

Control cost of degenerate parabolic equations

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OPTIMAL CONTROL FOR EVOLUTIONARY PDES AND RELATED TOPICS

INDAM Meeting

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Organized by P. COLLI, A. FAVINI, E. ROCCA, AND J. SPREKELS



Some pictures of beautiful Cortona



Figure: A peaceful evening at Palazzone





Figure: Breakfast at Palazzone





Figure: “None should sleep” from Puccini’s *Turandot*



Outline

- 1 Control of degenerate parabolic equations
 - Locally distributed control
 - Boundary control
- 2 Evaluation of the cost of null controllability
 - Exact controllability by the moment method
 - Optimal bounds for the null controllability cost
- 3 Concluding remarks



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Degenerate parabolic equations

$$Au = \sum_{i,j=1}^N a_{ij}(x) \partial_{x_i} \partial_{x_j} u + \sum_{i=1}^N b_i(x) \partial_{x_i} u + c(x)u \quad \text{degenerate elliptic}$$

$$u^f \leftrightarrow \begin{cases} \partial_t u - Au = f & \text{in } (0, T) \times \Omega \\ u = G & \text{on } (0, T) \times \partial\Omega \\ u(0, x) = u_0(x) & x \in \Omega \end{cases}$$

- A degenerate on $\partial\Omega$: $\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq \nu(x) |\xi|^2$ with $\nu(x) > 0$ in Ω
- A degenerate in Ω but hypoelliptic, for instance

$$Au(x) = \sum_{k=1}^K X_k^2(x)u \quad (1 \leq K \leq N)$$

where

$$X_k(x) = \sum_{i=1}^N \sigma_i^k(x) \partial_{x_i} \quad (1 \leq k \leq K)$$

satisfy Hörmander's bracket generating condition

$$\text{span} \{ X_{k_1}(x), [X_{k_1}, X_{k_2}](x), \dots \mid 1 \leq k_j \leq K \} = \mathbb{R}^N$$



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Boundary degeneracy in climate science





[INSAM]
Istituto Nazionale
di Alta Matematica




Workshop on "Mathematical Paradigms of Climate Science"

Overview Lectures

Jacques Blum, *Université de Nice*
 Jesús I. Díaz, *Universidad Complutense de Madrid*
 Hank Dijkstra, *Utrecht University*
 Franco Flandoli, *Università di Pisa*
 Michael Ghil, *UCLA & ENEC Paris*
 Christof Meile, *University of Georgia*
 Franco Molteni, *ICTP Trieste*
 Roberto Natalini, *IAC-CNR, Roma*
 Antonio Navarra, *CNCC Bologna*
 Enrique Zuazua, *Basque Center for Applied Mathematics*

Research Lectures

Fatima Alabau-Boussouira, *Université Paul Sabatier-Mont*
 Giuseppe Buttazzo, *Università di Pisa*
 Alberto Corral, *Institut Català de Ciències del Clima, Barcelona*
 Michel Cristofol, *Université d'Alsace-Marseille*
 Michel Crucifix, *Université catholique de Louvain*
 Hélène Frankowska, *Université Pierre et Marie Curie, Paris 6*
 Gary Froyland, *University of New South Wales, Sydney, Australia*
 Michael Högl, *Universität Potsdam*
 Carlene Liverani, *Università di Roma Tor Vergata*
 N.Sri Ramasubrahmanyam, *University of Illinois at Urbana-Champaign*
 Francesco Paparella, *Università del Salento*
 Ilya Pavlyukevich, *Universität Jena*
 Giovanni Russo, *Università di Catania*
 Susanna Terracini, *Università di Torino*
 Emmanuel Trelat, *Université Pierre et Marie Curie, Paris 6*
 Masahiro Yamamoto, *University of Tokyo*

Paradigms
of Climate
Science

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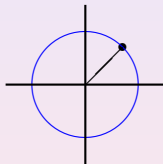
June 24 - 28, 2013
 Roma, Aula INDAM
www.mpcs2013.unito.it




