Control of the acoustic pressure in ultrasound linear propagation

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Irena Lasiecka (University of Memphis, USA)

Nonlinear acoustics: PDE models, motivations/applications

- High intensity (focused) ultrasound plays a central role in several medical treatments and industrial applications; to mention a few, lithotripsy, thermoterapy, ultrasonic welding, sonochemistry, ...

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- Classical PDE models of *nonlinear* acoustics are the Westervelt equation (formulated in terms of the acoustic pressure u),

$$\mathbf{u_{tt} - c^2\Delta u - b\Delta u_t = \frac{\beta_a}{\rho c^2}\frac{\partial^2}{\partial t^2}(u^2)} \qquad \text{in } \mathbf{(0,T)\times\Omega}$$

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and the Kuznetsov equation (formulated in terms of the acoustic velocity potential ψ),

$$\boldsymbol{\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \frac{\partial}{\partial t}\Big(\frac{\beta_a - 1}{c^2}\psi_t^2 + |\nabla \psi|^2\Big)} \qquad \text{in } \mathbf{(0,T) \times \Omega}$$

$c, b > 0$: *speed* and *diffusivity* of sound, $\rho > 0$: mass density; $\beta_a > 1$

[Westervelt, 1963], [Kuznetsov, 1971]

The linearization of both PDE is the strongly damped wave equation:

$$u_{tt} - c^2\Delta u - b\Delta u_t = 0 \qquad \text{in } (0,T)\times\Omega$$

Remark

If $b > 0$, the above equation has a parabolic-like behaviour

[Chen-Triggiani, 1987] (proving a conjecture due to Chen & Russell)

Maxwell-Cattaneo *vs* Fourier law

If in place of the Fourier law

$$\vec{q} = -\kappa \nabla \theta$$

for the heat flux q, the (*space-time*) Maxwell-Cattaneo law is utilized ([Cattaneo, 1946]), *viz.*

$$\tau \dot{q} + \vec{q} = -\kappa \nabla \theta \,,$$

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$$\tau \dot{\mathbf{q}} + \vec{q} = -\kappa \nabla \theta \,,$$

the following *model equation* – referred to as Jordan-Moore-Gibson-Thompson (JMGT) equation – has been derived:

$$\boxed{\boldsymbol{\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \frac{\partial}{\partial t}\Big(\frac{1}{c^2}\frac{B}{2A}\psi_t^2 + |\nabla \psi|^2\Big)}}$$

ψ is the *acoustic velocity potential*; then $-\nabla\psi$ is the acoustic *particle velocity*

[Moore-Gibson, 1960], [Thompson, 1972], [Jordan, 2009]

The (Stokes-)Moore-Gibson-Thompson equation

The *linearization* of the JMGT equation, that is

$$\boldsymbol{\tau\psi_{ttt} + \alpha\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = 0} \qquad \text{in } \boldsymbol{(0,T)\times\Omega}$$

has received a great deal of attention in the latest years; it is referred to as Moore-Gibson-Thompson (MGT) equation

[Stokes, 1851]

Remark

Unlike the Westervelt and Kuznetsov equations, whose linearization is a parabolic-like equation, the MGT equation has a hyperbolic character.

The MGT equation

The dynamical behaviour of the MGT equation

$$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = 0$$

varies dramatically, according to the equation parameters' values

Indeed, it ranges

- from *non-existence* of solutions ($b = 0$) to *well-posedness*, as well as
- from *instability* ($\gamma := \alpha - \tau \frac{c^2}{b} < 0$) to *uniform stability* ($\gamma > 0$)

The JMGT and MGT equations: some references

- Results on well-posedness and uniform stability properties of the MGT equation have been established in [Kaltenbacher-Lasiecka-Marchand, 2012] and in [Marchand-McDevitt-Triggiani, 2011]

  In the aforesaid works, the PDE is supplemented with Dirichlet BC

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- Further references: [Conejero-Lizama-Rodenas, 2014] (chaotic behaviour)

