{y} - \bar{y}_\sigma\|_{L^2(Q)} \leq C(\sqrt{\tau} + h).$$

A Numerical Example

$$\min_{u \in BV(0,T)} \frac{1}{2}\|y_u - y_d\|^2_{L^2(Q)} + \bar{\alpha}\|u'\|_{\mathcal{M}(0,1)} \quad \bar{\alpha} = \frac{1}{125\pi^2} \approx 8.1 \cdot 10^{-4}$$

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t}(x,t) - \Delta y(x,t) & = & u\chi_{(0,1)} & \text{in } Q = (-1,1) \times (0,2) \\ y(-1,t) = y(1,t) & = & 0 & \text{on } (0,2) \\ y(x,0) & = & 0 & \text{in } (-1,1) \end{array} \right.$$

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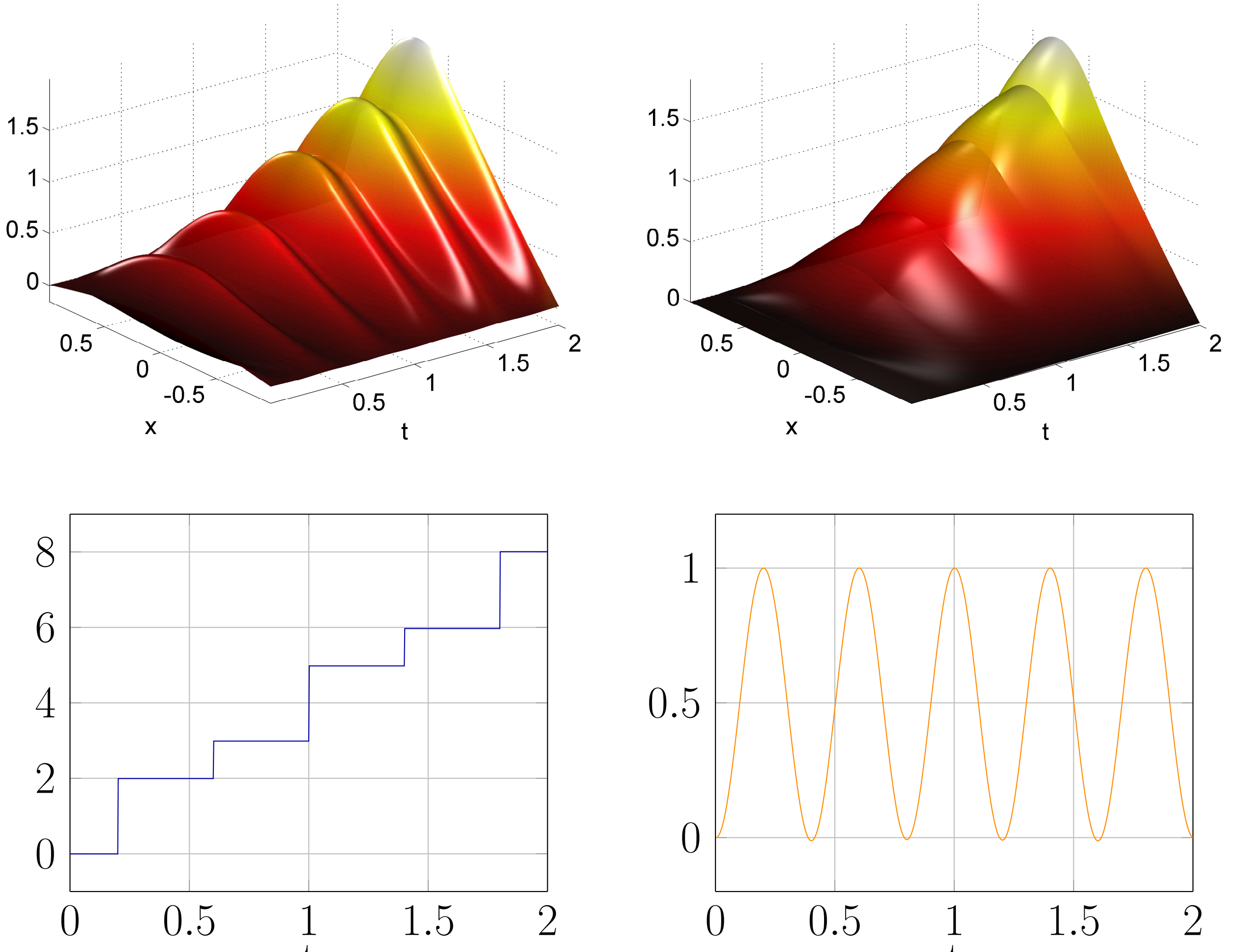
Figure 1: $y_{d,\sigma}$ and $\bar{y}_\sigma$, $\bar{u}_\tau$ and $\bar{\varphi}_\tau/\bar{\alpha}$



