Control cost of degenerate parabolic equations

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OPTIMAL CONTROL FOR EVOLUTIONARY PDES AND RELATED TOPICS

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Organized by P. COLLI, A. FAVINI, E. ROCCA, AND J. SPREKELS

Some pictures of beautiful Cortona

Figure: A peaceful evening at Palazzone

Figure: Breakfast at Palazzone

Figure: “None should sleep” from Puccini’s *Turandot*

Outline

1. Control of degenerate parabolic equations
   - Locally distributed control
   - Boundary control

2. Evaluation of the cost of null controllability
   - Exact controllability by the moment method
   - Optimal bounds for the null controllability cost

3. Concluding remarks

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Degenerate parabolic equations

$$\boxed{Au = \sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^{N} b_i(x)\partial_{x_i}u + c(x)u}$$ degenerate elliptic

$$u^f \leftrightarrow \begin{cases} \partial_t u - Au = f & \text{in} \quad (0,T)\times\Omega \\ u = G & \text{on } (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & x\in\Omega \end{cases}$$

- A degenerate on $\partial\Omega$: $\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant \nu(x)|\xi|^2$ with $\nu(x) > 0$ in Ω
- A degenerate in Ω but hypoelliptic, for instance

$$Au(x) = \sum_{k=1}^{K} X_k^2(x)u \qquad (1 \leqslant K \leqslant N)$$

where

$$X_k(x) = \sum_{i=1}^{N} \sigma_i^k(x)\partial_{x_i} \qquad (1 \leqslant k \leqslant K)$$

satisfy Hörmander's bracket generating condition

$$\text{span}\left\{X_{k_1}(x), \left[X_{k_1}, X_{k_2}\right](x), \dots \mid 1 \leqslant k_j \leqslant K\right\} = \mathbb{R}^N$$

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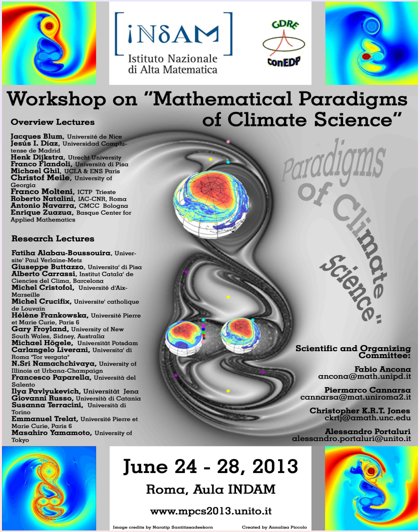
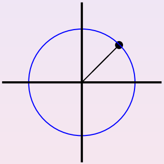
Boundary degeneracy in climate science



Budyko-Sellers model

$$\begin{cases} u_t - \left((1-x^2)u_x\right)_x = q(t,x)\,\beta(u) - \gamma(u) & x \in (-1,1) \\ (1-x^2)u_x|_{x=\pm 1} = 0 \end{cases}$$



- $u(t,x)$= sea-level zonally averaged temperature
- $q(t,x)$= solar input
- $\beta(u)$= co-albedo
- $\gamma(u)$= outgoing infrared radiation

Degenerate parabolic problems in 1D

$a \in C([0,1]) \cap C^1(]0,1])$ and $a > 0$ on $]0,1]$

$$\begin{cases} u_t - \big(a(x)u_x\big)_x = f & \text{in } Q_T = (0,T)\times(0,1) \\ u(0,x) = u_0(x) & u(t,1) = 0 \ + \ \text{b.c. at } \ x = 0 \end{cases}$$

$u_0 \in L^2(0,1)\,,\ f \in L^2(Q_T)$

weakly degenerate case $\boxed{1/a \in L^1(0,1)}$

$$u(t,0) = 0$$

strongly degenerate case $\boxed{1/a \notin L^1(0,1)}$

$$\lim_{x\downarrow 0} a(x)u_x(t,x) = 0$$

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Null controllability: locally distributed control

$$u^f \leftrightarrow \begin{cases} u_t - \big(a(x)u_x\big)_x = \chi_\omega f & \textit{in } Q_T = (0,T)\times(0,1) \\ u(0,x) = u_0(x) \\ u(t,1) = 0 \ \text{ and } \ \begin{cases} u(t,0) = 0 & \text{weakly degenerate} \\ au_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases} \qquad (C)$$