  [Liu-Triggiani, 2014] (inverse problems)

  [Caixeta-Lasiecka-Domingos Cavalcanti, 2016] (attractors)

  [Dell'Oro-Pata, 2016] (intrinsic relation with viscoelastic equations)

  [Lasiecka-Wang, 2016], [Dell'Oro-Lasiecka-Pata, 2016] (linear model comprising a *memory term*)

Control problems relative to the quasilinear models

- Optimal boundary control of the Westervelt and the Kuznetsov equations has been discussed in [Clason-Kaltenbacher-Veljović, 2009]

  These authors establish existence of optimal controls and derive *first order* necessary optimality conditions

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- Boundary observability as well as boundary stabilizability of the *undamped* Westervelt equation, *via* linear feedbacks, has been shown in [Kaltenbacher, 2010]

Control of the ultrasound excitation: appropriate BC

$\Omega \subset \mathbb{R}^n$ bounded, Γ_0 *excited* boundary, $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ (*artificial* boundary)
The MGT equation is supplemented with the following Boundary Conditions:

$$\begin{cases} \frac{\partial}{\partial\nu} u = g & \text{on } \Gamma_0 \quad (\textit{boundary control}) \\ u_t + c\frac{\partial}{\partial\nu} u = 0 & \text{on } \Gamma_1 \quad (\textit{absorbing BC}) \end{cases}$$

Artificial boundaries are introduced in the study of wave propagation phenomena in unbounded spatial domains, in order to limit the area of observation/computation; the absorbing BC are used to avoid reflections (no waves can 'come back')

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Artificial boundaries are introduced in the study of wave propagation phenomena in unbounded spatial domains, in order to limit the area of observation/computation; the absorbing BC are used to avoid reflections (no waves can 'come back')

Thus, the boundary control problem under investigation is as follows (in the unknown $u = u(t,x)$, i.e. the acoustic pressure here):

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} + c^2\Delta u - b\Delta u_t = 0 & \text{on } (0,T)\times\Omega \\ \frac{\partial u}{\partial\nu} = g & \text{on } (0,T)\times\Gamma_0 \\ u_t + c\frac{\partial u}{\partial\nu} = 0 & \text{on } (0,T)\times\Gamma_1 \\ + IC \end{cases} \tag{IBVP}$$

Which cost functional?

We note that

- Finite time horizon problems – in the absence of penalization of the state at final time –, are the most pertinent ones
- With u representing the acoustic pressure, a quantity to be minimized (under the action of a *surface force* g) is

$$\|u - u^r\|^2_{L^2(0,T;L^2(\Omega))}\,, \quad \text{where } u^r \text{ is a reference pressure}$$

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We consider the following quadratic functional to be minimized

$$J(g) = \int_0^T \Big(\int_\Omega |u(t,x)|^2\,dx + \int_{\Gamma_0} |g(t,x)|^2\,d\sigma \Big)\,dt\,, \tag{C}$$

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Problem (Optimal control problem)

Given initial data $(u(0,\cdot), u_t(0,\cdot), u_{tt}(0,\cdot)) =: y_0$ in an appropriate space H, seek a control function $g \in L^2(0,T;L^2(\Gamma_0))$ which minimizes the cost functional (C), where $u(\cdot) = u(\cdot\,; y_0, g)$ is the solution to (IBVP) corresponding to $g(\cdot)$.

