On a general framework analysis for indirect stabilization of coupled systems of PDE's

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Outline

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2. Direct control and stabilization : the example of the scalar wave equation
3. Indirect stabilization of coupled systems of PDE's
4. The method of higher order energy
5. Lack of exponential stability
6. Further classes of indirectly stabilized coupled PDEs

The control of PDE’s aims to impose to the solutions of an evolution PDE a prescribed behavior,

under the action of controls that appear in the equation either as a source term (open loop): this is exact controllability

or as a feedback (closed loop, the control depends on the solution): this is stabilization

or to estimate the initial data in terms of the dual PDE without control, that is for the free dual PDE: this is observability

$\vdots$

The subject of control, stabilization and observability is well-mastered in the ***direct*** case, that is when each equation (or unknown) is directly controlled, stabilized, or observed

Our main concern here is what we call ***indirect*** control or stabilization for coupled systems of PDE's.

The terminology ***indirect*** is used whenever the number of controls, feedbacks, or observations is strictly less than the number of equations (or unknowns) of the system.

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Linear abstract formulation:

$$\begin{cases} y'(t) = Ay(t) + Bg(t), \quad t > 0, \\ y(0) = y^0 \end{cases}$$

- $A : D(A) \subset H \to H$ generates a $\mathcal{C}^0$ semigroup in the Hilbert space$H$. A describes the dynamics of the underlying physical model.
- $B \in \mathcal{L}(U, D(A^*)')$ is the control operator, which can be bounded or unbounded. B is said to be bounded if $B \in \mathcal{L}(U, H)$.
- U is a Hilbert space, namely the control space.
- $g \in L^2_{loc}([0, \infty), U)$ is the control
- y^0 is the initial data
- y is the state variable
- $'$ stands for the time derivative.

When $g = 0$, i.e. when the system is not controlled,

and when the dynamics A of the system A is fixed,

the state of the system $y(T)$ at time $T > 0$ is completely determined by the initial state of the system y^0, cthis for all $T > 0$. Thus, we cannot impose a prescribed behavior to the state of the system.

When $g \neq 0$, one says that the system $y'(t) = Ay(t) + Bg(t)$ is a controlled system.

Exact controllability :

Definition

The controlled system $y'(t) = Ay(t) + Bg(t)$ is said to be exactly controllable at time $T > 0$ if for all $y^0 \in H$ and all $y^T \in H$, there exists a control $g \in L^2([0, T], U)$ such that the solution y of the controlled system satisfies: $y(0) = y^0$ and $y(T) = y^T$.

One also says in an equivalent way that the pair (A, B) is exactly controlable in time T.

• Such a control, if it exists, drives the initial state of the system to a desired state at time $T > 0$.

• Such a control is not unique in general. Uniqueness is obtained when one adds additional conditions of optimal control type such as for instance, to choose among all possible controls the one which has the minimal L^2 norm. One may also add constraints on the state vector...

Closed loop control or stabilization by a feedback :

One can look for a control that "drives asymptotically" the system to a targeted state, for instance an equilibrium state, when the time goes to infinity, this through a control which depends on the state.

Such a closed loop control takes in general the form $g(t) = Ky(t)$. The previous system then becomes

$$y'(t) = Ay(t) + BKy(t), \quad t > 0.$$

Definition

One says that the control $g = Ky$ strongly stabilizes the dynamicl system if for every initial data $y^0 \in H$, the solution $y(t) \to 0$ in H when $t \to \infty$

Definition

One says that the control $g = Ky$ *exponentially stabilizes the dynamical system if there exists* $M \geqslant 0$ *and* $\gamma > 0$ *such that*

$$|y(t)|_H \leqslant Me^{-\gamma t}|y^0|_H, \quad \forall\, y^0 \in H, \forall\, t \geqslant 0.$$

Such a control is called a damping term of a feedback.

It allows to modify the system in such a way that it auto-regulates.

One can prove that in finite dimension, there is equivalence between exact controllability, strong stabilization and exponential stabilization.

The situation is much more delicate and complex in infinite dimension.

Let us consider the case where A is skew-adjoint – the scalar wave equation for instance –

One can then choose for this example $K = -B^*$.

Then the energy of the solutions y satisfies the following dissipation relation

$$\frac{d}{dt}\left(\frac{|y(t)|_H^2}{2}\right) = -|B^* y(t)|_H^2 \leqslant 0, \quad \forall\, t > 0.$$

where $E_y(t) = \frac{1}{2}|y(t)|_H^2$ is the energy of the solution y.

One can also prove that the admissibility and observability at time $T > 0$ of B^* for the dual free problem, then the closed loop system for the feedback operator $K = -B^*$ is exponentially stabilizable.

