Existence theorems for optimal control problems with convex-concave dynamics and unbounded controls: two models from economics.

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Optimal Control for Evolutionary PDEs and Related Topics
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Main target

The goal of this talk is to describe an original method to deal with the problem of the existence of solutions to an *infinite horizon* optimal control problem, with ODE state equation, in which controls may be *unbounded* in finite time intervals, and the dynamics has *convex-concave* behaviour.

We show two applications, both motivated by economic optimization problems:

- the **Ramsey-Skiba model** (Growth Theory)
- the **Shallow lake model** (Environmental Economics)

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Why such problems may require a specific method?

Even for simple one dimensional problems of this kind, existence of an optimum may be non trivial. Indeed:

- The infinite horizon setting plus the unboundedness of the controls prevent from having good a priori estimates, so the application of any compactness results is not straightforward
- Non-concavity and linear growth of the production function
- Possible presence of a state constraint (Ramsey-Skiba)

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Outline

Economic Growth: the Ramsey-Skiba model

Infinite horizon utility maximization problem for an endogenous growth model with **convex-concave** production function, introduced first by Skiba (1973) and studied by various authors (e.g. Askenazy - Le Van, Nishimura et al., Santos et al., Fiaschi and Gozzi)

Why study this kind of problems?

1a. **Why study this kind of problems, from a general viewpoint?**

- Important part of economic growth literature (i.r.s.)
- Lack of generality in recent works facing convex-concave technology (technical challenges are present)

1b. **In particular, what is the state of the art for the existence question?**

- Seemingly, the question is (positively) answered only for the **concave dynamics** case (Ekeland 2009, 2010, Ekeland Long Zhou 2013, Asheim and Ekeland 2016)
- In latter works, the optimum is obtained via a feedback relation; this approach requires to assume more regularity on the admissible states and on the value function
- For the same case, there is also a direct method in Freni, Gozzi and Salvadori 2010

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Qualitative features of the model

2. **Qualitative features of the model**

- $x(t) =$ capital at time and $c(t) =$ consumption rate, at $t \geq 0$
- $u(c) =$ instantaneous utility from consumption, increasing and concave
- $F(x) =$ per capita production function increasing, not concave
- **state constraint:** $x(t) \geq 0$, for $t \geq 0$

Quantitative description of the model

3. Quantitative description of the model

- Let $x(\cdot;x_0,c)$ be the unique solution of
$$\begin{cases} \dot{x}(t) = F(x(t)) - c(t) & t \geq 0 \\ x(0) = x_0 \geq 0 \end{cases}$$

- Goal: maximize
$$U(x_0;c) := \int_0^{+\infty} e^{-\rho t} u(c(t))\,dt$$
for all $c \in \Lambda(x_0)$, where

- $\Lambda(x_0) := \left\{ c \in L^1_{loc}([0,+\infty),\mathbb{R}) / c \geq 0 \text{ a.e.}, x(\cdot;x_0,c) \geq 0 \right\}$

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Main assumptions

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- $\begin{cases} \dot{x}(t) = F(x(t)) - c(t) & t \geq 0 \\ x(0) = x_0 \end{cases}$

$$\begin{cases} F \in \mathscr{C}^1(\mathbb{R},\mathbb{R}) \\ F' > 0 \text{ in } \mathbb{R} \\ F(0) = 0 \\ \lim_{x\to+\infty} F(x) = +\infty \\ \lim_{x\to+\infty} F'(x) =: L > 0 \\ F \text{ is convex in } [0,\overline{x}] \\ F \text{ is concave in } [\overline{x},+\infty) \end{cases}$$

$$\begin{cases} u \in \mathscr{C}^2((0,+\infty),\mathbb{R}) \\ u \in \mathscr{C}^0([0,+\infty),\mathbb{R}) \\ u(0) = 0, \ \lim_{x\to+\infty} u(x) = +\infty \\ u' > 0,\ u'' < 0 \\ \lim_{x\to0^+} u'(x) = +\infty, \ \lim_{x\to+\infty} u'(x) = 0 \\ \lim_{t\to+\infty} e^{\varepsilon_0 t} e^{-\rho t} u\left(e^{(L+\varepsilon_0)t}\right) = 0, \\ \text{for some } \varepsilon_0 > 0 \end{cases}$$

