# Invisible Control of Self-Organizing Agents Leaving Unknown Environments

(joint work with G. Albi, E. Cristiani, and D. Kalise)

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# Pedestrian dynamics: multiscale models and control

Rising interest towards modeling and control of large groups of individuals:

- Models for pedestrian dynamics: Hughes '02, Maury-Chupin-Santambrogio '11, Di Francesco-Markowich-Pietschmann-Wolfram '11;
- Control of multiagent systems via external agents: B.-Buttazzo '16 (theoretical), Albi-Pareschi-Zanella '14, Pinnau-Totzeck-Tse-Burger '16 (numerical), Butail-Bartolini-Porfiri '13 (practical);
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List is far from exhaustive, but shows two central issues:

- dimensionality reduction of micro models via mesoscopic ones when number of agents explodes;
- influencing the behavior of the crowd via controlled external agents.

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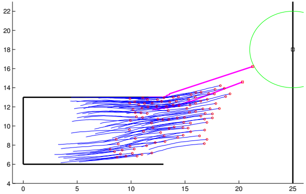
# The evacuation problem




*The crowd find its way to the exit thanks to the two fuchsia leaders.*

- Our goal is to evacuate a crowd of individuals from an environment they don't know under limited visibility.
- We show that invisible sparse strategies (i.e., by means of few, unrecognized, external agents) influence the crowd effectively and improve evacuation.
- Through Boltzmann equation we propose a mesoscopic description of this dynamics when the number of pedestrians is large.
- Develop a set of numerical techniques for computing optimal exit strategies in both cases.

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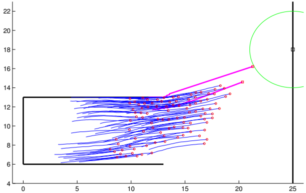


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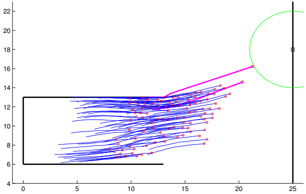


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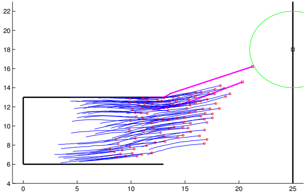


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# Model guidelines: followers (i.e., evacuees)

The non-controlled agents of the crowd are called followers, and are subject to a second-order dynamics with

- an isotropic metric short-range repulsion force;
- a relaxation term toward a given characteristic speed;
- if the exit is not visible
  - an isotropic topological alignment force, i.e., given $\mathcal{N} \in \mathbb{N}$, the $i$-th agent aligns with those inside $\mathcal{B}_{\mathcal{N}}(x_i, t)$, the smallest ball at time $t$ containing at least $\mathcal{N}$ agents.
  - a random walk, in order to explore the unknown environment;
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# Model guidelines: leaders (i.e., stewards)

The controlled agents of the crowd are called leaders. They are less than followers ($N^{\mathrm{L}} \ll N^{\mathrm{F}}$) and evolve according to a first-order dynamics with

- an isotropic metric short-range repulsion force;
- an optimal force which is the result of an offline optimization procedure, minimizing some cost functional.

First vs. second-order model: for followers a second-order model is necessary since they must perceive velocities to align. The bigger inertia is compensated by stronger forces w.r.t. the ones in leaders' dynamics.

Metric vs. topological interaction: alignment is topological since empirical evidence suggests that only close neighbors play a role, (see Ballerini et al. '08).

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# Microscopic model

For $i = 1, \dots, N^{\mathrm{F}}$ and $k = 1, \dots, N^{\mathrm{L}}$

$$\begin{cases} \dot{x}_i &= v_i, \\ \dot{v}_i &= A(x_i, v_i) + \sum_{j=1}^{N^{\mathrm{F}}} H(x_i, v_i, x_j, v_j) + \sum_{\ell=1}^{N^{\mathrm{L}}} H(x_i, v_i, y_\ell, w_\ell) \\ \dot{y}_k &= w_k = \sum_{j=1}^{N^{\mathrm{F}}} R_{\zeta,r}(y_k, x_j) + \sum_{\ell=1}^{N^{\mathrm{L}}} R_{\zeta,r}(y_k, y_\ell) + {\color{red} u_k}, \end{cases}$$

- Let $\theta(x)$ represents the characteristic function of the target's visibility zone and

$$A(x, v) := (1 - \theta(x))C^{\mathrm{Z}}(z - v) + \theta(x)C^{\mathrm{D}}\left(\frac{x^{\mathrm{D}} - x}{|x^{\mathrm{D}} - x|} - v\right) + C^{\mathrm{V}}(\alpha^2 - |v|^2)v,$$

  where $z \sim \mathcal{N}(0, \sigma^2)$, $\alpha$ is the characteristic speed.

