New phenomena for the null controllability of parabolic systems

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Controllability of systems: The finite dimensional case

$$(1) \qquad \begin{cases} \partial_t y = Ly + Bv \\ y(0) = y^0 \end{cases}$$

$L \in \mathcal{M}_n(\mathbb{R}), B \in \mathcal{M}_{n,m}(\mathbb{R}), m \leq n.$

Definition

System (1) is controllable at time $T > 0$ if

$$\forall y^0, y^1 \in \mathbb{R}^n, \exists\, v \in L^2(0,T)^m \text{ such that } y(T; y^0, v) = y^1$$

Controllability of systems: The finite dimensional case

$$(2) \qquad \begin{cases} \partial_t y = Ly + Bv \\ y(0) = y^0 \end{cases}$$

$L \in \mathcal{M}_n(\mathbb{R}), B \in \mathcal{M}_{n,m}(\mathbb{R})$.

Proposition (The Fattorini-Hautus test)

System (2) (or (L, B)) is controllable if and only if

$$\ker(s - L^*) \cap \ker(B^*) = \{0\}, \quad \forall s \in \mathbb{C}.$$

Proposition

System (2) is controllable at time $T > 0$ if and only if it is controllable at any time.

Controllability of systems: The infinite dimensional case

$$(3) \qquad \begin{cases} \partial_t y = Ly + Bv \\ y(0) = y^0 \end{cases}$$

- $(L, D(L))$ generator of a semigroup of class $\mathcal{C}_0$ on H, Hilbert space ,
- $B : V \to H$ is a bounded operator, V is also a Hilbert space

Definition

1. *System (3) is exactly controllable at time $T > 0$ if*

$$\forall y^0, y^1 \in H, \ \exists v \in L^2((0,T); V) \text{ such that } y(T; y^0, v) = y^1$$

2. *System (3) is approximately controllable at time $T > 0$ if*

$$\forall y^0, y^1 \in H, \ \forall \varepsilon > 0, \ \exists v \in L^2((0,T); V) \text{ such that } \|y(T; y^0, v) - y^1\| < \varepsilon$$

3. *System (3) is null controllable at time $T > 0$ if*

$$\forall y^0 \in H, \exists v \in L^2((0,T); V) \text{ such that } y(T; y^0, v) = 0$$

Wave equation : Exact controllability

$\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, $\gamma \subseteq \partial\Omega$

$$(4) \qquad \boxed{\begin{cases} \partial_{tt}^2 y = \Delta y, & \text{in } \Omega \times (0,T) \\ y = v 1_\gamma, & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y^{01},\ \partial_t y(\cdot,0)) = y^{02}, & \text{in } \Omega \end{cases}}$$

Theorem

Let $x_0 \in \mathbb{R}^N$ and $m(x) = x - x_0$. Let $\gamma = \{x \in \partial\Omega \,;\, m(x).\nu(x) > 0\}$, $R_0 = \max_{x\in\overline{\Omega}} |m(x)|$, $T_0 = 2R_0$. For all $T > T_0$, for all $(y^{01}, y^{02}) \in L^2(\Omega) \times H^{-1}(\Omega)$, $(y^{11}, y^{12}) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists $v \in L^2(\partial\Omega \times (0,T))$ such that y solution of (4) satisfies:

$$y(\cdot,T) = y^{11}(\cdot), \quad \partial_t y(\cdot,T) = y^{12}(\cdot), \qquad on\ \Omega.$$

Hyperbolic behavior in control of Hyperbolic equations

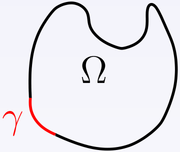
Control of hyperbolic equations requires

1. Minimal time of control.
2. Geometrical condition on the location of the control.

Heat equation: Approximate and null controllability

$\Omega \subset \mathbb{R}^N$ a bounded domain, $N \geq 1$, $\gamma \subseteq \partial\Omega$ a relative open subset, $T > 0$

(5)
$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (0,T), \\ y = v 1_\gamma \text{ on } \partial\Omega \times (0,T), \quad y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases}$$



Theorem (Boundary controllability : results)

1. *For any $y^0, y^1 \in H^{-1}(\Omega)$, any $\varepsilon > 0$, there exists $v \in L^2(\partial\Omega \times (0,T))$ s.t. the solution y to (5) satisfies*
$$\|y(\cdot,T) - y^1\|_{H^{-1}(\Omega)} \leq \varepsilon.$$
2. *For any $y^0 \in H^{-1}(\Omega)$ there exists $v \in L^2(\partial\Omega \times (0,T))$ s.t. the solution y to (5) satisfies*
$$y(\cdot,T) = 0 \text{ in } \Omega.$$

Heat equation: Approximate and null controllability

1. No exact controllability (due to the regularizing effect).
2. Approximate controllability $\Longleftrightarrow$ null controllability.
3. No minimal time, no geometric condition on the location of the control.