Image credits by Ilsevier-Einführungsbuch

Budyko-Sellers model

$$\begin{cases} u_t - ((1 - x^2)u_x)_x = q(t, x) \beta(u) - \gamma(u) & x \in (-1, 1) \\ (1 - x^2)u_x|_{x=\pm 1} = 0 \end{cases}$$



- $u(t, x)$ = sea-level zonally averaged temperature
- $q(t, x)$ = solar input
- $\beta(u)$ = co-albedo
- $\gamma(u)$ = outgoing infrared radiation



Degenerate parabolic problems in 1D

$$a \in C([0, 1]) \cap C^1(]0, 1]) \quad \text{and} \quad a > 0 \quad \text{on} \quad]0, 1]$$

$$\begin{cases} u_t - (a(x)u_x)_x = f & \text{in } Q_T = (0, T) \times (0, 1) \\ u(0, x) = u_0(x) & u(t, 1) = 0 + \text{b.c. at } x = 0 \end{cases}$$

$$u_0 \in L^2(0, 1), \quad f \in L^2(Q_T)$$

weakly degenerate case

$$1/a \in L^1(0, 1)$$

$$u(t, 0) = 0$$

strongly degenerate case

$$1/a \notin L^1(0, 1)$$

$$\lim_{x \downarrow 0} a(x)u_x(t, x) = 0$$



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Null controllability: locally distributed control

$$u^f \leftrightarrow \begin{cases} u_t - (a(x)u_x)_x = \chi_\omega f & \text{in } Q_T = (0, T) \times (0, 1) \\ u(0, x) = u_0(x) \\ u(t, 1) = 0 \text{ and } \begin{cases} u(t, 0) = 0 & \text{weakly degenerate} \\ au_x(t, \cdot)|_{x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases} \quad (C)$$

- null controllable in time $T > 0$

$$\forall u_0 \in L^2(0, 1) \quad \exists f \in L^2(Q_T) : \begin{cases} u^f(T, \cdot) \equiv 0 \\ \int_{Q_T} |f|^2 \leq C_T \int_0^1 |u_0|^2 \end{cases}$$

- observability on $(0, T) \times \omega$

$$\begin{cases} v_t + (a(x)v_x)_x = 0 & \text{in } Q_T \\ v(t, 1) = 0 \text{ and } \begin{cases} v(t, 0) = 0 & \text{weakly degenerate} \\ av_x(t, \cdot)|_{x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases} \quad (C^*)$$

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The simplest example of degeneracy

$$\omega = (a, b) \subset\subset (0, 1)$$

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Theorem (C – Martinez – Vancostenoble, 2008)

$$\text{null controllability} \begin{cases} \text{false} & \alpha \geq 2 \quad (\rightarrow \text{regional null controllability}) \\ \text{true} & 0 \leq \alpha < 2 \quad \begin{cases} \text{any b.c.} & 0 \leq \alpha < 1 \quad \text{weak} \\ \text{Neumann b.c.} & 1 \leq \alpha < 2 \quad \text{strong} \end{cases} \end{cases}$$



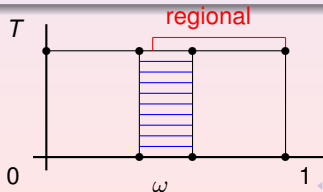
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Carleman estimate $0 < \alpha < 2$

$$\begin{cases} w_t + (x^\alpha w_x)_x = f & \text{in } Q_T \\ w(t, 1) = 0 \text{ and } \begin{cases} w(t, 0) = 0 \\ x^\alpha w_x(t, \cdot)|_{x=0} = 0 \end{cases} \end{cases} \quad \begin{array}{l} \text{weakly degenerate} \\ \text{strongly degenerate} \end{array}$$

let $\varphi(t, x) = \theta(t)\psi(x)$ where

$$\theta(t) = \left(\frac{1}{t(T-t)} \right)^4 \quad \psi(x) = \frac{x^{2-\alpha} - 2}{(2-\alpha)^2}$$