- null controllable in time $T > 0$

$$\forall u_0 \in L^2(0,1) \quad \exists f \in L^2(Q_T) \ : \ \begin{cases} u^f(T,\cdot) \equiv 0 \\ \int_{Q_T} |f|^2 \leqslant C_T \int_0^1 |u_0|^2 \end{cases}$$

- observability on $(0,T)\times\omega$

$$\begin{cases} v_t + \big(a(x)v_x\big)_x = 0 & \textit{in } Q_T \\ v(t,1) = 0 \ \text{ and } \ \begin{cases} v(t,0) = 0 & \text{weakly degenerate} \\ av_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases} \qquad (C^*)$$

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The simplest example of degeneracy

$\omega = (a,b) \subset\subset (0,1)$

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = \chi_\omega f \quad \textit{in } Q_T = (0,T)\times(0,1) \\ u(0,x) = u_0(x) \\ u(t,1) = 0 \ \text{ and } \begin{cases} u(t,0) = 0 & \text{weakly degenerate} \\ x^\alpha u_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases}$$

Theorem (C – Martinez – Vancostenoble, 2008)

$$\textit{null controllability} \begin{cases} \textit{false} & \alpha \geq 2 & (\to \textit{regional null controllability}) \\ \textit{true} & 0 \leq \alpha < 2 & \begin{cases} \textit{any b.c.} & 0 \leq \alpha < 1 & \textit{weak} \\ \textit{Neumann b.c.} & 1 \leq \alpha < 2 & \textit{strong} \end{cases} \end{cases}$$

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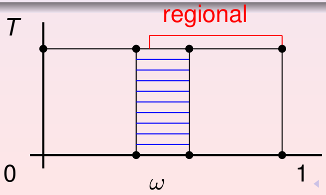


Carleman estimate $0 < \alpha < 2$

$$\begin{cases} w_t + \left(x^\alpha w_x\right)_x = f \quad \textit{in } Q_T \\ w(t,1) = 0 \ \text{ and } \begin{cases} w(t,0) = 0 & \text{weakly degenerate} \\ x^\alpha w_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases}$$

let $\boxed{\varphi(t,x) = \theta(t)\psi(x)}$ where

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^4 \qquad \psi(x) = \frac{x^{2-\alpha} - 2}{(2-\alpha)^2}$$

Theorem (C – Martinez – Vancostenoble, 2008)

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\tau\theta x^\alpha w_x^2 + \tau^3\theta^3 x^{2-\alpha} w^2\right) e^{2\tau\varphi}\, dxdt$$

$$\leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}\, dxdt + C\int_0^T \left\{\tau\theta w_x^2 e^{2\tau\varphi}\right\}_{|x=1} dt$$

Further results in 1D: locally distributed control

For general a

$$\limsup_{x\downarrow 0} \frac{x|a'(x)|}{a(x)} < 2$$

- divergence form $\boxed{\exists\, K \in [0,2) \text{ such that } xa'(x) \le Ka(x)}$
  - Martinez – Vancostenoble (2006) $\partial_t u - \partial_x\big(a(x)\partial_x u\big) = \chi_\omega f$
  - Alabau – C – Fragnelli (2006) $\partial_t u - \partial_x\big(a(x)\partial_x u\big) + g(u) = \chi_\omega f$
  - Flores – de Teresa (2010) $\partial_t u - \partial_x\big(x^\alpha \partial_x u\big) + x^\beta b(t,x)\partial_x u = \chi_\omega f$
- nondivergence form C – Fragnelli – Rocchetti (2007, 2008)

$$\partial_t u - a(x)\partial_x^2 u - b(x)\partial_x u = \chi_\omega f$$

- degenerate/singular Vancostenoble – Zuazua (2008), Vancostenoble (2009)

$$\partial_t u - \partial_x\big(x^\alpha \partial_x u\big) - \frac{\lambda}{x^\beta}\, u = \chi_\omega f$$