Abstract setup

To incorporate the absorbing BC and boundary control, we introduce the following *harmonic extensions* N_i, $i = 0, 1$:

$N_0 \colon \varphi \mapsto w$, where w is such that

$$\begin{cases} \Delta w - w = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial \nu} = \varphi & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases}$$

while $N_1 \colon \psi \mapsto w$, with

$$\begin{cases} \Delta w - w = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} = \psi & \text{on } \Gamma_1 . \end{cases}$$

Abstract setup (*cont'd*)

Let $\mathcal{A}$ be the *realization* of $-\Delta$ in $L^2(\Omega)$ with Neumann BC:

$$\mathcal{A} = -\Delta\,, \quad \mathcal{D}(\mathcal{A}) = \Big\{ f \in H^2(\Omega) : \ \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = 0 \Big\}\,.$$

The following trace results hold true, for any $v \in \mathcal{D}(A)$:

$$\begin{aligned} N_0^*(\mathcal{A}+I)v &= v|_{\Gamma_0} && \text{on } \Gamma_0; \\ N_1^*(\mathcal{A}+I)v &= v|_{\Gamma_1} && \text{on } \Gamma_1. \end{aligned}$$

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Several computations yield

$$\begin{aligned} &u_{ttt} + \alpha u_{tt} + c^2\mathcal{A}u + c(I+\mathcal{A})N_1N_1^*(I+\mathcal{A})u_t + b\mathcal{A}u_t + \\ &\qquad + \frac{b}{c}(I+\mathcal{A})N_1N_1^*(I+\mathcal{A})u_{tt} = c^2(I+\mathcal{A})N_0 g + b(I+\mathcal{A})N_0 g_t\,. \qquad (1) \end{aligned}$$

First order control system

The third order abstract equation is rewritten as the following first order control system

$$\boxed{\frac{d}{dt}\begin{pmatrix} u \\ u_t \\ u_{tt} \end{pmatrix} = A\begin{pmatrix} u \\ u_t \\ u_{tt} \end{pmatrix} + B_0 g + \underbrace{B_1 g_t}_{strange(r)}}$$

where

$$A = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\tau^2 c^2 \mathcal{A} & -\tau^{-1}\big[-b\mathcal{A} + \gamma c(\mathcal{A}+I)N_1 N_1^*(\mathcal{A}+I)\big] & -\tau^{-1}\big[\alpha I + \frac{b}{c}(\mathcal{A}+I)N_1 N_1^*(\mathcal{A}+I)\big] \end{pmatrix}$$

and

$$B_0 = -\begin{pmatrix} 0 \\ 0 \\ c^2(\mathcal{A}+I)N_0 \end{pmatrix}, \qquad B_1 = -\begin{pmatrix} 0 \\ 0 \\ b(\mathcal{A}+I)N_1 \end{pmatrix}.$$

The *uncontrolled* equation

With $g \equiv 0$ the equation (1) is

$$u_{ttt} + \alpha u_{tt} + c^2 A u + c(A+I)N_1 N_1^*(A+I)u_t + bAu_t + \\ + \frac{b}{c}(A+I)N_1 N_1^*(A+I)u_{tt} = 0\,.$$

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Following [Kaltenbacher *et al.*, 2012] and [Marchand *et al.*, 2011], we rewrite the equation as

$$(u_t + \alpha u)_{tt} + bA\Big(u_t + \frac{c^2}{b}u\Big) + \frac{b}{c}(A+I)N_1 N_1^*(A+I)\Big(u_{tt} + \frac{c^2}{b}u_t\Big) = 0\,,$$

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which suggests the introduction of the new variable

$$z := u_t + \frac{c^2}{b}u\,.$$

(to be continued)

The *uncontrolled* equation (*cont'd*)

We are allowed now to rewrite $u_t + \alpha u$ in terms of z and u:

$$u_t + \alpha u = z + \gamma u\,, \qquad \gamma = \alpha - \frac{c^2}{b}\,,$$

which gives

$$z_{tt} + bAz + \frac{b}{c}(A+I)N_1 N_1^*(A+I)z_t + \gamma u_{tt} = 0\,. \tag{2}$$

On the other hand, by the definition of variable z one has $u_t = z - c^2/b\,u$, which gives

$$u_{tt} = z_t - \frac{c^2}{b}u_t = z_t - \frac{c^2}{b}z + \Big(\frac{c^2}{b}\Big)^2 u\,,$$

which inserted in (2) yields the following equation:

$$z_{tt} + bAz + \frac{b}{c}(A+I)N_1 N_1^*(A+I)z_t + \gamma z_t - \gamma\frac{c^2}{b}z + \Big(\frac{c^2}{b}\Big)^2 u = 0\,.$$