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Let us start with the wave equation set in a bounded open set Ω with a smooth boundary Γ.

The case of internal feedback:

$$\begin{cases} w_{tt}(t,x) - \Delta w(t,x) + b(x)w_t(t,x) = 0, \quad t \in]0,T[, x \in \Omega, \\ w(t,x) = 0, \quad t \in]0,T[, x \in \Gamma, \\ (w, w_t)(0,x) = (w^0, w^1), \quad x \in \Omega. \end{cases}$$

where $b \in L^\infty(\Omega)$, $b \geqslant 0$ pp. in Ω, is the feedback coefficient.

It models the deformation of an elastic membrane under the effect of an initial deformation, and the action of a feedback.

w is the displacement (deformation of the membrane), w_t is the velocity.

It is a model problem. One can replace the operator $-\Delta$ with Dirichlet boundary conditions by the bilaplacian Δ^2 with clamped or hinged boundary conditions, the elasticity operator, the beam equation, Schrödinger equation, or the transport equation ...

This problem (and the other problems) can be reformulated as:

$$y' = Ay, y(0) = y^0.$$

setting $y = (w, w_t)$, $H = H_0^1(\Omega) \times L^2(\Omega)$ equipped with the norm $|y|_H$ defined by $|y|_H = \left(|\nabla w|^2_{L^2(\Omega)^n} + |w_t|^2_{L^2(\Omega)} \right)^{1/2}$ and **defining**

the operator A by:

$$D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega), \quad A(w, q) = (q, \Delta w + bq), \quad \forall\, (w, q) \in D(A)$$

Note that the norm defined on the energy space H is related to the natual energy, defined for the wave equation by

$$E_w(t) = \frac{1}{2}\left(\int_\Omega (w_t^2 + |\nabla w(t)|^2)dx\right).$$

$\rightsquigarrow$ it keeps trace of the invariance properties of the free wave system (conservation of the energy) when no feedback is acting.

$\rightsquigarrow$ here, one can show that the energy of the strong solutions satisfies the dissipation relation:

$$E'(t) = -\int_\Omega b(x) w_t(t,x)^2 dx \leqslant 0,$$

for all $t \geqslant 0$.

The wave equation with boundary feedback:

$$\begin{cases} w_{tt}(t,x) - \Delta w(t,x) = 0, \quad t \in]0,T[, x \in \Omega, \\ w(t,x) = 0, \quad t \in]0,T[, x \in \Gamma_0, \\ \dfrac{\partial w}{\partial \nu} + b(x) w_t + \eta w = 0 \quad t \in]0,T[, x \in \Gamma_1, \\ (w, w_t)(0,x) = (w^0, w^1), \quad x \in \Omega. \end{cases}$$

where Γ_0, Γ_1 is a partition of Γ, $b \in L^\infty(\Gamma_1)$, $b \geqslant 0$ pp. in Γ_1.

Abstract form:

$$y' = Ay, y(0) = y^0.$$

setting $y = (w, w_t)$, with a suitable choice of the energy space H, and the operator A (in general by duality). It will not detail this point here.

The energy of solutions is defined as

$$E_w(t) = \frac{1}{2}\left(\int_\Omega (w_t^2 + |\nabla w(t)|^2)dx + \int_{\Gamma_1} b w_t^2 d\sigma\right).$$

The energy of strong solutions satisfies the dissipation relation

$$E_w'(t) = -\int_{\Gamma_1} b w_t^2 d\sigma \leqslant 0,$$

for all $t \geqslant 0$.

Are these two closed loop systems exponentially stable ?

The exponential stabilization for both localized and boundary feedback can be proved by different methods: microlocal analysis, multipliers, frequency domain analysis, non harmonic analysis, . . .

- under geometric conditions on the support of the stabilization coefficient b, smoothness properties (for microlocal analysis), ...
- and for T sufficiently large (due to the finite propagation speed for the wave equation)

Russell, J.-L. Lions, Ho, Bardos & Lebeau & Rauch, Lasiecka & Triggiani, Zuazua, Kormornik, Burq & Gérard . . .

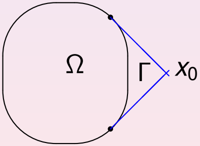
• Geometric Control Condition (GCC) Bardos Lebeau Rauch 1992 :

One says that a subset ω of Ω satisfies (GCC) if there exists a time $T > 0$ such that every geodesics travelling at speed 1 meets ω at a time $t \in (0, T)$ (moreover in a non diffractive point if the control region is on the boundary). It is almost a necessary and sufficient condition, under sufficient smoothness conditions. It is not in general an explicite condition.