Further results

- C – de Teresa (2009) cascade 2×2

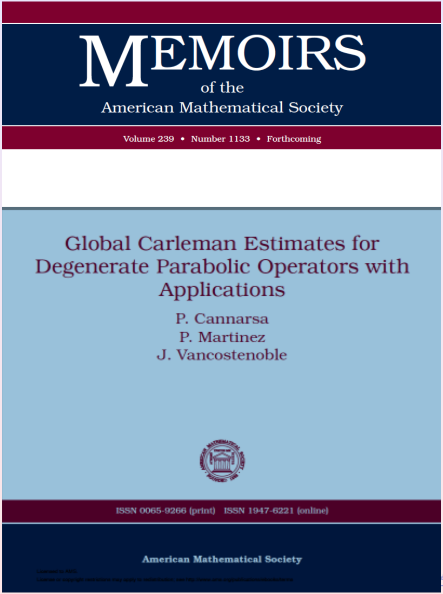
$$\omega \cap \mathcal{O} \neq \varnothing \qquad \begin{cases} \partial_t u - \partial_x(x^\alpha \partial_x u) + c(t,x)u = \xi + \chi_\omega h \\ \partial_t v - \partial_x(x^\alpha \partial_x v) + d(t,x)v = \chi_{\mathcal{O}} u \end{cases}$$

- Ben Hassi – Ammar Khodja – Hajjaj – Maniar (2011, 2013)

$$\omega \cap \mathcal{O} \neq \varnothing \qquad \begin{cases} \partial_t u - \partial_x\big(a_1(x)\partial_x u\big) + c(t,x)u = \xi + \chi_\omega h \\ \partial_t v - \partial_x\big(a_2(x)\partial_x v\big) + d(t,x)v = \chi_{\mathcal{O}} u \end{cases}$$

- inverse problems C – Tort – Yamamoto (2010)
- interior degeneracy Fragnelli – Mugnai (2013)
- Neumann boundary conditions and inverse problems
  Boutaayamou – Fragnelli – Maniar (2014)

Higher dimension



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Boundary control at $x = 1$

$0 \leqslant \alpha < 2\,,\ T > 0$

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 \quad \text{in} \quad (0,T)\times(0,1) \\ u(t,1) = G(t) \quad \text{and} \quad \begin{cases} u(t,0) = 0 & \text{weakly degenerate} \\ x^\alpha u_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \\ u(0,x) = u_0(x) \end{cases}$$

follows from locally distributed result, but can also be derived by

- moment Fattorini – Russel (1971)
- flatness Martin – Roiser – Rouchon (2014), Moyano (2015)

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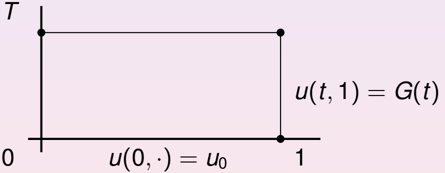

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Boundary control at $x = 0$

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- approximate controllability C – Tort – Yamamoto (2011)
- null controllability Gueye (2014)
  by transmutation and spectral analysis for the wave equation
- moment method C – Martinez – Vancostenoble

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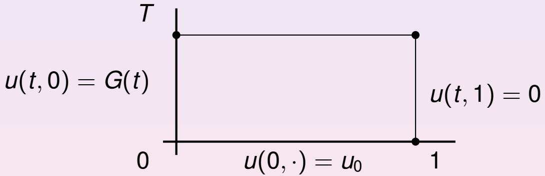

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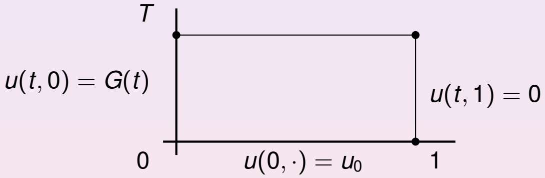

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Spectral analysis of degenerate elliptic operators