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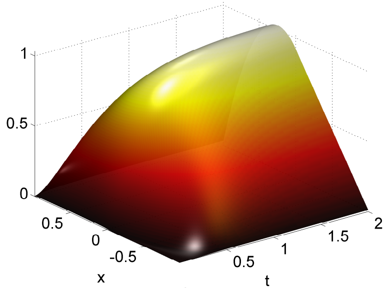
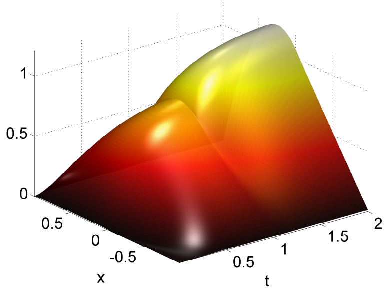
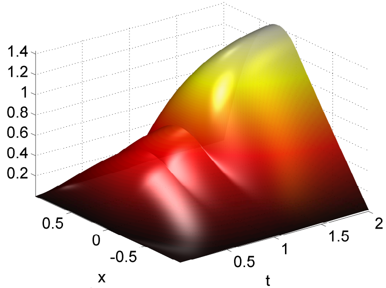


$\theta = 100$



$\theta = 50$



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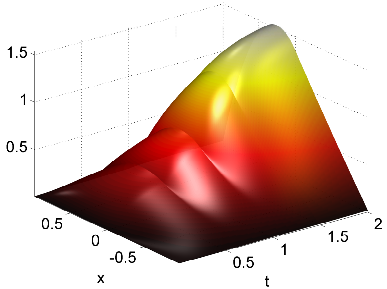
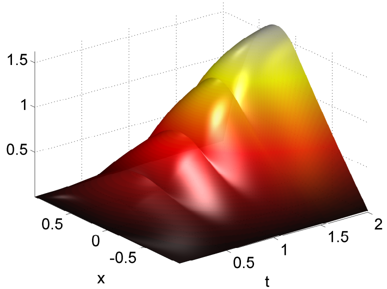
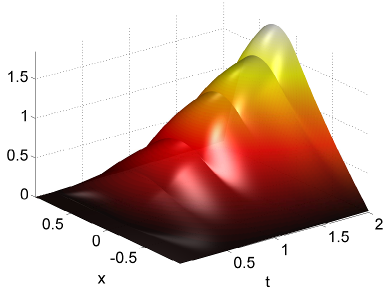