The *uncontrolled* equation (*cont'd*)

Summarizing, we are led to the following coupled system of (second-order in time) equations in the unknowns (u, z):

$$\begin{cases} u_{tt} = z_t - \frac{c^2}{b} u_t \\ z_{tt} + bAz + \frac{b}{c}(A+I)N_1 N_1^*(A+I)z_t + \gamma z_t - \gamma \frac{c^2}{b} u_t = 0 \end{cases}$$

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or, equivalently, to the following coupled ODE-PDE system:

$$\boxed{\begin{cases} u_t = -\frac{c^2}{b} u + z \\ z_{tt} = -bAz - \frac{b}{c}(A+I)N_1 N_1^*(A+I)z_t - \gamma z_t + \gamma \frac{c^2}{b} z - \gamma \left(\frac{c^2}{b}\right)^2 u \,. \end{cases}} \qquad (3)$$

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We need to show well-posedness of the Cauchy problems associated with system (3) in appropriate function spaces

Semigroup well-posedness results

We established semigroup well-posedness in three different function spaces.

Theorem (Well-posedness, I)

The (first order in time) system (3) *in the unknown* (u, z, z_t) *is well-posed in the space*

$$Y = \underbrace{H^1(\Omega)}_{u} \times \underbrace{H^1(\Omega) \times L^2(\Omega)}_{(z, z_t)} .$$

Its dynamics is described by a closed operator $\tilde{A} : \mathcal{D}(\tilde{A}) \subset Y \to Y$ *which is the generator of a* C_0*-semigroup* $e^{\tilde{A}t}$ *on* Y, $t \geq 0$.

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Theorem (Well-posedness. State space)

The original (first order) uncontrolled *problem is well-posed in the function space*

$$H = \underbrace{H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)}_{(u, u_t, u_{tt})} .$$

Well-posedness results (*cont'd*)

Proof of Well-posedness, I. (*sketch*) The ODE-PDE system (3) is equivalent to the following first-order system

$$\begin{pmatrix} u \\ z \\ z_t \end{pmatrix}_t = \begin{pmatrix} -\frac{c^2}{b} I & I & 0 \\ 0 & 0 & I \\ -\gamma\left(\frac{c^2}{b}\right)^2 I & -b\mathcal{A} + \gamma\frac{c^2}{b} I & -\gamma I - \frac{b}{c_1}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I) \end{pmatrix} \begin{pmatrix} u \\ z \\ z_t \end{pmatrix};$$

the dynamics operator $\tilde{A}$ is readily decomposed as the sum of three operators, namely

$$\tilde{A} = \tilde{A}_1 + C_1 + K_1\,,$$

where

$$\tilde{A}_1 = \begin{pmatrix} -\frac{c^2}{b} I & 0 & 0 \\ 0 & 0 & I \\ 0 & -b(\mathcal{A}+I) & -\gamma I - \frac{b}{c_1}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I) \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ -\gamma\left(\frac{c^2}{b}\right)^2 I & 0 & 0 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (\gamma\frac{c^2}{b}+b)\, I & 0 \end{pmatrix},$$

where

(to be continued)

Proof of Well-posedness, I. (*cont'd*)

- $\tilde{A}_1$ is a (maximally) dissipative operator on

$$H^1(\Omega) \times \underbrace{\mathcal{D}((\mathcal{A}+I)^{1/2}) \times L^2(\Omega)}_{Y_z}$$

  and hence it is the generator of a C_0-semigroup of *contractions* $e^{\tilde{A}_1 t}$ on Y (which, however, is not *analytic*);

- C_1 is a bounded operator from Y into itself;