Want to determine the eigenvalues $\{\lambda_{\alpha,n}\}$ and normalized eigenfunctions $\{\Phi_{\alpha,n}\}$ of

$$\begin{cases} -(x^\alpha \phi'(x))' = \lambda\phi(x) & x \in (0,1) \\ \phi(0) = 0, \quad \phi(1) = 0 \end{cases}$$

Introduce parameters $\nu_\alpha = \dfrac{1-\alpha}{2-\alpha}$ and $\kappa_\alpha = \dfrac{2-\alpha}{2}$ Then

$$\lambda_{\alpha,n} = \kappa_\alpha^2 j_{\nu_\alpha,n}^2 \quad \text{and} \quad \Phi_{\alpha,n}(x) = \frac{\sqrt{2\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} x^{(1-\alpha)/2} J_{\nu_\alpha}(j_{\nu_\alpha,n} x^{\kappa_\alpha}) \qquad x \in (0,1)$$

where $j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \dots$ are the positive zeros of the Bessel function J_ν

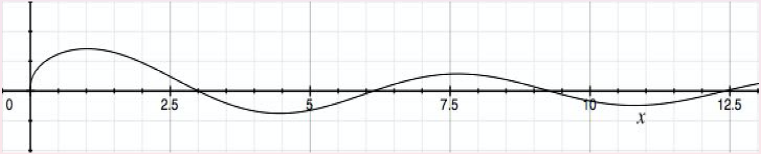

Figure: the Bessel function $J_{2/5}$

An exact controllability result

For any $v \in L^2(0,1)$ let $\widehat{v}_{\alpha,n} = \int_0^1 v(x)\Phi_{\alpha,n}(x)\,dx$

For any $K > 0$ define $\mathcal{P}_{\alpha,K} = \left\{ v \in L^2(0,1) \ :\ \sum_{n\geq 1} n^{3/2}|\widehat{v}_{\alpha,n}|e^{K\kappa_\alpha \pi n} < \infty \right\}$

Theorem (C–Martinez-Vancostenoble)

$\exists K^* > 0$ such that $\forall \alpha \in [0,1)$, $T > 0$, $u_0 \in L^2(0,1)$, and $u_T \in \mathcal{P}_{\alpha,K^*}$ one can find a control $G \in H^1(0,T)$ such that the solution of

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & \text{in } (0,1)\times(0,T) \\ u(t,0) = G(t), & u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

satisfies $u(T,\cdot) = u_T$

- Heat equation Fattorini-Russell (1971)
- Theorem implies approximate and null controllability
- $\mathcal{P}_{\alpha,K^*}$ is independent of T (Seidman 1979)
- Multi-D approach by Ervedoza-Zuazua (2011)

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$\exists K^* > 0$ such that $\forall \alpha \in [0,1)$, $T > 0$, $u_0 \in L^2(0,1)$, and $u_T \in \mathcal{P}_{\alpha,K^*}$ one can find a control $G \in H^1(0,T)$ such that the solution of

$$\begin{cases} u_t - (x^\alpha u_x)_x = 0 & in \ \ (0,1)\times(0,T) \\ u(t,0) = G(t), & u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

satisfies $u(T,\cdot) = u_T$

- Heat equation Fattorini-Russell (1971)
- Theorem implies approximate and null controllability
- $\mathcal{P}_{\alpha,K^*}$ is independent of T (Seidman 1979)
- Multi-D approach by Ervedoza-Zuazua (2011)

Proof by moment method

- For $u_0 = \sum_{n=1}^{\infty} \mu_n^0 \Phi_n$ and $u_T = \sum_{n=1}^{\infty} \mu_n^T \Phi_n$ let

$$u(t,x) = \sum_{n=1}^{\infty} \mu_n(t)\Phi_n(x) \quad \text{so that} \quad \mu_n(T) = \mu_n^T$$

$\Rightarrow$ Use adjoint problem to show boundary control G must satisfy

$$r_n \int_0^T G(t)\, e^{\lambda_n t} dt = \mu_n^T\, e^{\lambda_n T} - \mu_n^0 \quad \text{where} \quad r_n := x^\alpha \Phi_n'(x)_{|x=0} \sim (1-\alpha)R_n$$