Heat equation: Approximate and null controllability

Four IMPORTANT REFERENCES

1. H.O. Fattorini, D.L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.
2. S. Dolecki , *Observability for the one-dimensional heat equation*, Studia Mathematica. 48 (1973), 292-305.
3. G. Lebeau, L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E. 20 (1995), no. 1-2, 335–356.
4. A. Fursikov, O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

Hyperbolic phenomena in control:

1. Minimal time > 0.
2. Condition on the location of the domain of control.
3. Approximate controllability $\not\Leftrightarrow$ Null controllability.

Issue:

Do " Hyperbolic phenomena" occur in control of parabolic equations?

Contents

Pointwise control of the heat equation

Let $x_0 \in (0, \pi)$

$$\begin{cases} \partial_t y - \partial^2_{xx} y = \delta\,(x - x_0)\, v(t) & \text{in } (0,\pi) \times (0,T) \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T) \\ y(\cdot,0) = y^0 & \text{in } (0,\pi) \end{cases}$$

S. DOLECKI , *Observability for the one-dimensional heat equation*, Studia Math. **48** (1973), 291–305.

Pointwise control of the heat equation

Theorem (Pointwise control for parabolic systems-S. Dolecki, 73')

Let

$$T(x_0) := \limsup \frac{-\log|\sin(kx_0)|}{k^2}$$

Then:

1. *System* (6) *is null controllable for* $T > T(x_0)$
2. *System* (6) *is not null controllable for* $T < T(x_0)$

$$(6) \qquad \begin{cases} \partial_t y - \partial^2_{xx} y = \delta(x - x_0)\, v(t) & \text{in } (0,\pi)\times(0,T) \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T) \\ y(\cdot,0) = y^0 & \text{in } (0,\pi) \end{cases}$$

Boundary control for parabolic systems

F. Ammar Khodja, A. Benabdallah, M. González-Burgos, L. de Teresa, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal. **267** (2014), no. 7, 2077–2151.

$$(7) \qquad \begin{cases} \partial_t y - \left(D\partial_{xx}^2 + A\right) y = 0 & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi), \end{cases}$$

where $T > 0$ is a given time,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \text{ (with } d > 0), \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{M}_{2,1}(\mathbb{R})$$

Results

Theorem (F. Ammar Khodja, A. Benabdallah, M. González-Burgos, L. de Teresa, 2014)

Let $d \neq 1$

1. $\forall T > 0$: *Approximate controllability if and only if* $\sqrt{d} \notin \mathbb{Q}$
2. $\exists T_0 = c(\Lambda) \in [0, +\infty]$ *such that*
   1. *The system is null controllable at time* T *if* $\sqrt{d} \notin \mathbb{Q}$ *and* $T > T_0$
   2. *Even if* $\sqrt{d} \notin \mathbb{Q}$, *if* $T < T_0$, *the system is not null controllable at time* T

$c(\Lambda)$ is the index of condensations of the sequence $\Lambda = (k^2, dk^2)_{k \geq 1}$.

Index of condensation : some background

- The *index of condensation* of a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}$ is a real number $c(\Lambda) \in [0, +\infty]$ associated with this sequence and which "measures" the condensation at infinity.
- This notion has been :
  - introduced by V.l. Bernstein in 1933: Leçons sur les progrès récents de la théorie des séries de Dirichlet for real sequences,
  - extended by J. R. Shackell in 1967 for complex sequences.