Theorem (C – Martinez – Vancostenoble, 2008)

There exists $\tau_0, C > 0$ such that $\forall \tau \geq \tau_0$

$$\begin{aligned} \iint_{Q_T} \left(\tau \theta x^\alpha w_x^2 + \tau^3 \theta^3 x^{2-\alpha} w^2 \right) e^{2\tau\varphi} dx dt \\ \leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi} dx dt + C \int_0^T \left\{ \tau \theta w_x^2 e^{2\tau\varphi} \right\}_{|x=1} dt \end{aligned}$$

Further results in 1D: locally distributed control

For general a

$$\limsup_{x \downarrow 0} \frac{x|a'(x)|}{a(x)} < 2$$

- divergence form $\exists K \in [0, 2)$ such that $xa'(x) \leq Ka(x)$

- Martinez – Vancostenoble (2006) $\partial_t u - \partial_x(a(x)\partial_x u) = \chi_\omega f$
- Alabau – C – Fragnelli (2006) $\partial_t u - \partial_x(a(x)\partial_x u) + g(u) = \chi_\omega f$
- Flores – de Teresa (2010) $\partial_t u - \partial_x(x^\alpha \partial_x u) + x^\beta b(t, x)\partial_x u = \chi_\omega f$

- nondivergence form C – Fragnelli – Rocchetti (2007, 2008)

$$\partial_t u - a(x)\partial_x^2 u - b(x)\partial_x u = \chi_\omega f$$

- degenerate/singular Vancostenoble – Zuazua (2008), Vancostenoble (2009)

$$\partial_t u - \partial_x(x^\alpha \partial_x u) - \frac{\lambda}{x^\beta} u = \chi_\omega f$$



Further results

- C – de Teresa (2009) cascade 2×2

$$\omega \cap \mathcal{O} \neq \emptyset \quad \begin{cases} \partial_t u - \partial_x(x^\alpha \partial_x u) + c(t, x)u = \xi + \chi_\omega h \\ \partial_t v - \partial_x(x^\alpha \partial_x v) + d(t, x)v = \chi_{\mathcal{O}} u \end{cases}$$

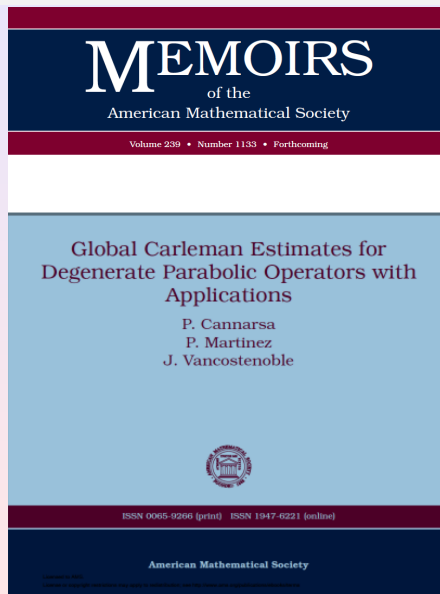
- Ben Hassi – Ammar Khodja – Hajjaj – Maniar (2011, 2013)

$$\omega \cap \mathcal{O} \neq \emptyset \quad \begin{cases} \partial_t u - \partial_x(a_1(x) \partial_x u) + c(t, x)u = \xi + \chi_\omega h \\ \partial_t v - \partial_x(a_2(x) \partial_x v) + d(t, x)v = \chi_{\mathcal{O}} u \end{cases}$$

- inverse problems C – Tort – Yamamoto (2010)
- interior degeneracy Fragnelli – Mugnai (2013)
- Neumann boundary conditions and inverse problems
Boutaayamou – Fragnelli – Maniar (2014)