- K_1 is a compact operator: in fact, with $f \in \mathcal{D}((\mathcal{A}+I)^{1/2})$ one has

$$\gamma(\frac{c^2}{b}+b)\,f = \gamma(\frac{c^2}{b}+b)(\mathcal{A}+I)^{-1/2}[(\mathcal{A}+I)^{1/2}f]$$

$\Longrightarrow$ The generation on Y is guaranteed. □

Further well-posedness results for the ODE-PDE system

Theorem (Well-posedness, II)

System (3) *is well-posed also in the space*

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Theorem (Well-posedness, II)

System (3) *is well-posed also in the space*

$$Y_1 = \underbrace{H^2(\Omega)}_{u} \times \underbrace{H^1(\Omega) \times L^2(\Omega)}_{(z,z_t)} .$$

Thus, in view of the definition of the domain of the generator A,

$$\mathcal{D}(A) = \Big\{(u,z,z_t) \in H^1(\Omega) \times H^2(\Omega) \times H^1(\Omega) : \ \frac{\partial z}{\partial \nu}\Big|_{\Gamma_0} = 0\,, \ z_t + c_1 \frac{\partial z}{\partial \nu}\Big|_{\Gamma_1} = 0\Big\}\,,$$

we are able to infer the following result

Theorem (Well-posedness, III)

The ODE-PDE system (3) *is well-posed in*

$$Y_0 \sim \underbrace{H^1(\Omega)}_{u} \times \underbrace{L^2(\Omega) \times [H^1(\Omega)]'}_{(z,z_t)} .$$

Summary

Theorem (State space)

The uncontrolled *system* $y' = Ay$ *is well-posed in the function space*

$$H = \underbrace{H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)}_{y=(u,u_t,u_{tt})} .$$

The operator $A\colon \mathcal{D}(A) \subset H \to H$ *is the infinitesimal generator of a* C_0*-semigroup* e^{At} *in* H, $t \geq 0$. *Thus, for any* $y_0 \in H$, *the weak solution of the uncontrolled equation is given by* $y(t) = e^{At} y_0$, *and* $y(\cdot) \in C([0, T]; H)$.

State space: $H = H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$

Control space: $U = L^2(\Gamma_0)$

Control operators: it can be shown that

$$\mathbf{B_i} \in \mathcal{L}(\mathbf{U}, \mathcal{D}(\mathbf{A}^{*\gamma_0})) , \quad \gamma_0 < 1$$

Unusual yet recurrent pattern

The obtained abstract control system:

$$\begin{cases} y' = Ay + B_0 g + B_1 g_t \\ y(0) = y_0 \in H \end{cases} \tag{weird}$$

The cost functional:

$$J(g) = \int_0^T \left(|Ry(t)|_H^2 + |g(t)_U^2 \right) dt \tag{cost}$$

Remark

The abstract equation in (weird) *brings us back to a* pattern *and to* issues *encountered in the study of optimal boundary control for* strongly damped *wave/plate equations*

[B., 1992],
[Triggiani, 1994], [Lasiecka-Lukes-Pandolfi, 1995], [Lasiecka-Pandolfi-Triggiani, 1997]

Major issue

Remark

The boundary control system is not well posed, with $g \in L^2(0, T; U)$.

Former approach

Along the lines of the former work [B., 1992], one might

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Then, in the absence of penalization of $g_t \equiv v$, one would need to deal with the challenges of an optimal control problem with non-coercive functional

[B.-Pandolfi, 2000] (hyperbolic case, *finite* horizon)

Subsequent approach: auxiliary, parametrized problem

In the approach of [Triggiani, 1994], [Lasiecka-Lukes-Pandolfi, 1995], etc., integration by parts gives the solution formula

$$y(t) = e^{At}[y_0 - B_1 g_0] + \int_0^t e^{A(t-s)} B_0 g(s)\, ds + \\ + A \int_0^t e^{A(t-s)} B_1 g(s)\, ds + B_1 g(t)$$

where

- $g_0 = g(0)$ is initially an *arbitrary* element of U
- the operators B_i, $i = 1, 2$, (*intrinsically*) take values in $[\mathcal{D}(A^*)]'$; then, AB_1 maps into $[\mathcal{D}({A^*}^2)]'$
- smoothing properties of the *observation* operator R are required