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Finiteness of the value function

4. **Remark**

The value function is $V(x_0) := \sup_{c \in \Lambda(x_0)} U(x_0; c) \quad \forall x_0 \geq 0.$

Note: Condition

$$\exists \varepsilon_0 > 0 : \lim_{t \to +\infty} e^{\varepsilon_0 t} e^{-\rho t} u\left(e^{(L+\varepsilon_0)t}\right) = 0$$

is an *if and only if* condition for the value function to be finite for every $x_0 \geq 0$ in the case:

$$u(c) = c^{1-\sigma}, \quad F(x) = Lx$$

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Goals of the analysis

5.1 Ultimate goal of the analysis

- Properties of optimal paths (x^*, c^*), improving the previous results.
  For example: conditions for the monotonicity of c^* are not known, while monotonicity of x^* is proved in Askenazy - Le Van and in Nishimura et al.

5.2 Main Tool

- Deep study of the properties of the value function, in particular $\mathscr{C}^1$ regularity and presence of singularities in the first derivative.

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Results up to now

6 **Up to now, the following results have been proven (Acquistapace, B. 2016)**

- **Existence of the optimal controls** (first implementation of the discussed method)
- Finiteness, asymptotic behaviour, Lipschitz-continuity, “strong” monotonicity of the value function
- Dynamic programming: the value function solves the Hamilton-Jacobi-Bellman equation, in the viscosity sense

Outline

1. Description of the models
   - The Ramsey-Skiba model
   - The Shallow Lake model
2. Description of the method

Environmental Economics: the Shallow Lake model

Social benefit maximization problem taking into account the avail of the farming activities near a shallow lake and the utility of a clear lake.

- Obviously, productive activities near the lake increase the employment level...
- but also the pollution degree of the water.

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The model is introduced in Mäler et al. (2000), and has been studied mostly via **dynamic programming** like in Kossioris - Zohios (2012), or from the **dynamical systems** viewpoint (e.g. Wagener (2003), Kiseleva - Wagener (2010, 2015)).

Why study existence for this model?

1. **Why study the existence problem for this model?**

- The model has developed his own literature during the last 15 years.
- The main approach up to now has been the analysis of the system obtained coupling the state equation with the adjoint equation given by the **Pontryagin Maximum Principle.**
- As it is well known, such Principle provides conditions for optimality which in general are merely **necessary**. Indeed...
- **No direct existence result** seems to have been proven up to now.

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Main qualitative features of the model

2. Main qualitative features of the model

- $x(t) =$ amount of phosphorus (pollution) in the water
  $c(t) =$ input of phosphorus due to farming, at $t \geq 0$
- welfare function of the form $(c, x) \to \log c - \gamma x^2$
- $F(x) =$ biological dynamics of phosphorus in the lake, decreasing, not concave

Quantitative description of the model

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- Goal: maximize
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for every admissible control c, that is to say
$$c \in L^1_{loc}[0,+\infty), \quad c > 0 \quad \text{a.e. in } [0,+\infty)$$

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The classical example of such F is:

$$F(x) = -bx + \frac{x^2}{1 + x^2}$$

- $-bx$ = loss of phosphorus (sedimentation etc..)
- $x^2/(1 + x^2)$ = biological production

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Boundedness of the value function

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The value function is $V(x_0) := \sup_{\substack{c \in L^1_{loc} \\ c>0 \text{ a.e.}}} \mathscr{B}(x_0; c) \quad \forall x_0 \geq 0.$

Theorem

The value function is bounded; indeed

$$V(x_0) \leq \frac{1}{\rho} \log\left(\frac{\rho + b_0}{\sqrt{2e\gamma}}\right) \quad \forall x_0 \geq 0.$$

Remark. This is done essentially by concavity and ODE comparison arguments (the sublinearity of F is used)

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Remark

Both models include dynamics showing convex-concave behaviour and concave functional, but...