$\Rightarrow$ Construct biorthogonal family $\{\sigma_n\}$

$$\int_0^T \sigma_m(t)\, e^{\lambda_n t} dt = \delta_{mn} \quad \text{satisfying} \quad \|\sigma_n\|_{L^2(0,T)} \leqslant B_T e^{K\sqrt{\lambda_n} - \lambda_n T}$$

$\Rightarrow$ Solve above equations taking

$$G(t) = \int_0^t g(s)\, ds \quad \text{with} \quad g(t) = \sum_{n=1}^{\infty} \frac{\lambda_n}{r_n} \left(\mu_n^0 - \mu_n^T\, e^{\lambda_n T}\right)\sigma_n(t)$$

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Uniformly attainable set

For every $\alpha \in [0,1)$,

$$\mathcal{P}_{\alpha,K^*} = \left\{ v \in L^2(0,1) \ : \ \sum_{n\geq 1} n^{3/2} |\widehat{v}_{\alpha,n}| e^{K^* \kappa_\alpha \pi n} < \infty \right\}$$

is attanable in any $T > 0$

Proposition

$$\bigcap_{\alpha\in[0,1)} \mathcal{P}_{\alpha,K^*} = \{0\}$$

Open problem

Is zero the only target that can be attained $\forall \alpha \in [0,1)$?

Uniformly attainable set

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Regularity of attainable states

Proof of $\cap_{\alpha\in[0,1)}\mathcal{P}_{\alpha,K^*} = \{0\}$ where

$$\mathcal{P}_{\alpha,K^*} = \Big\{ v \in L^2(0,1) \ : \ \sum_{n\geq 1} n^{3/2}|\widehat{v}_{\alpha,n}|e^{K^*\kappa_\alpha \pi n} < \infty \Big\}$$

uses regularity result

Proposition

If $v \in \mathcal{P}_{\alpha,K^*}$, then there exists an even function F_α, **holomorphic** *in the strip $\big\{z\in\mathbb{C} \ : \ |\Im z| < \frac{K^*}{\pi}\big\}$ such that*

$$v(x) = x^{1-\alpha}F_\alpha(x^{\kappa_\alpha}) \qquad \forall x \in [0,1]$$

Heat equation Fattorini-Russell (1971)

Outline

1. Control of degenerate parabolic equations
   - Locally distributed control
   - Boundary control
2. Evaluation of the cost of null controllability
   - Exact controllability by the moment method
   - Optimal bounds for the null controllability cost
3. Concluding remarks

The cost of null controllability

$\forall u_0 \in L^2(0,1)$ define $\mathcal{G}(\alpha, u_0) := \{G \in H^1(0,T) \ : \ u(T,\cdot) = 0\}$ where

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & \text{in } (0,T)\times(0,1) \\ u(t,0) = G(t), & u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

$\boxed{\inf_{G\in\mathcal{G}(\alpha,u_0)} \|G\|_{H^1(0,T)}}$ gives the control cost needed to steer u_0 to 0

Theorem (C–Martinez-Vancostenoble)

(a) *There exists $M_1(u_0)$ and M_2, independent of α, such that*

$$\frac{M_1(u_0)}{1-\alpha} \leqslant \inf_{G\in\mathcal{G}(\alpha,u_0)} \|G\|_{H^1(0,T)} \leqslant \frac{M_2}{1-\alpha}\|u_0\|_{L^2}$$

(b) *There exist $M_1, M_2 > 0$, independent of α, such that*

$$\frac{M_1}{1-\alpha} \leqslant \sup_{\|u_0\|=1} \inf_{G\in\mathcal{G}(\alpha,u_0)} \|G\|_{H^1(0,T)} \leqslant \frac{M_2}{1-\alpha}$$

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Proof of bounds for control cost