Index of condensation : some background

Let $\Lambda = (\lambda_k) \subset \mathbb{C}$ be a sequence with pairwise distinct elements and:

$$\exists \delta > 0 : \Re(\lambda_k) \geq \delta |\lambda_k| > 0, \ \forall k \geq 1,$$

such that the sequence Λ is measurable: there exists $D \in [0, \infty[$ (the density of Λ) such that

$$\lim_{n \longrightarrow \infty} \frac{n}{\lambda_n} = D,$$

Index of condensation: some background

Definition

The index of condensation of Λ is:

$$c(\Lambda) = \limsup_{k\to\infty} \frac{-\ln\left|E'(\lambda_k)\right|}{\Re\lambda_k} \in [0, +\infty].$$

$$E'(\lambda_k) = -\frac{2}{\lambda_k} \prod_{j\neq k}^{\infty} \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right)$$

Is it possible to have a minimal time of control > 0?

$T_0 > 0$?

For $\Lambda = (k^2, dk^2)_{k\geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda) > 0$?

Theorem (F. Ammar Khodja, A. Benabdallah, M. González-Burgos, L. de Teresa, 2014)

Let $\Lambda = \left(k^2,\ d\, k^2\right)$. *For any* $\delta \in [0, \infty]$, *there exists* $\sqrt{d} \in \mathbb{R} \backslash \mathbb{Q}$ *such that* $c(\Lambda) = \delta$

Distributed controllability problem

$$\begin{cases} y_t - y_{xx} + q(x)Ay = Bv1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

F. Ammar Khodja, A. Benabdallah, M. González-Burgos, L. de Teresa, *New phenomena for the null controllability of parabolic systems: Minimal time and geometrical dependence* to appear in Jour. Math. Anal. App (2016).

F. Boyer and G. Olive, *Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients*, Math. Control Relat. Fields **4** (2014), no 3, 263–287.

Theorem (F. Boyer and G. Olive)

Let $\omega = (a, b)$ and $q \in L^\infty(0, \pi)$, a function satisfying

$$\operatorname{Supp} q \cap \omega = \emptyset. \tag{8}$$

Then, system is approximately controllable at time $T > 0$ if and only if

$$|I_k(q)| + |I_{1,k}(q)| \neq 0 \quad \forall k \geq 1. \tag{9}$$

$$I_k(q) := \frac{2}{\pi}\int_0^\pi q(x)|\sin(kx)|^2\,dx, \quad I_{1,k}(q) := \frac{2}{\pi}\int_0^a q(x)|\sin(kx)|^2\,dx$$

Fattorini-Russell method

- Let φ be a solution of the adjoint problem:

$$\begin{cases} -\partial_t \varphi = \left(D\partial_{xx}^2 + A^*\right)\varphi, & \text{in } Q = (0,\pi)\times(0,T), \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi^0 \in L^2(0,\pi;\mathbb{R}^2). & \end{cases}$$

- If y is a solution of the direct problem, then

$$\left\langle y(T),\varphi^0\right\rangle - \left\langle y^0,\varphi(0)\right\rangle = \int_0^T v(t) B^* D\partial_x\varphi(0,t)\,dt$$

Thus $y(T)=0$ if, and only if, there exists v such that

$$\boxed{\int_0^T\int_\omega v\,B^*\varphi\,dx\,dt = -\left\langle y^0,\varphi(0)\right\rangle,\ \forall\varphi^0\in L^2(0,\pi;\mathbb{R}^2)}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Fattorini-Russell method

Material at our disposal

$$L := -\frac{d^2}{dx^2} Id + q(x)A^*, \quad \sigma(L^*) = \{k^2 : k \geq 1\}.$$

Given $k \geq 1$, if

$$\Phi_{1,k}^* := \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}, \quad \Phi_{2,k}^* := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix},$$

where ψ_k is the unique solution of the non-homogeneous Sturm-Liouville problem:

$$(10) \qquad \begin{cases} -\psi_{xx} - k^2\psi = [I_k(q) - q(x)]\,\varphi_k \text{ in } (0,\pi), \\ \psi(0) = 0, \quad \psi(\pi) = 0, \\ \displaystyle\int_0^\pi \psi(x)\varphi_k(x)\,dx = 0, \end{cases}$$

Fattorini-Russell method

Material at our disposal

$$\left(L^* - k^2 I_d\right)\Phi^*_{1,k} = I_k(q)\Phi^*_{2,k} \quad \text{and} \quad \left(L^* - k^2 I_d\right)\Phi^*_{2,k} = 0).$$

$$\begin{cases} \Lambda_1 := \{k \geq 1 : I_k(q) \neq 0\}, \\ \Lambda_2 := \{k \geq 1 : I_k(q) = 0\}, \end{cases}$$

In particular, if $k \in \Lambda_1$ then k^2 is a simple eigenvalue and $\Phi^*_{2,k}$ and $\Phi^*_{1,k}$ are, respectively, an eigenfunction and a generalized eigenfunction of the operator L^* associated to k^2, while if $k \in \Lambda_2$ then $\Phi^*_{1,k}$ and $\Phi^*_{2,k}$ are both eigenfunctions of L^* associated to k^2.