... as far as the applicability of the method is concerned, the role of convexity-concavity in the dynamics and in the functional is different.

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Step 0

Basic uniform estimates

Fact

In the RS model:

$$\int_0^t c(s)\,ds \leq x_0 e^{\overline{M}t}$$
$$x(t;k_0,c) \leq x_0 e^{\overline{M}t}$$

for every admissible control c.

Fact

In the SL model:

$$\int_0^{+\infty} e^{-\rho t} c(t)\,dt \leq M_1(x_0),$$
$$\int_0^{+\infty} e^{-\rho t} x(t)\,dt \leq M_2(x_0).$$

for every maximizing control c.

Step 0

Basic uniform estimates

Fact

In the RS model:

$$\int_0^t c(s)\,ds \leq x_0 e^{\overline{M}t}$$
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Step 0

Basic uniform estimates

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In the RS model:

$$\begin{aligned} \int_0^t c(s)\,ds &\leq x_0 e^{\overline{M}t} \\ x(t;k_0,c) &\leq x_0 e^{\overline{M}t} \end{aligned}$$

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Step 1: Localization Lemmas

$$J(c) = \int_0^{+\infty} e^{-\rho t} u(c(t))\,dt \text{ or } \int_0^{+\infty} e^{-\rho t}\left[\log c(t) - \gamma x^2(t)\right] dt$$

Theorem

There exists a function $N : [0,+\infty)^2 \to (0,+\infty)$*, increasing in both variables, such that:*
for every $(x_0, T) \in [0,+\infty)^2$ *and every admissible (at least, maximizing)* c*, there exists an admissible* c^T *satisfying*

$$J\left(x_0; c^T\right) \geq J\left(x_0; c\right)$$
$$c^T = c \wedge N(x_0, T) \text{ almost everywhere in } [0, T]$$

In particular, c^T *is bounded above, in* $[0, T]$*, by a quantity which does not depend on the original control* c*, but only on* T *and on the initial status* x_0*.*

Note: This is useful also in Dynamic Programming.

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There exists a function $N:[0,+\infty)^2 \to (0,+\infty)$, *increasing in both variables, such that:*
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$$J\left(x_0;c^T\right)\geq J\left(x_0;c\right)$$
$$c^T=c\wedge N\left(x_0,T\right) \text{ almost everywhere in } [0,T].$$

- N **not** dependent on c is fundamental in order to apply compactness
- A careful choice of β_T gives $x^T \geq x$ for $\dot{\mathrm{x}}=F(\mathrm{x})-c$ **(RS)**
  $x^T \leq x$ for $\dot{\mathrm{x}}=F(\mathrm{x})+c$ **(SL)**

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- N **not** dependent on c is fundamental in order to apply compactness
- A careful choice of β_T gives $x^T \geq x$ for $\dot{\mathrm{x}} = F(\mathrm{x}) - c$ **(RS)**
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Step 1: Localization Lemmas (SL model)

$$J(c) = \int_0^{+\infty} e^{-\rho t} \left[\log c(t) - \gamma x^2(t)\right] \mathrm{d}t$$

Theorem

There exists a function $\eta : [0,+\infty)^2 \to (0,+\infty)$*, continuous and strictly decreasing in the second variable, such that* $\eta < N$ *and: for every* $x_0 \geq 0$ *and every* $T \geq 1$*, if* c *is maximizing, there exists an admissible* c^T *such that*

$$J\left(x_0; c^T\right) \geq J(x_0; c)$$
$$c^T = (c \wedge N(x_0, T)) \vee \eta(x_0, T) \quad a.\ e.\ in\ [0, T].$$