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$$H(x, v, y, w) := -C^{\mathrm{R}}_{\mathrm{F}} R_{\gamma,r}(x, y) + (1 - \theta(x))\frac{C^{\mathrm{A}}}{\mathcal{N}^*}(w - v)\chi_{B_{\mathcal{N}}(x,t)}(y)$$

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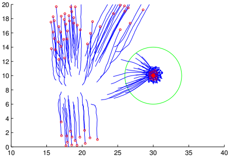


*Followers dynamics without leaders*

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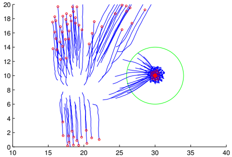


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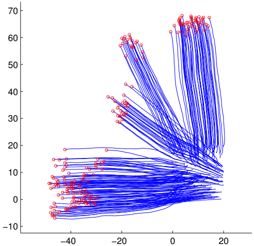
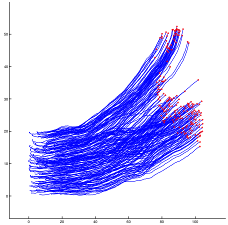



*If not influenced, the random term + topological alignment splits the followers*

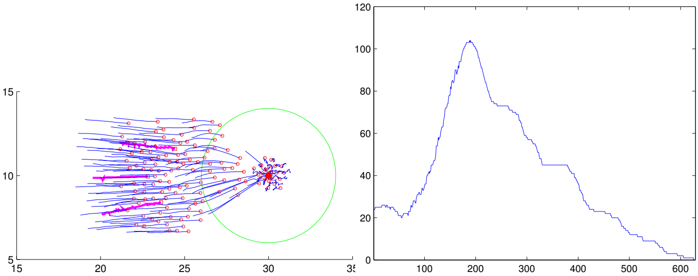



*Influence of external agents with preferred direction: when removed, the group splits*

# Control strategy: “dumb”

The control $u_k : [0,T] \to \mathbb{R}^d$, $\quad k = 1, \ldots, N^{\mathrm{L}}$ as “dumb” strategy:

$$u_k(t) = \left( \frac{x^{\mathrm{D}} - y_k(t)}{|x^{\mathrm{D}} - y_k(t)|} - y_k(t) \right)$$




*(Left) Dynamics with “dumb” strategy and (Right) occupancy of the exit’s visibility zone*

# Smart strategy: Modified Compass Search

The control $u_k : [0,T] \to \mathbb{R}^d, \quad k = 1, \dots, N^{\mathrm{L}}$, $u_k$ minimizes the functional

$$J(u) = \int_0^T \left( P(t) + \nu \sum_{k=1}^{N^{\mathrm{L}}} \|u_k(t)\|^2 \right) dt,$$

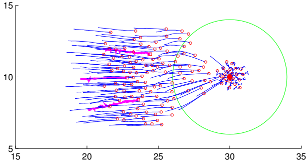
Where $P(t)$ is the number of followers outside exit at time $t$.

We proceed via Modified Compass Search[1]:

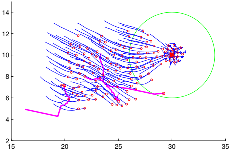
- *leaders' trajectories are piecewise constant;*
- *starting from an initial guess, at each iteration we modify the current best strategy with small random variations;*
- *we keep the variation if the evaluated cost is smaller than before;*
- *the method generates a sequence converging to a local minimum.*

[1] Audet-Dang-Orban '14

# Clog up effect around exit




*Dynamics with “dumb” strategy*

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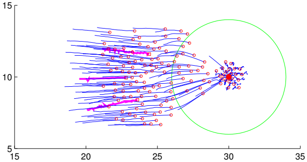


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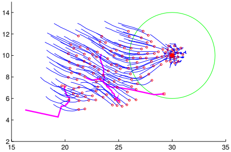


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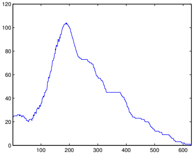
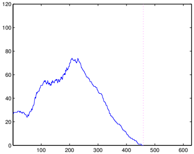