$\{\Phi^*_{i,k},\ i = 1, 2, \quad k \geq 1\}$ is a Riesz basis of $L^2(0, \pi; \mathbb{R}^2)$.

Fattorini-Russell method

- $$y(T)=0 \Longleftrightarrow \left\langle y(T),\Phi^*_{i,k}\right\rangle = 0, \quad \forall k\geq 1,\ i=1,2$$
- Choosing $\varphi^0=\Phi^*_{2,k}$, we have $\varphi_{2,k}\,(x,t)=e^{-k^2(T-t)}\Phi^*_{2,k}(x)$ and
$$\varphi_{2,k}(x,0)=e^{-k^2T}\Phi^*_{2,k}(x)$$
- Choosing $\varphi^0=\Phi^*_{1,k}$, we have
$\varphi_{1,k}\,(x,t)=e^{-k^2(T-t)}(\Phi^*_{1,k}(x)-(T-t)I_k(q)\Phi^*_{2,k})$ and
$$\varphi_{1,k}(x,0)=e^{-k^2T}(\Phi^*_{1,k}(x)-(T-t)I_k(q)\Phi^*_{2,k})$$
- Thus $y(T)=0$ if, and only if, there exists $v\in L^2(\Omega\times(0,T))$ such that
$$\iint_{Q_T} v(x,t)1_\omega B^*\varphi_{i,k}(x,t)\,dx\,dt = -\left\langle y_0,\varphi_{i,k}(\cdot,0)\right\rangle,\ \forall i=1,2,\ \forall k\geq 1$$

The first main idea

- Search controls under the particular form

$$v(x,t) = f_1(x)v_1(T-t) + f_2(x)v_2(T-t), \quad \mathrm{Supp} f_1, \mathrm{Supp} f_2 \subseteq \omega = (a,b)$$

- The moment problem leads to

$$\begin{cases} f_{1,k} \int_0^T v_1(t)e^{-k^2 t}\,dt + f_{2,k} \int_0^T v_2(t)e^{-k^2 t}\,dt = -e^{-k^2 T}\left\langle y_0, \Phi^*_{2,k}\right\rangle \\ \widetilde{f}_{1,k} \int_0^T v_1(t)e^{-k^2 t}\,dt + \widetilde{f}_{2,k} \int_0^T v_2(t)e^{-k^2 t}\,dt \\ \qquad - I_k(q) f_{1,k} \int_0^T v_1(t)\,te^{-k^2 t}\,dt - I_k(q) f_{2,k} \int_0^T v_2(t)\,te^{-k^2 t}\,dt \\ \qquad = -e^{-k^2 T}\left(\left\langle y_0, \Phi^*_{1,k}\right\rangle - T I_k(q)\left\langle y_0, \Phi^*_{2,k}\right\rangle\right), \end{cases}$$

- $$f_{i,k} := \int_0^\pi f_i(x)\varphi_k(x)\,dx, \quad \widetilde{f}_{i,k} := \int_0^\pi f_i(x)\psi_k(x)\,dx, \quad i = 1,2,\ k \geq 1.$$

A vectorial moment method

The moment problem can be written as

$$A_k V_k + \widetilde{A}_k \widetilde{V}_k = F_k \quad \forall k \geq 1,$$

$$A_k = \begin{pmatrix} f_{1,k} & f_{2,k} \\ \widetilde{f}_{1,k} & \widetilde{f}_{2,k} \end{pmatrix}, \; \widetilde{A}_k = \begin{pmatrix} 0 & 0 \\ -I_k(q) f_{1,k} & -I_k(q) f_{2,k} \end{pmatrix}$$