In particular the norms $\left\|c^T\right\|_{L^\infty([0,T])}, \left\|\log c^T\right\|_{L^\infty([0,T])}$ *are bounded above by a quantity which does not depend on* c*.*

Step 2: double-sequence diagonalization

- For $x_0 > 0$ fixed, take a maximizing sequence

$$\lim_{n \to +\infty} J(x_0; c_n) = V(x_0)$$

- For every $T > 0$, use the localization lemma 1 or 2 and then the Dunford-Pettis criterion, infinitely many times:

$$(c_n)_n \overset{Loc.\,Lemma}{\dashrightarrow} (c_n^1)_n \overset{D.P.}{\supseteq} (\overline{c}_n^1)_n \overset{Loc.\,Lemma}{\dashrightarrow} (c_n^2)_n \overset{D.P.}{\supseteq} (\overline{c}_n^2)_n \dashrightarrow \dots$$

- $\overline{c}_n^1 \rightharpoonup c^1$ in $L^1([0,1])$, $\quad \overline{c}_n^2 \rightharpoonup c^2$ in $L^1([0,2])\dots$

- Prove that c^{T+1} extends c^T.

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The sequence $\left(\bar{c}_n^T\right)$ is extracted from $\left(c_n^T\right)$, but

a priori, $\left(c_n^{T+1}\right)$ is not extracted from $\left(\bar{c}_n^T\right)$.

- Using the monotonicity of the functions

$$T \to N(T, x_0), \qquad T \to \eta(T, x_0)$$

it is possible to prove that $\left(c_n^{T+1}\right)$ coincides, a.e. in $[0, T]$, with a subsequence of $\left(\bar{c}_n^T\right)$.

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Step 2: double-sequence diagonalization

$(\gamma_n)_n =$ first approximating sequence, $\gamma =$ sub-optimal control

- Find $(\gamma_n)_{n\geq 0} \subseteq \Lambda_{ad}(k_0)$, $\gamma \in \mathscr{L}^1_{loc}([0,+\infty),\mathbb{R})$ such that

  $$\gamma_n \rightharpoonup \gamma \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T > 0$$
  $$0 \leq \gamma_n, \gamma \leq N(k_0, T) \quad \forall T, \text{ a.e. in } [0,T], n \geq T \quad \textbf{(RS)}$$
  $$\eta(x_0, T) \leq \gamma_n, \gamma \leq N(k_0, T) \quad \forall T, \text{ a.e. in } [0,T], n \geq T \quad \textbf{(SL)}$$

Note: $(\gamma_n)_n$ coincides a. e. in $[0,T]$ with a subsequence of $\left(\overline{c}_n^T\right)_n$, hence both $(\gamma_n)_n$ and γ inherit the local boundedness properties of sequences $\left(\overline{c}_n^T\right)_n$, $T \in \mathbb{N}$.

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$$\implies 0 \leq x_{\gamma_n} \xrightarrow{n\to+\infty} x_\gamma \text{ pointwise in } [0,+\infty).$$

In particular, this guarantees the admissibility of γ (also) for the **RS** problem.

Step 3: simple diagonalization and functional convergence

$$J(c) = \int_0^{+\infty} e^{-\rho t} u(c(t))\,dt \text{ or } \int_0^{+\infty} e^{-\rho t} \left[\log c(t) - \gamma x^2(t)\right] dt$$

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Consider the images of the γ_n's through the utility function:

$$f_n = u(\gamma_n) \text{ for } \mathbf{RS} \text{ or } f_n = \log(\gamma_n) \text{ for } \mathbf{SL}$$

- The sequence $(f_n)_n$ is locally uniformly bounded by the above relations
- Standard diagonalization: $(f_{n,n})_n$ extracted from $(f_n)_n$ such that

$$f_{n,n} \rightharpoonup f \text{ in } L^1([0, T], \mathbb{R}) \quad \forall T > 0$$