Subsequent approach (*cont'd*)

Focus is placed on the following auxiliary equation – depending on a parameter α:

$$y_\alpha(t) = e^{At}\alpha + (Lg)(t)\,, \tag{4}$$

where

$$(Lg)(t) := \int_0^t e^{A(t-s)} B_0 g(s)\,ds + A\int_0^t e^{A(t-s)} B_1 g(s)\,ds + B_1 g(t)\,.$$

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Accordingly, in place of the original optimal control problem, we will consider the following one:

Problem (Auxiliary optimal control problem)

Given $\alpha \in [\mathcal{D}(A^{*2}]'$, *seek a control function* $g \in L^2(0,T;U)$ *which minimizes the cost functional* (cost), *where* $y_\alpha(\cdot)$ *is the solution to* (4) *corresponding to the control* $g(\cdot)$.

The present case: negative/positive features, starting point

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Proposition

The following functional-analytic properties hold true.

i) RA^2 *can be extended to a* bounded *operator on the state space* H*;*

ii) $RB_1 = 0$*;*

iii) $A^{-1}B_i$ *are* compact *operators from* $L^2(\Gamma_0)$ *into* H*,* $i = 0, 1$*;*

iv) RL *is a* compact *operator from* $L^2(0, T; L^2(\Gamma_0))$ *into* $C([0, T]; H)$*.*

One needs to introduce, for any $s \in [0, T)$, the dynamics described by the equation

$$y(t, s; \alpha) = e^{A(t-s)}\alpha + L_s g(t) , \qquad s \le t \le T , \tag{5}$$

as well as the cost functional

$$J_{s,T}(g;\alpha) \equiv \int_s^T \left(|Ry(t)|_H^2 + |g(t)|_U^2\right) dt \tag{6}$$

where $L_{s,T}$ – L_s, in short – is the operator defined by

$$(L_s g)(t) = \underbrace{\int_s^t e^{A(t-\tau)} B_0 g(\tau)\, d\tau + A \int_s^t e^{A(t-\tau)} B_1 g(\tau)\, d\tau}_{(L_s^0 g)(t)} + B_1 g(t) , \quad t \in [s, T] .$$

Theorem (Main results)

With reference to the optimal control problem (5)-(6), *the following statements are valid:*

S1. **(Optimal pair)** *Given* $\alpha \in [\mathcal{D}(A^{*2})]'$, *there exists a* unique optimal pair

$$(\hat{y}(t,s;\alpha), \hat{g}(t,s;\alpha))$$

such that

$$\hat{g}(t,s;\alpha) = [I + L_s^* R^* R L_s]^{-1} L_s^* R^* R e^{A(\cdot - s)} \alpha \in C([s,T];U)\,,$$

$$\hat{y}(t,s;\alpha) = e^{A(t-s)}\alpha + \{L_s \hat{g}(\cdot,s;\alpha)\}(t) \in C([s,T];[\mathcal{D}(A^{*2})]')\,,$$

$$R\hat{y}(t,s;\alpha) = [I + R L_s L_s^* R^*]^{-1} R e^{A(\cdot - s)} \alpha \in C([s,T];H)\,.$$

Theorem *(continued)*

S2. **(Riccati operator)** *Let* $\Phi(t,s)\colon [\mathcal{D}(A^{*2})]' \to [\mathcal{D}(A^{*2})]'$ *the operator defined – for any couple* (t,s) *such that* $0 \le s \le t \le T$ *– as follows:*

$$\Phi(t,s)\alpha := \hat{y}(t,s;\alpha) - B_1\hat{g}(t,s;\alpha)\,, \qquad \alpha \in [\mathcal{D}(A^{*2})]'\,. \tag{7}$$