*Occupancy of the exit's visibility zone with "dumb" strategy*




*Occupancy of the exit's visibility zone with "smart" strategy*

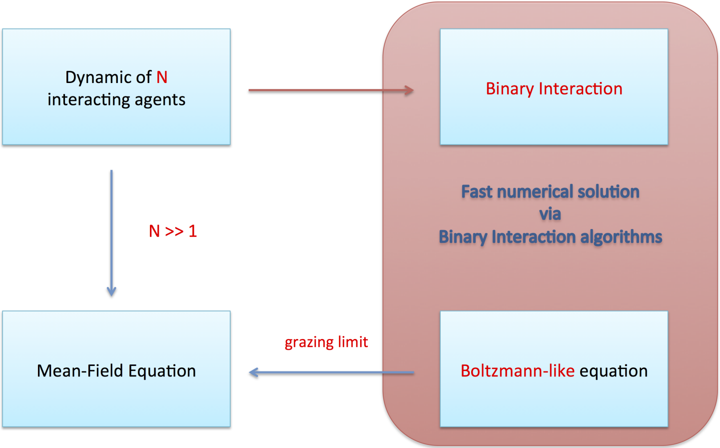
Good strategies avoid exit's clog up, hence congestion drop.

# Mean-field approximation

- The number of interacting agents can be rather large, thus we need to solve a very large system of ODEs, which can constitute a serious difficulty.
- A natural way to tackle this problem is to approximate the problem with a kinetic equation.
- The problem of rigorously derive a mean-field equation from a system of interacting particles in the case of swarming and flocking models is well studied[2] also for an optimal control problem[3] .
- We consider here binary interaction *Boltzmann-type* and *mean-field* approximation of the microscopic dynamic[4].
- This approach is related with the development of Boltzmann collision algorithms in plasma physics[5]

[2] Canizo-Carrillo-Rosado '10, Carrillo-Y. P. Choi-Hauray-Salem '15
[3] Fornasier-Solombrino '13, Fornasier-Piccoli-Rossi '14, B.-Fornasier-Rossi-Solombrino '15
[4] Carrillo-Fornasier-Toscani-Vecil '10, Pareschi-Toscani '13
[5] Bird '94, Bobylev-Nanbu '00, Caflisch-Pareschi-Dimarco '10

# Boltzmann approximation



# Mean-field controlled dynamic

The "mean-field limit" of the microscopic model describes the evolution of the continuous density of followers $f = f(t,x,v)$ coupled with the evolution of the microscopic leaders

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = -\nabla_v \cdot ((S + \mathcal{H}[f] + \mathcal{H}[g])f) + \frac{1}{2}\sigma^2(\theta C^{\mathrm{Z}})^2 \Delta_v f, \\ \dot{y}_k = \int_{\mathbb{R}^{2d}} R_{\zeta,r}(y_k, x) f(x,v)\; dx\; dv + \sum_{\ell=1}^{N^{\mathrm{L}}} R_{\zeta,r}(y_k, y_\ell) + u_k, \\ \qquad k = 1, \dots, N^{\mathrm{L}} \end{cases} \tag{MF}$$

- Where the non-linear interactions terms are

$$\mathcal{H}[f](x,v) = \int_{\mathbb{R}^{2d}} H(x,\hat{x},v,\hat{v}) f(\hat{x},\hat{v})\; d\hat{x}\; d\hat{v},$$

$$S(x,v) = -\theta(x) C^{\mathrm{Z}} v + (1-\theta(x)) C^{\mathrm{D}} \left( \frac{x^{\mathrm{D}} - x}{|x^{\mathrm{D}} - x|} - v \right) + C^{\mathrm{V}}(\alpha^2 - |v|^2) v.$$

- Density $g = g(t,x,v) = \sum_{k=1}^{N^{\mathrm{L}}} \delta_{y_k(t), w_k(t)}(x,v)$, represents the empirical measure of leaders and $u_k$ is given by the solution of an optimal control problem.

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# Kinetic approximation

We assume that $f, g$ satisfy the following Boltzmann-Povzner dynamics[6] + the ODEs of leaders

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \lambda^{\mathrm{F}} Q^F(f,f) + \lambda^{\mathrm{L}} Q^L(f,g) \\ \dot{y}_k = \displaystyle\int_{\mathbb{R}^{2d}} R_{\zeta,r}(y_k, x) f(x,v)\, dx\, dv + \sum_{\ell=1}^{N^{\mathrm{L}}} R_{\zeta,r}(y_k, y_\ell) + u_k, \\ \qquad k = 1, \dots, N^{\mathrm{L}} \end{cases} \tag{BP}$$

where $\lambda^{\mathrm{F}}$ and $\lambda^{\mathrm{L}}$ are the interaction frequencies and