UB follows from our construction of G

$$G(t) = \int_0^t g(s)\, ds \quad \text{with} \quad g(t) = \sum_{n=1}^{\infty} \frac{\lambda_{\alpha,n}}{r_{\alpha,n}}\, \mu^0_{\alpha,n}\, \sigma_{\alpha,n}(t)$$

using $r_{\alpha,n} = x^\alpha \Phi'_{\alpha,n}(x)_{|x=0} \sim (1-\alpha) R_n$

LB let G be any control such that

$$r_{\alpha,n} \int_0^T G(t)\, e^{\lambda_{\alpha,n} t} dt = -\mu^0_{\alpha,n} = -\int_0^1 u_0(x) \Phi_{\alpha,n}(x)\, dx$$

show $\mu^0_{\alpha,n} \xrightarrow{\alpha \uparrow 1} \int_0^1 u_0(x) \Phi_{1,n}(x)\, dx$ with $\Phi_{1,n}(x) = \frac{J_0(j_{0,n}\sqrt{x})}{|J_0'(j_{0,n})|}$

since $\{\Phi_{1,n}\}$ orthonormal basis and $u_0 \neq 0$ deduce $|\mu^0_{\alpha,n_0}| \geqslant m(u_0) > 0$

conclude

$$\frac{m(u_0)}{(1-\alpha) R_{\alpha,n_0}} \leqslant \frac{m(u_0)}{r_{\alpha,n_0}} \leqslant \Big(\int_0^T e^{2\lambda_{\alpha,n_0} t} dt \Big)^{\frac{1}{2}} \|G\|_{L^2(0,T)}$$

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Open problem: boundary control for general a

$a \in C([0,1]) \cap C^1(]0,1])$ and $a > 0$ on $]0,1]$

$$\begin{cases} u_t - \big(a(x)u_x\big)_x = 0 & \text{in } (0,T)\times(0,1) \\ u(t,0) = G(t) \\ u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

assume

$$\exists\, \mu \in [0,1) \quad \text{such that} \quad x\, a'(x) \leq \mu\, a(x)$$

study

- null controllability
- behaviour of the control cost as $\mu \uparrow 1$

Evaluation of distributed control cost

Joint work with **P. Martinez** and **J. Vancostenoble**

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = \chi_\omega f \quad in\ Q_T = (0,T)\times(0,1) \\ u(0,x) = u_0(x) \\ u(t,1) = 0 \ \text{ and } \ \begin{cases} u(t,0) = 0 & \text{weakly degenerate} \\ x^\alpha u_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases}$$

with $\omega = (a,b) \subset\subset (0,1)$

$$\mathcal{F}(\alpha,u_0) := \left\{ f \in L^2(Q_T) \ : \ u(T,\cdot) = 0 \right\}$$

Then

$$\frac{M_1}{(2-\alpha)^2 T} \leqslant \log\left(\sup_{\|u_0\|=1} \inf_{f\in\mathcal{F}(\alpha,u_0)} \|f\|_{H^1(0,T)} \right) \leqslant \frac{M_2}{(2-\alpha)^2 T}$$

Degenerate wave equation

Joint work with **F. Alabau-Boussouira** and **G. Leugering**

$$\begin{cases} u_{tt} - \big(a(x)u_x\big)_x = 0 & t > 0 \quad x \in (0,1) \\ u(0,x) = u_0(x) \quad u_t(0,x) = u_1(x) & x \in (0,1) \end{cases}$$

Define

$$E_u(t) := \frac{1}{2}\int_0^1 \left\{u_t^2(t,x) + a(x)u_x^2(t,x)\right\}dx$$

Theorem

Assume $\mu_a := \sup\limits_{0<x\leqslant 1} \dfrac{x|a'(x)|}{a(x)} < 2$ *Then* $\forall T \geqslant 0$ *the solution satisfies*

$$a(1)\int_0^T u_x^2(t,1)\,dt \geqslant \left\{(2-\mu_a)T - \frac{4}{\min\{1,a(1)\}} - 2\,\mu_a\sqrt{C_a}\right\}E_u(0)$$

Thus the wave system is observable for

$$T > T_a := \frac{4}{(2-\mu_a)\min\{1,a(1)\}} + 2\,\mu_a\sqrt{C_a}$$

Thank you for your attention