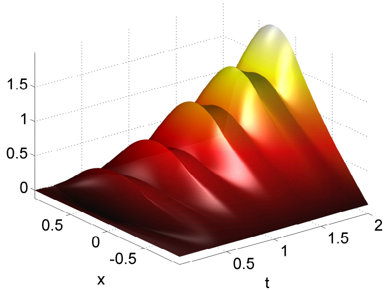
$\theta = 10$



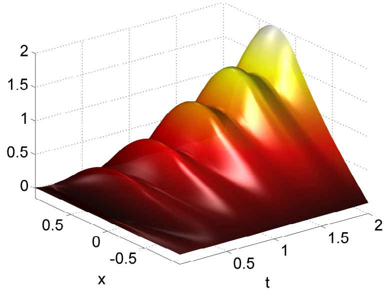
$\theta = 5$



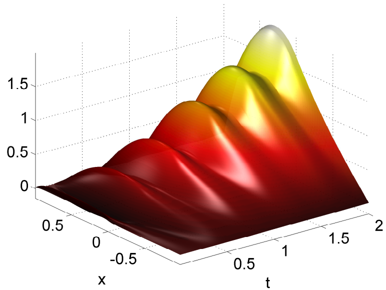
$\theta = 1$



$\theta = 0.1$



$\theta = 0.01$



$\theta = 0.001$

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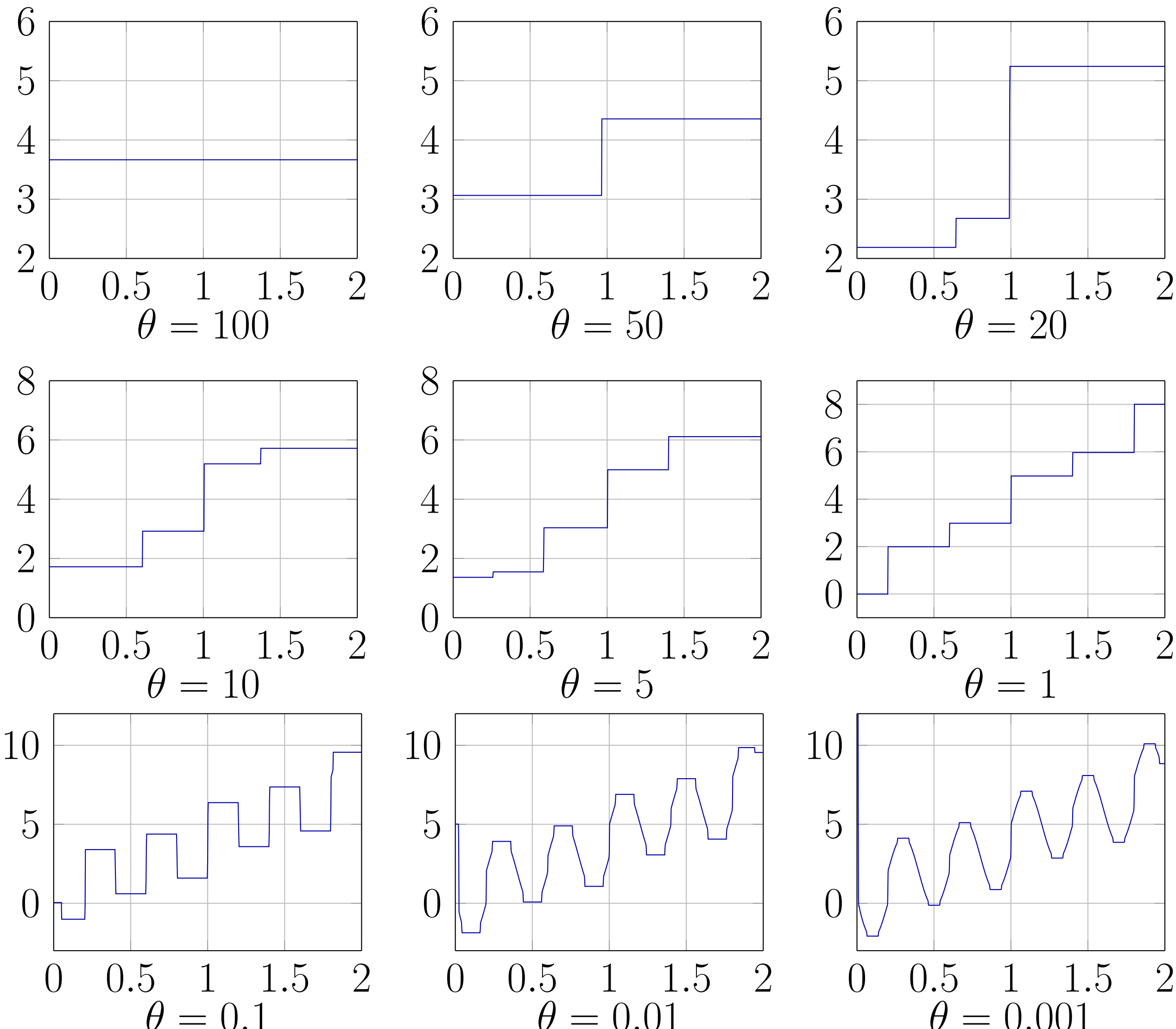