$$Q^F(f,f) = \mathbb{E}\left[\int_{\mathbb{R}^{2d}} \left(\frac{1}{J_{\mathrm{FF}}} f(x, v_*) f(\hat{x}, \hat{v}_*) - f(x,v) f(\hat{x}, \hat{v})\right) d\hat{x}\, d\hat{v}\right],$$

$$Q^L(f,g) = \mathbb{E}\left[\int_{\mathbb{R}^{2d}} \left(\frac{1}{J_{\mathrm{FL}}} f(x, v_{**}) g(\tilde{x}, \tilde{v}_{**}) - f(x,v) g(\tilde{x}, \tilde{v})\right) d\tilde{x}\, d\tilde{v}\right].$$

with $v_*, v_{**}$ indicates the pre-interaction velocities and $\mathbb{E}[\cdot]$ is the expected value w.r.t. $\xi$.

[6] Povzner '62

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# Binary interactions[7]

- When a follower $(x, v)$ interacts with another follower $(\hat{x}, \hat{v})$, they update their state variables according to

$$
(\mathrm{FF}) \begin{cases} v^* &= v + \eta^{\mathrm{F}} [\underbrace{\theta(x) C^{\mathrm{Z}} \xi + S(x, v)}_{A(x,v)} + N^{\mathrm{F}} H(x, v, \hat{x}, \hat{v})] \\ \hat{v}^* &= \hat{v} + \eta^{\mathrm{F}} \left[\theta(\hat{x}) C^{\mathrm{Z}} \xi + S(\hat{x}, \hat{v}) + N^{\mathrm{F}} H(\hat{x}, \hat{v}, x, v)\right] \end{cases}
$$

where $\eta^{\mathrm{F}}$ is the strength of the interaction follower-follower, and $\xi \sim \mathcal{N}(0, \varsigma^2)$.

- When a follower $(x, v)$ interacts with a leader $(\tilde{x}, \tilde{v})$,

$$
(\mathrm{FL}) \begin{cases} v^{**} &= v + \eta^{\mathrm{L}} N^{\mathrm{L}} H(x, v, \tilde{x}, \tilde{v}) \\ \tilde{v}^{**} &= \tilde{v} \end{cases}
$$

where $\eta^{\mathrm{L}}$ is the strength of the interaction follower-leader.

[7] Cercignani-Illner-Pulvirenti '94

# Theorem: Grazing collision limit

*Suppose that for some* $\delta \in (0,1)$

- $\mathbb{E}\left(\|\xi\|^{2+\delta}\right)$ *is finite,*
- *the functions* $S$ *and* $H$ *are in* $L^p_{loc}$ *for* $p = 2, 2+\delta$.

*Fix the control* $u$. *Let consider the following scaling*

$$\eta^{\mathrm{F}} = \eta^{\mathrm{L}} = \varepsilon, \quad \lambda^{\mathrm{F}} = \lambda^{\mathrm{L}} = \frac{1}{\varepsilon}, \quad \varsigma^2 = \frac{\sigma^2}{\varepsilon}$$

*and define* $(f^\varepsilon, y^\varepsilon)$ *be a solution of* (BP). *Then, as* $\varepsilon \to 0$, $(f^\varepsilon, y^\varepsilon)$ *converges pointwise to a solution of* (MF), *the "mean-field limit" of the microscopic model.*

# Idea of the proof[8]

- Let $\mathcal{T}$ be a function space with suitably smooth test functions. Then for $\varphi \in \mathcal{T}_\delta$, the weak formulation of (BP) reads

$$\lambda \langle Q(f,f), \varphi \rangle = \lambda \mathbb{E} \left( \int_{\mathbb{R}^{4d}} (\varphi(x, v^*) - \varphi(x,v)) f(x,v) f(\hat{x}, \hat{v}) \; dx dv d\hat{x} d\hat{v} \right).$$

- We derive the Taylor expansion around $v^* - v$ up to the second order of the weak interaction operator, obtaining

$$\lambda \langle Q(f,f), \varphi \rangle = \lambda \mathbb{E} \Bigg( \int_{\mathbb{R}^{4d}} \nabla_v \varphi(x,v) \cdot (v^* - v) f(x,v) f(\hat{x},\hat{v}) \; dx dv d\hat{x} d\hat{v} +$$

$$+ \frac{\lambda}{2} \int_{\mathbb{R}^{4d}} \left[ \sum_{i,j=1}^{d} \partial_v^{(i,j)} \varphi(x,v) \left(v^* - v\right)_i \left(v^* - v\right)_j \right] f(x,v) f(\hat{x},\hat{v}) \; dx \; dv \; d\hat{x} \; d\hat{v} \Bigg) + \lambda R_\varphi$$

- Thanks to the boundedness of $S$ and $H$ we have that the remainder $R_\varphi$ is bounded.
- Therefore using the grazing collision scaling, and taking the limit for $\varepsilon \to 0$ for any $\varphi \in \mathcal{T}_\delta$ we obtain the mean-field equation (MF).