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$$f_{n,n} \rightharpoonup f \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T > 0$$

- Inversion:

$$c^* := u^{-1}(f) \quad \textbf{(RS)}$$
$$c^* := e^f \quad \textbf{(SL)}$$

- Hence by construction, in $L^1([0,T])$ for every $T > 0$:

$$\gamma_n \rightharpoonup \gamma$$
$$u(\gamma_{n,n}) \rightharpoonup u(c^*) \quad \textbf{(RS)}$$
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$$x^* \geq x_\gamma \text{ pointwise in } [0,+\infty) \quad \textbf{(RS)} \tag{1}$$
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using concavity arguments.
In particular, relation (1) implies that c^* if admissible for the **RS** problem.

Step 3: simple diagonalization and functional convergence

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  using concavity arguments.
  In particular, relation (1) implies that c^* if admissible for the **RS** problem.

Step 3: simple diagonalization and functional convergence

Convergence of the functionals for Ramsey-Skiba problem:

$$
\begin{aligned}
V(x_0) &= \lim_{n\to+\infty} U(x_0;\gamma_{n,n}) \\
&= \lim_{n\to+\infty} \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(\gamma_{n,n}(s))\,\mathrm{d}s\mathrm{d}t \\
&= \rho \int_0^{+\infty} e^{-\rho t} \lim_{n\to+\infty} \int_0^t u(\gamma_{n,n}(s))\,\mathrm{d}s\mathrm{d}t \\
&= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c^*(s))\,\mathrm{d}s\mathrm{d}t = U(c^*;k_0)
\end{aligned}
$$

Step 3: simple diagonalization and functional convergence

Convergence of the functionals for Shallow Lake problem:

$$
\begin{aligned}
V(x_0) &= \lim_{n\to+\infty} \mathscr{B}(x_0, \gamma_{n,n}) \\
&\leq \rho \lim_{n\to+\infty} \int_0^{+\infty} e^{-\rho t} \left(\int_0^t \log \gamma_{n,n}(s)\,\mathrm{d}s - \frac{c}{\rho} x^2_{\gamma_{n,n}}(t) \right) \mathrm{d}t \\
&\leq \rho \int_0^{+\infty} e^{-\rho t} \limsup_{n\to+\infty} \left(\int_0^t \log \gamma_{n,n}(s)\,\mathrm{d}s - \frac{c}{\rho} x^2_{n,n}(t) \right) \mathrm{d}t \\
&= \rho \int_0^{+\infty} e^{-\rho t} \left(\int_0^t \log c^*(s)\,\mathrm{d}s - \frac{c}{\rho} x^2_{\gamma}(t) \right) \mathrm{d}t \\
&\leq \rho \int_0^{+\infty} e^{-\rho t} \left(\int_0^t \log c^*(s)\,\mathrm{d}s - \frac{c}{\rho} (x^*)^2(t) \right) \mathrm{d}t \\
&= \mathscr{B}(x_0; c^*).
\end{aligned}
$$

Conclusive remark

Remark: by

$$0 \le \gamma_{n,n} \le N(k_0; T) \text{ a.e. in } [0,T] \quad \forall T > 0$$
$$u(\gamma_{n,n}) \rightharpoonup u(c^*) \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T > 0$$

and

$$\eta(x_0; T) \le \gamma_{n,n} \le N(k_0; T) \text{ a.e. in } [0,T] \quad \forall T > 0$$
$$\log(\gamma_{n,n}) \rightharpoonup \log(c^*) \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T > 0$$

it follows that the control c^* which is *optimal* at k_0 is actually **locally bounded** in $[0,+\infty)$ (while the *admissible* controls could be unbounded in some compact subset of $[0,+\infty)$).

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it follows that the control c^* which is *optimal* at k_0 is actually **locally bounded** in $[0, +\infty)$ (while the *admissible* controls could be unbounded in some compact subset of $[0, +\infty)$).

Thank you for the attention!

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