Higher dimension



Outline

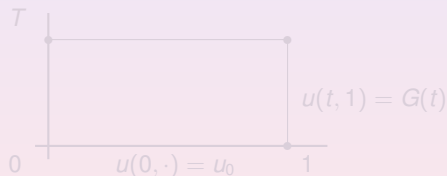
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Boundary control at $x = 1$

$$0 \leq \alpha < 2, \quad T > 0$$

$$\begin{cases} u_t - (x^\alpha u_x)_x = 0 & \text{in } (0, T) \times (0, 1) \\ u(t, 1) = G(t) & \text{and } \begin{cases} u(t, 0) = 0 \\ x^\alpha u_x(t, \cdot)|_{x=0} = 0 \end{cases} & \begin{array}{l} \text{weakly degenerate} \\ \text{strongly degenerate} \end{array} \\ u(0, x) = u_0(x) \end{cases}$$



follows from **locally distributed result**, but can also be derived by

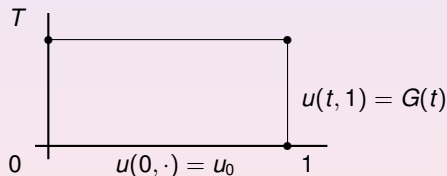
- **moment** Fattorini – Russel (1971)
- **flatness** Martin – Roiser – Rouchon (2014), Moyano (2015)



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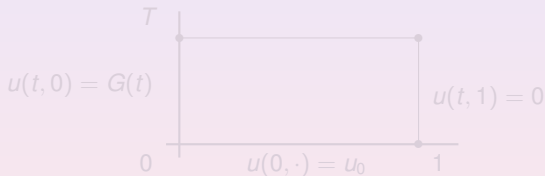
- [moment](#) Fattorini – Russel (1971)
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Boundary control at $x = 0$

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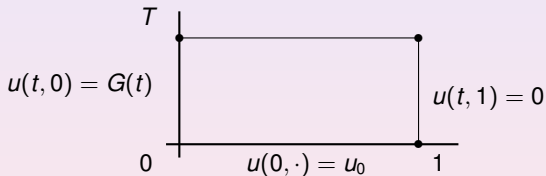
- approximate controllability C – Tort – Yamamoto (2011)
- null controllability Gueye (2014)
by transmutation and spectral analysis for the wave equation
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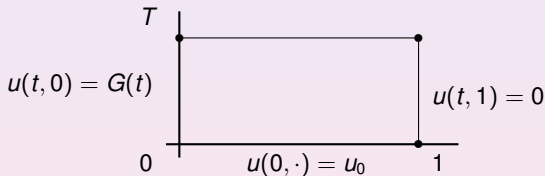
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Spectral analysis of degenerate elliptic operators

Want to determine the eigenvalues $\{\lambda_{\alpha,n}\}$ and normalized eigenfunctions $\{\Phi_{\alpha,n}\}$ of

$$\begin{cases} -(x^\alpha \phi'(x))' = \lambda \phi(x) & x \in (0, 1) \\ \phi(0) = 0, \quad \phi(1) = 0 \end{cases}$$

Introduce parameters $\nu_\alpha = \frac{1-\alpha}{2-\alpha}$ and $\kappa_\alpha = \frac{2-\alpha}{2}$ Then

$$\lambda_{\alpha,n} = \kappa_\alpha^2 j_{\nu_\alpha,n}^2 \quad \text{and} \quad \Phi_{\alpha,n}(x) = \frac{\sqrt{2\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} x^{(1-\alpha)/2} J_{\nu_\alpha}(j_{\nu_\alpha,n} x^{\kappa_\alpha}) \quad x \in (0, 1)$$

where $j_{\nu,1} < j_{\nu,2} < \dots < j_{\nu,n} < \dots$ are the positive zeros of the Bessel function J_ν

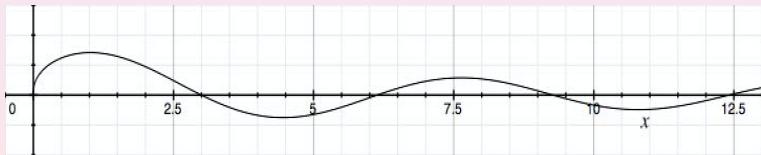


Figure: the Bessel function $J_{2/5}$



An exact controllability result

For any $v \in L^2(0, 1)$ let $\widehat{v}_{\alpha, n} = \int_0^1 v(x) \Phi_{\alpha, n}(x) dx$