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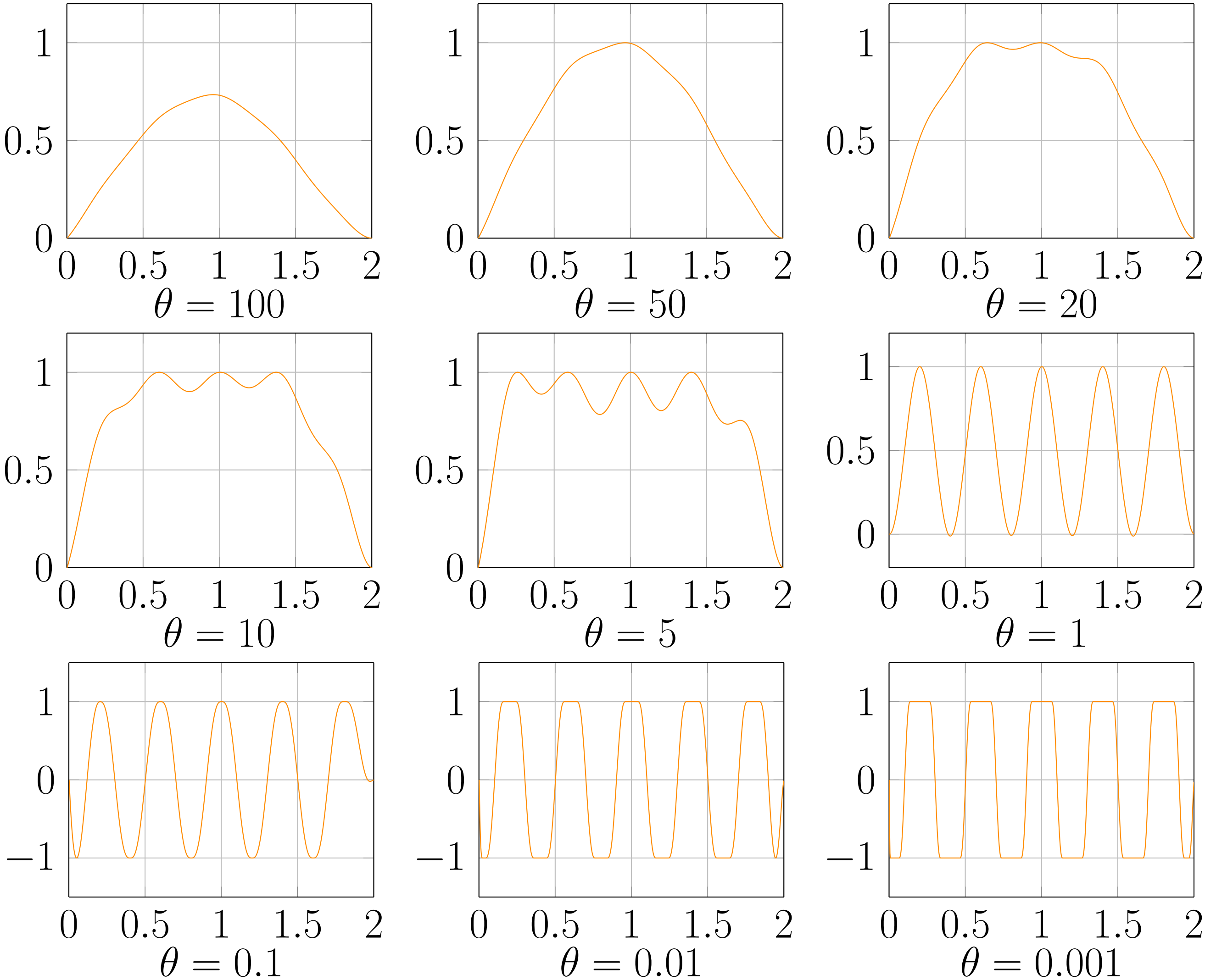