$$V_k := \begin{pmatrix} \int_0^T v_1(t) e^{-k^2 t}\, dt \\ \int_0^T v_2(t) e^{-k^2 t}\, dt \end{pmatrix}, \quad \widetilde{V}_k := \begin{pmatrix} \int_0^T v_1(t) t e^{-k^2 t}\, dt \\ \int_0^T v_2(t) t e^{-k^2 k^2 t}\, dt \end{pmatrix},$$

$$F_k = e^{-k^2 T} \begin{pmatrix} -\left\langle y_0, \Phi_{2,k}^* \right\rangle \\ -\left(\left\langle y_0, \Phi_{1,k}^* \right\rangle - T I_k(q) \left\langle y_0, \Phi_{2,k}^* \right\rangle \right) \end{pmatrix}.$$

A vectorial moment method

Can we find v_1, v_2 solving the previous system of moments such that

$$v_1, v_2 \in L^2(0,T)?$$

Construction of the functions f_1, f_2

There exist functions $f_1, f_2 \in L^2(0,\pi)$ satisfying $\operatorname{Supp} f_1, \operatorname{Supp} f_2 \subseteq \omega$ and such that there exists positive constants C_1 and C_2 (only depending on f_1 and f_2) such that

$$|\det A_k| \geq C_1 \frac{|I_{1,k}(q)|}{k^6} - C_2 \frac{|I_k(q)|}{k}, \quad \forall k \geq 1.$$

$$I_{1,k}(q) := \int_0^a q(x)|\varphi_k(x)|^2\,dx, \quad I_k(q) := \int_0^\pi q(x)|\varphi_k(x)|^2\,dx.$$

A minimal time

Let

$$T_0(q) := \limsup_{k\to\infty} \frac{\min\{-\log|I_{1,k}(q)|, -\log|I_k(q)|\}}{k^2}.$$

Theorem

The moment problem leads to

$$\begin{cases} \displaystyle\int_0^T v_i(t) e^{-k^2 t}\, dt = e^{-k^2 T} M_{1,i}^{(k)}(y_0), \\ \displaystyle\int_0^T v_i(t)\, t e^{-k^2 t}\, dt = e^{-k^2 T} M_{2,i}^{(k)}(y_0), \end{cases} \qquad i = 1, 2\ k \geq 1$$

$$\left| M_{i,j}^{(k)}(y_0) \right| \leq C_\varepsilon e^{k^2 (T_0(q) + 2\varepsilon)} \|y_0\|_{L^2(0,\pi;\mathbb{R}^2)}, \quad \forall k \geq 1, \quad 1 \leq i,j \leq 2.$$

Biorthogonal family

E. Fernández-Cara, M. González-Burgos, L. de Teresa, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. **259** (2010), no. 7, 1720–1758.

The sequence

$$\left\{e_{1,k} := e^{-k^2 t}, e_{2,k} := te^{-k^2 t}\right\}_{k\geq 1}$$

admits a biorthogonal family $\{q_{1,k}, q_{2,k}\}_{k\geq 1}$ in $L^2(0,T)$

$$\int_0^T e_{r,k} q_{s,j}(t)\, dt = \delta_{kj}\delta_{rs}, \quad \forall k, j \geq 1, \quad 1 \leq r, s \leq 2, \tag{11}$$

which satisfies that for every $\varepsilon > 0$ there exists a constant $C_{\varepsilon,T} > 0$ such that

$$\|q_{i,k}\|_{L^2(0,T)} \leq C_{\varepsilon,T}\, e^{\varepsilon k^2}, \quad \forall k \geq 1, \quad i = 1, 2. \tag{12}$$

Conclusion $T > T_0(q)$

$$\begin{cases} \displaystyle\int_0^T v_i(t) e^{-k^2 t}\, dt = e^{-k^2 T} M_{1,i}^{(k)}(y_0), \\ \displaystyle\int_0^T v_i(t)\, t e^{-k^2 t}\, dt = e^{-k^2 T} M_{2,i}^{(k)}(y_0), \end{cases}$$

$$\left| M_{i,j}^{(k)}(y_0) \right| \leq C_\varepsilon e^{k^2 (T_0(q) + 2\varepsilon)} \|y_0\|_{L^2(0,\pi;\mathbb{R}^2)}, \quad \forall k \geq 1, \quad 1 \leq i,j \leq 2.$$