For any $K > 0$ define $\mathcal{P}_{\alpha, K} = \left\{ v \in L^2(0, 1) : \sum_{n \geq 1} n^{3/2} |\widehat{v}_{\alpha, n}| e^{K \kappa_{\alpha} \pi n} < \infty \right\}$

Theorem (C–Martinez–Vancostenoble)

$\exists K^* > 0$ such that $\forall \alpha \in [0, 1)$, $T > 0$, $u_0 \in L^2(0, 1)$, and $u_T \in \mathcal{P}_{\alpha, K^*}$ one can find a control $G \in H^1(0, T)$ such that the solution of

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satisfies $u(T, \cdot) = u_T$

- Heat equation Fattorini-Russell (1971)
- Theorem implies approximate and null controllability
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- Multi-D approach by Ervedoza-Zuazua (2011)



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Proof by moment method

- For $u_0 = \sum_{n=1}^{\infty} \mu_n^0 \Phi_n$ and $u_T = \sum_{n=1}^{\infty} \mu_n^T \Phi_n$ let

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⇒ Use adjoint problem to show boundary control G must satisfy

$$r_n \int_0^T G(t) e^{\lambda_n t} dt = \mu_n^T e^{\lambda_n T} - \mu_n^0 \quad \text{where} \quad r_n := x^\alpha \Phi_n'(x)|_{x=0} \sim (1 - \alpha) R_n$$

⇒ Construct biorthogonal family $\{\sigma_n\}$

$$\int_0^T \sigma_m(t) e^{\lambda_n t} dt = \delta_{mn} \quad \text{satisfying} \quad \|\sigma_n\|_{L^2(0,T)} \leq B_T e^{K\sqrt{\lambda_n} - \lambda_n T}$$

⇒ Solve above equations taking

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Uniformly attainable set

For every $\alpha \in [0, 1)$,

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is attainable in any $T > 0$

Proposition

$$\bigcap_{\alpha \in [0, 1)} \mathcal{P}_{\alpha, K^*} = \{0\}$$

Open problem

Is **zero** the only target that can be attained $\forall \alpha \in [0, 1)$?



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Regularity of attainable states

Proof of $\cap_{\alpha \in [0,1)} \mathcal{P}_{\alpha, K^*} = \{0\}$ where

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uses regularity result

Proposition

If $v \in \mathcal{P}_{\alpha, K^*}$, then there exists an even function F_{α} , **holomorphic** in the strip $\{z \in \mathbb{C} : |\Im z| < \frac{K^*}{\pi}\}$ such that

$$v(x) = x^{1-\alpha} F_{\alpha}(x^{\kappa_{\alpha}}) \quad \forall x \in [0, 1]$$

Heat equation Fattorini-Russell (1971)



Outline

- 1 Control of degenerate parabolic equations
 - Locally distributed control
 - Boundary control
- 2 Evaluation of the cost of null controllability
 - Exact controllability by the moment method
 - Optimal bounds for the null controllability cost
- 3 Concluding remarks



The cost of null controllability

$\forall u_0 \in L^2(0, 1)$ define $\mathcal{G}(\alpha, u_0) := \{G \in H^1(0, T) : u(T, \cdot) = 0\}$ where

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$\inf_{G \in \mathcal{G}(\alpha, u_0)} \|G\|_{H^1(0, T)}$ gives the **control cost** needed to steer u_0 to 0

Theorem (C–Martinez–Vancostenoble)

(a) *There exists $M_1(u_0)$ and M_2 , independent of α , such that*

$$\frac{M_1(u_0)}{1 - \alpha} \leq \inf_{G \in \mathcal{G}(\alpha, u_0)} \|G\|_{H^1(0, T)} \leq \frac{M_2}{1 - \alpha} \|u_0\|_{L^2}$$