$\theta = 100$
$\theta = 50$
$\theta = 20$
$\theta = 10$
$\theta = 5$
$\theta = 1$
$\theta = 0.1$
$\theta = 0.01$
$\theta = 0.001$

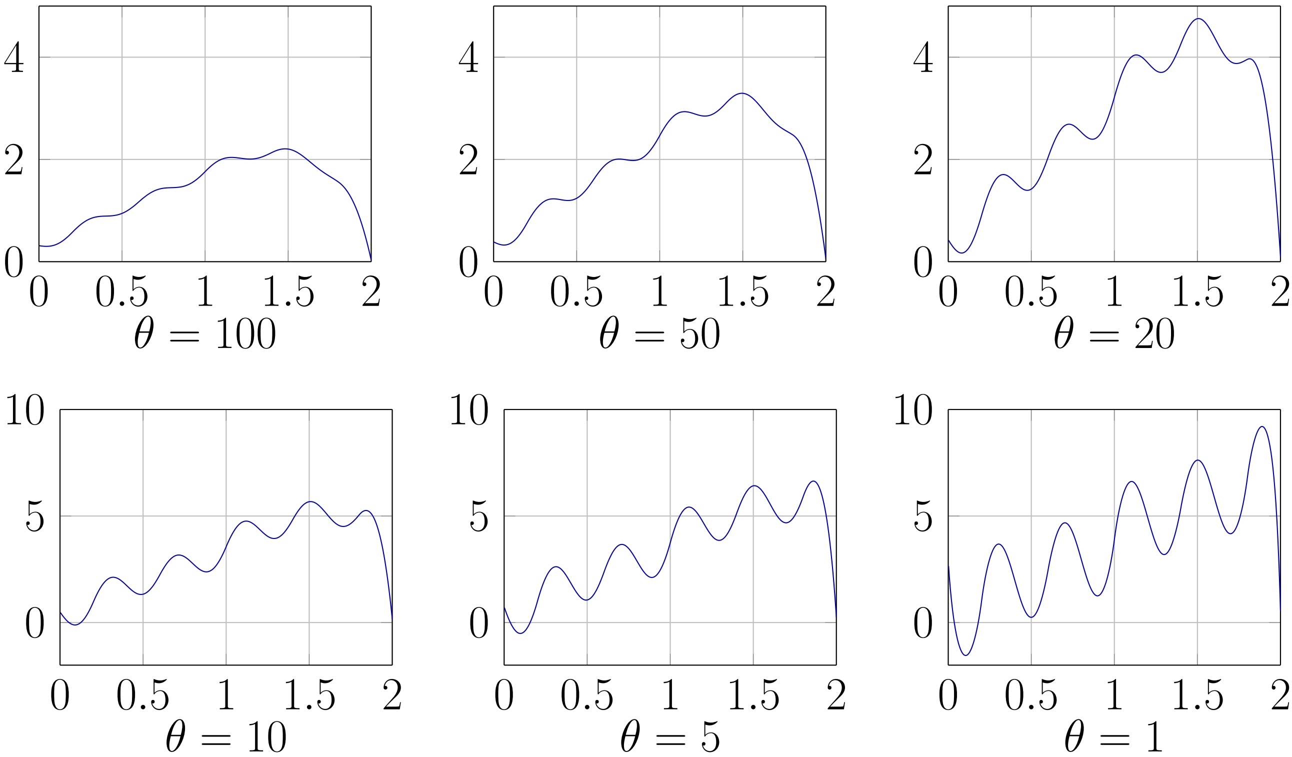

Figure 5: $\bar{u}^{\theta}_{\tau,L^2}$ for different values of θ.

THANK YOU FOR YOUR ATTENTION