$$v_i(t) = \sum_{k \geq 1} e^{-k^2 T} \left(M_{1,i}^{(k)}(y_0) q_{1,k}(t) + M_{2,i}^{(k)}(y_0) q_{2,k}(t) \right), \quad i = 1, 2.$$

$$\left\| e^{-k^2 T} M_{\ell,i}^{(k)}(y_0) q_{\ell,k}(t) \right\|_{L^2(0,T)} \leq C_{\varepsilon,T}\, e^{-k^2 (T - T_0(q) - 3\varepsilon)} \|y_0\|_{L^2(0,\pi;\mathbb{R}^2)},$$

for any $k \geq 1$ and $\ell, i : 1 \leq \ell, i \leq 2$.

$$\varepsilon \in \left(0, \frac{T - T_0(q)}{3} \right) \Rightarrow v_i \in L^2(0,T), \ i = 1, 2.$$

The controllability result

We have proved

There exists $T_0(q) \in [0, +\infty]$ such that the system is null controllable at time $T > T_0(q)$

$T > T_0(q))$ a necessary condition?

What happens if $T < T_0(q)$?

Is it possible to have $T_0(q) > 0$?

Theorem

For any $\tau_0 \in [0, \infty]$, there exists a function $q \in L^\infty(0, \pi)$ satisfying

$$|I_k(q)| + |I_{1,k}(q)| \neq 0 \quad \forall k \geq 1,$$

such that

$$T_0(q) = \tau_0.$$

A non controllability result

- The null controllability property is equivalent to the following observability inequality:

$$\|\varphi(0)\|_{L^2(0,\pi;\mathbb{R}^2)} \leq C_T \int_0^T \int_\omega \|B^*\varphi(x,t)\|^2 \, dt.$$

for all φ solution to

$$\begin{cases} -\partial_t \varphi = \left(\partial_{xx}^2 + A^*\right)\varphi, & \text{in } Q = (0,\pi)\times(0,T), \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The minimal time of control depends on ω

M. González-Burgos, L. de Teresa, *Controllability results for cascade systems of m coupled parabolic PDEs by one controll force*, Port. Math. **67** (2010), no. 1, 91–113.

Assume that there exist an open subset $\omega_0 \subset \omega$ and $\sigma > 0$ such that

$$|q| \geq \sigma > 0 \quad \text{in} \quad \omega_0,$$

then the null controllability occurs at any time $T > 0$.

The moment method gives a sufficient condition of null controllability

This is due to the fact that, in this method, we have restricted the control to a particular form

$$v(x,t) = f_1(x)v_1(T-t) + f_2(x)v_2(T-t), \quad \text{Supp} f_1, \text{Supp} f_2 \subseteq \omega = (a,b)$$

$\operatorname{Supp} q \cap \omega = \emptyset$: The negative controllability result

Assume that $T_0(q) > 0$, otherwise there is nothing to prove.
Argue by contradiction and suppose the observability estimate

$$\|\varphi(0)\|_{L^2(0,\pi;\mathbb{R}^2)} \leq C_T \int_0^T \int_\omega \|B^* \varphi(x,t)\|^2 \, dt$$

holds true for all φ solution to

$$\begin{cases} -\partial_t \varphi = \left(\partial_{xx}^2 + A^*\right) \varphi, & \text{in } Q = (0,\pi) \times (0,T), \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Supp $q \cap \omega = \emptyset$: The negative controllability result

For $\theta_0 = a_k \Phi^*_{1,k} + b_k \Phi^*_{2,k}$, with $(a_k, b_k) \in \mathbb{R}^2$ and $\Phi^*_{i,k}$ the eigenvectors of the operator $L := \partial_{xx} + qA$, the previous inequality reads as

$$A_{1,k} \leq C A_{2,k},$$

$$A_{1,k} := e^{-2k^2 T} \left\{ |a_k|^2 + \left[|a_k|^2 | \, \|\psi_k\|^2_{L^2(0,\pi)} + (b_k - T a_k I_k(q))^2 \right] \right\}$$

$$A_{2,k} := \int_0^T \int_\omega e^{-2k^2 t} \left| a_k \psi_k(x) + b_k \varphi_k(x) - t a_k I_k(q) \varphi_k(x) \right|^2 dx.$$

$$\begin{cases} -\psi_{xx} - k^2 \psi = [I_k(q) - q(x)] \, \varphi_k \text{ in } (0, \pi), \\ \psi(0) = 0, \quad \psi(\pi) = 0, \\ \displaystyle\int_0^\pi \psi(x) \varphi_k(x) \, dx = 0, \end{cases}$$