*The (*a posteriori*, optimal cost, or Riccati) operator defined by*

$$P(t)\alpha = \int_t^T e^{A^*(\tau-t)}R^*R\Phi(t,s)\alpha\, d\tau \equiv \int_t^T e^{A^*(\tau-t)}R^*R\hat{y}(\tau,t;\alpha)\, d\tau\,, \tag{8}$$

is positive and selfadjoint; $P(t)\in\mathcal{L}(H)$ *for any* $t\in[s,T]$ *(its regularity properties are detailed separately below).*

S3. **(Implicit feedback formula)** *The optimal control satisfies*

$$\hat{g}(t,s;\alpha) = -[B_0^* + B_1^*A^*]P(t)\Phi(t,s)\alpha$$

that is the following implicit equation

$$\hat{g}(t,s;\alpha) = -[B_0^* + B_1^*A^*]P(t)\hat{y}(t,s;\alpha) + [B_0^* + B_1^*A^*]P(t)B_1\hat{g}(t,s;\alpha)\,,$$

with $\Phi(t,s)$ *defined in* (7).

Theorem *(continued)*

S4. **(Optimal cost)** *The optimal cost is given by*

$$J_s(\hat{g};\alpha) = \int_s^T \left(|R\hat{y}(t)|_H^2 + |\hat{g}(t)|_U^2\right) dt = \|[I + RL_sL_s^*R^*]^{-1/2} \, Re^{A(\cdot-s)}\alpha\|_{L^2(s,T;H)}^2$$

which is rewritten in terms of the Riccati operator as follows:

$$J_s(\hat{g}) = (P(s)\alpha,\alpha) = \left([I + RL_sL_s^*R^*]^{-1} \, Re^{A(\cdot-s)}\alpha, Re^{A(\cdot-s)}\alpha\right)_{L^2(s,T;H)},$$

thereby providing

$$P(s)\alpha = e^{A^*(\cdot-s)}R^* \, [I + RL_sL_s^*R^*]^{-1} \, Re^{A(\cdot-s)}\alpha \quad \forall\alpha \in [\mathcal{D}(A^{*2})]' .$$

Key property

Proposition (Bounded 'gain' operator)

The optimal cost operator $P(t)$ *defined by* (8) *satisfies the following regularity properties:*

For any given $t \in [0, T]$, *one has*

$$A^{*2} P(t) A^2 \in \mathcal{L}(H)\,;$$

equivalently,

$$P(t) \in \mathcal{L}([\mathcal{D}(A^{*\gamma_1})]', \mathcal{D}(A^{*\gamma_2})) \qquad \forall \gamma_1, \gamma_2 \le 2\,.$$

As a consequence, $B_i^* P(\cdot) A^2 \in \mathcal{L}(H, U)$, $i = 1, 2$, *and*

$$[B_0^* + B_1^* A^*] P(t) \in \mathcal{L}([\mathcal{D}(A^{*2})]', U))\,.$$

Last (but not least) issues

- final form of the 'Riccati' equation

$$
\begin{aligned}
&(\frac{d}{dt}P(t)x,y) + (A^*P(t)x,y) + (P(t)Ax,y) + (R^*Rx,y) \\
&\quad - (B_0^*P(t)x, B_0^*P(t)y) - (B_1^*A^*P(t)x, B_0^*P(t)y) - (B_0^*P(t)x, B_1^*A^*P(t)y) \\
&\qquad - (B_1^*A^*P(t)x, B_1^*A^*P(t)y) = 0\,, \qquad x,y,\in [\mathcal{D}(A^*)]'
\end{aligned}
$$

- invertibility of the operator

$$
I - [{B_0}^* + {B_1}^*A^*]P(t)B_1\,,
$$

which will bring about the *explicit* *feedback* representation of the optimal control

- *matching condition* for the optimal control at $t = 0$:

there exists $g_0 \in U$ such that $\hat{g}_{y_0 - B_1 g_0}(0) \equiv g_0$

This work is *ongoing*

Thank you for your attention!