(b) *There exist $M_1, M_2 > 0$, independent of α , such that*

$$\frac{M_1}{1 - \alpha} \leq \sup_{\|u_0\|=1} \inf_{G \in \mathcal{G}(\alpha, u_0)} \|G\|_{H^1(0, T)} \leq \frac{M_2}{1 - \alpha}$$

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Proof of bounds for control cost

UB follows from our construction of G

$$G(t) = \int_0^t g(s) ds \quad \text{with} \quad g(t) = \sum_{n=1}^{\infty} \frac{\lambda_{\alpha,n}}{r_{\alpha,n}} \mu_{\alpha,n}^0 \sigma_{\alpha,n}(t)$$

using $r_{\alpha,n} = x^\alpha \Phi'_{\alpha,n}(x)|_{x=0} \sim (1 - \alpha)R_n$

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show $\mu_{\alpha,n}^0 \xrightarrow{\alpha \uparrow 1} \int_0^1 u_0(x) \Phi_{1,n}(x) dx$ with $\Phi_{1,n}(x) = \frac{J_0(j_{0,n} \sqrt{x})}{|J_0'(j_{0,n})|}$

since $\{\Phi_{1,n}\}$ orthonormal basis and $u_0 \neq 0$ deduce $|\mu_{\alpha,n}^0| \geq m(u_0) > 0$
conclude

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Open problem: boundary control for general a

$a \in C([0, 1]) \cap C^1(]0, 1])$ and $a > 0$ on $]0, 1]$

$$\begin{cases} u_t - (a(x)u_x)_x = 0 & \text{in } (0, T) \times (0, 1) \\ u(t, 0) = G(t) \\ u(t, 1) = 0 \\ u(0, x) = u_0(x) \end{cases}$$

assume

$$\exists \mu \in [0, 1) \text{ such that } x a'(x) \leq \mu a(x)$$

study

- null controllability
- behaviour of the control cost as $\mu \uparrow 1$



Evaluation of distributed control cost

Joint work with **P. Martinez** and **J. Vancostenoble**

$$\begin{cases} u_t - (x^\alpha u_x)_x = \chi_\omega f & \text{in } Q_T = (0, T) \times (0, 1) \\ u(0, x) = u_0(x) \\ u(t, 1) = 0 \text{ and } \begin{cases} u(t, 0) = 0 & \text{weakly degenerate} \\ x^\alpha u_x(t, \cdot)|_{x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases}$$

with $\omega = (a, b) \subset\subset (0, 1)$

$$\mathcal{F}(\alpha, u_0) := \{f \in L^2(Q_T) : u(T, \cdot) = 0\}$$

Then

$$\frac{M_1}{(2-\alpha)^2 T} \leq \log \left(\sup_{\|u_0\|=1} \inf_{f \in \mathcal{F}(\alpha, u_0)} \|f\|_{H^1(0, T)} \right) \leq \frac{M_2}{(2-\alpha)^2 T}$$



Degenerate wave equation

Joint work with **F. Alabau-Boussouira** and **G. Leugering**

$$\begin{cases} u_{tt} - (a(x)u_x)_x = 0 & t > 0 \quad x \in (0, 1) \\ u(0, x) = u_0(x) \quad u_t(0, x) = u_1(x) & x \in (0, 1) \end{cases}$$

Define

$$E_u(t) := \frac{1}{2} \int_0^1 \{u_t^2(t, x) + a(x)u_x^2(t, x)\} dx$$

Theorem

Assume $\mu_a := \sup_{0 < x \leq 1} \frac{x|a'(x)|}{a(x)} < 2$ Then $\forall T \geq 0$ the solution satisfies

$$a(1) \int_0^T u_x^2(t, 1) dt \geq \left\{ (2 - \mu_a)T - \frac{4}{\min\{1, a(1)\}} - 2\mu_a \sqrt{C_a} \right\} E_u(0)$$

Thus the wave system is **observable** for

$$T > T_a := \frac{4}{(2 - \mu_a) \min\{1, a(1)\}} + 2\mu_a \sqrt{C_a}$$



Thank you for your attention