Use the assumption on ω

Theorem

Let $\omega = (a,b) \subset (0,\pi)$ and $q \in L^\infty(0,\pi)$ be a function satisfying $\operatorname{Supp} q \cap \omega = \emptyset$. *Then, for any $k \geq 1$:*

$$\psi_k(x) = \tau_k \varphi_k(x) + g_k(x), \quad \forall x \in \omega,$$

with τ_k a positive constant and

$$g_k(x) = -\frac{I_k(q)}{k} \int_0^x \sin(k(x-\xi))\varphi_k(\xi)\, d\xi - \sqrt{\frac{\pi}{2}} \frac{I_{1,k}(q)}{k} \cos(kx), \quad \forall x \in \omega,$$

Use the assumption on ω

By choosing $a_k = 1$ and $b_k = -\tau_k$, we get:

$$a_k\psi_k(x) + b_k\varphi_k(x) = g_k(x)$$

$$g_k(x) = -\frac{I_k(q)}{k}\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\,d\xi - \sqrt{\frac{\pi}{2}}\frac{I_{1,k}(q)}{k}\cos(kx)$$

The observability estimate becomes

$$\begin{cases} 1 \leq Ce^{2k^2T}\left(|I_{1,k}(q)|^2 + |I_k(q)|^2\right) \\ \quad \leq Ce^{-2k^2\left[\frac{1}{k^2}\min\left(-\log\left|I_{1,k}(q)\right|, -\log|I_k(q)|\right)-T\right]}. \end{cases}$$

The negative null controllability result

From the definition of $T_0(q)$ there exists a subsequence of indices $\{k_n\}_{n\geq 1} \subseteq \mathbb{N}^*$ satisfying:

$$T_0(q) = \lim_{n\to\infty} \frac{\min\left(-\log\left|I_{1,k_n}(q)\right|, -\log\left|I_{k_n}(q)\right|\right)}{k_n^2}.$$

If $T_0(q) < \infty$, as a consequence, we deduce that for any $\varepsilon > 0$ there is $n_\varepsilon \geq 1$ such that

$$\frac{\min\left(-\log\left|I_{1,k_n}(q)\right|, -\log\left|I_{k_n}(q)\right|\right)}{k_n^2} \geq T_0(q) - \varepsilon, \quad \forall n \geq n_\varepsilon.$$

Coming back to the previous inequality, we obtain

$$1 \leq Ce^{-2k_n^2[T_0(q)-\varepsilon-T]}, \quad \forall n \geq n_\varepsilon,$$

which gives a contradiction if we take $\varepsilon \in (0, T_0(q) - T)$. In the case in which $T_0(q) = \infty$, the reasoning is easier and we also get a contradiction.

Is it possible to have a positive minimal time of control?

$T_0(q) > 0$?

For $\omega = (a, b) \subset (0, \pi)$ and $q \in L^\infty(0, \pi)$, is it possible that $T_0(q) > 0$?

Is it possible to have a positive minimal time of control?

$T_0(q) > 0$?

For $\omega = (a,b) \subset (0,\pi)$ and $q \in L^\infty(0,\pi)$, is it possible that $T_0(q) > 0$?

Theorem

For any $\tau_0 \in [0,\infty]$, there exists a function $q \in L^\infty(0,\pi)$ satisfying $\operatorname{Supp} q \cap \omega = \emptyset$ and

$$|I_k(q)| + |I_{1,k}(q)| \neq 0 \quad \forall k \geq 1,$$

such that

$$T_0(q) = \tau_0.$$

Comments

F. ALABAU-BOUSSOUIRA, M. LÉAUTAUD, *Indirect controllability of locally coupled wave-type systems and applications*, J. Math. Pures Appl. (9) **99** (2013), no. 5, 544–576.

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$$q \not\equiv 0 \quad \text{and} \quad q \geq 0 \quad \text{or} \quad q \leq 0 \quad \text{in } (0, \pi).$$

These results have been obtained as a consequence of the corresponding hyperbolic results by using the transmutation strategy.

Comments

This minimal time also arises in other parabolic problems

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Thank you for your attention!!