[8] Toscani '06

# Binary Interaction algorithm

The main idea is to simulate the *binary interaction dynamic* for small values of $\varepsilon$ in order to approximate the Fokker-Planck equation model[9].

- We use a *splitting method*, for transport and collisional part of the scaled Boltzmann equation (BP).

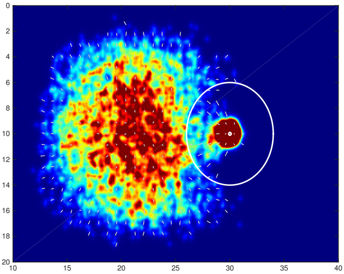
$$\partial_t f = -v \cdot \nabla_x f \tag{T}$$

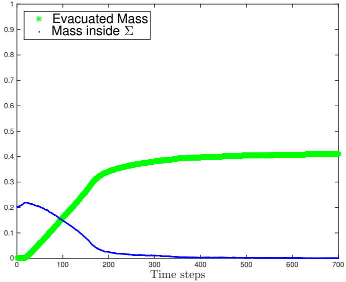
$$\partial_t f = \frac{1}{\varepsilon} Q_\varepsilon^F(f,f) + \frac{1}{\varepsilon} Q_\varepsilon^L(f,g) \tag{C}$$

- In order to simulate the evolution of (C) we use a Monte-Carlo method.
- The algorithms cost is linear, $O(N_s)$, w.r.t. to the number of sample particles $N_s$ used to reconstruct the density $f$.
- The resulting algorithm is fully *meshless* since the binary interactions are non local.

[9] Albi-Pareschi '13

# Uncontrolled case

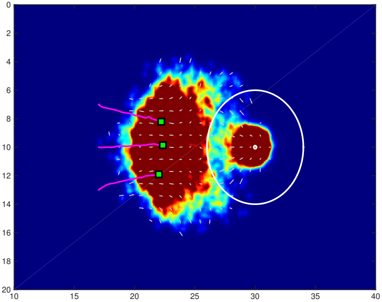


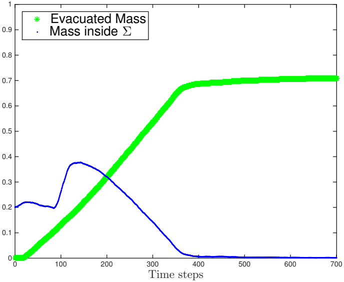




Uncontrolled evolution of followers' density.

# Dumb strategy

Control $u : [0,T] \to \mathbb{R}^d$ as a "dumb" strategy: $u_k(t) = \left( \frac{x^{\mathrm{D}} - y_k(t)}{|x^{\mathrm{D}} - y_k(t)|} - y_k(t) \right)$;



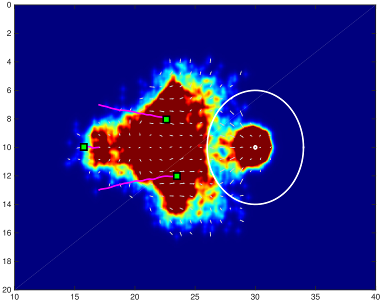

Evolution of followers' density under microscopic leaders' fixed strategy.

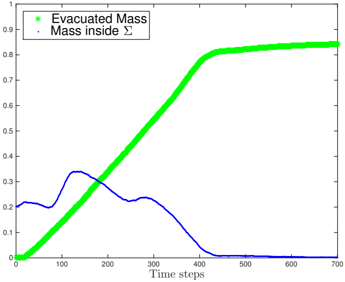
# Smart strategy

Control $u : [0,T] \to \mathbb{R}^d$ as a "smart" strategy such that

$$\min_u J(u) = \int_0^T \left( P(t) + \nu \sum_{k=1}^{N^{\mathrm{L}}} |u_k(t)|^2 \right) dt,$$

where $P(t)$ represents the mass of followers outside exit at time $t$.




Evolution of followers' density under microscopic leaders' smart strategy.

# A few info

- **WWW:** http://www-m15.ma.tum.de/Allgemeines/MattiaBongini

- **References**: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