- Multiplier Geometric Condition (MGC) J.-L. Lions 1988, Ho, Zuazua. :

$\exists\ x_0 \in \mathbb{R}^n$, such that $\omega \supset \Gamma(x_0)$ where $\Gamma(x_0)$ is the part of Γ that an observator placed at x_0 does not see. It is thus sufficient to control the part of the boundary that the observator does not see. It is a sufficent condition, but it is not a necessary one in general. It requires less regularity thatn (GCC). It is explicite.

Generalized to multiple observators by K. Liu 1997 (for the internal control case only).
(MGC) :



Sommaire

Several phenomenas in engineering, mechanics, physics, biology ... are modelized by dynamical systems coupling PDE's.

It is moreover important either for cost reasons, feasibility or for applications such as

- **insensitizing control** (J.-L. Lions 1989)
- **simultaneous control**
- **synchronization of systems** (Rao & Li Ta Tsien) ...

to control these systems acting only on certain equations and not on all of them.

Hence, we have in this case only a limited number of controls. This is indirect control (and by duality, indirect observability). This can also be studied from the closed loop system point of view (stabilization).

A general issue for these classes of coupled systems, is to determine, whether it is possible :

- **under the action of a reduced number of feedbacks**
- to stabilize the full system

...

The answers and the quality of these answers to this question will depend on the hypotheses made on:

1. degree of generality of the concerned PDEs

2. same dynamics or not for the considered systems: for instance, does one have $A_1 = A_2$ or not, i.e. do we couple wave equation with the same speeds or not, waves with plates ...?

3. bounded or unbounded operators of stabilization/control/observation : i.e. is the control globally distributed (B^* is then bounded and coercive in H), locally distributed (B^* is then bounded in H but is not coercive, ot localized on the boundary (B^* is then unbounded in H)

4. Properties of the coupling operator: coercive or not

5. differential order of the coupling operator?

6. geometric localization of the respective active regions of the coupling and feedback operators

$\vdots$

Indirect stabilization and the method of higher order energy : a research axis initiated in 1999 in Strasbourg

Model problems : boundary stabilization of coupled systems :

Let Ω be a bounded open subset of $\mathbb{R}^n$ with a sufficiently smooth boundary Γ, $\Gamma_1 \subset \Gamma$, $b, \eta \in L^\infty(\Gamma_1)$, $b \geqslant b_- > 0, \eta \geqslant \eta_- > 0$ p.p. in Γ_1, $\alpha \in (0, \infty)$.

We consider the following coupled system :

$$\begin{cases} u_{tt} - \Delta u + \alpha v = 0 \quad \text{in } \Omega \times (0, +\infty), \\ v_{tt} - \Delta v + \alpha u = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u = 0 \quad \text{in } \Gamma_0 \times (0, +\infty), \ v = 0 \quad \text{in } \Gamma \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} + \eta(x) u + b(x) u_t = 0 \quad \text{in } \Gamma_1 \times (0, +\infty), \\ (u, u_t)(0) = (u^0, u^1), \ (v, v_t)(0) = (v^0, v^1) \quad \text{in } \Omega\,. \end{cases}$$

This system can be reformulated in abstract form

$$\begin{cases} U' + \mathcal{A}_\alpha U = 0, \\ U'(0) = (u, v, u_t, v_t)(0), \end{cases}$$

with $U = (u, v, u_t, v_t)$ and where U' stands for the time derivative of U.

The total energy of a solution is given by:

$$E_U(t) = \frac{1}{2}\left(\int_\Omega (u_t^2 + |\nabla u|^2 + v_t^2 + |\nabla v|^2)dx + \int_{\Gamma_1} \eta u^2 d\sigma\right) + \alpha \int_\Omega uvdx$$

The strong solutions satisfy the dissipation relation

$$\frac{dE_U}{dt} = -\int_{\Gamma_1} b u_t^2 d\sigma \leqslant 0 \rightsquigarrow \text{ dissipation of } E$$

• The total energy is dissipated thanks to the boundary feedback in the equation fo u

• However, let us note that: no feedback appears in the v equation

• The equations are coupled if $\alpha \neq 0$, and decoupled otherwise

• Hence if $\alpha = 0$, one cannot hope to stabilize the full system since the v equation is then conservative

• If $\alpha \neq 0$ does the single present boundary feedback on the u equation can stabilized the full system, at least under certain conditions?

• If the answer is yes, it cannot be due to a perturbation argument with respect to the case $\alpha = 0$, and if yes, which type of decay one can hope?

Applying the Lasalle invariance principle, we could deduce the strong stabilization via a unique continuation argument, that is if we can prove that for every solution of

$$\begin{cases} u_{tt} - \Delta u + \alpha v = 0 \quad \text{in } \Omega \times (0, +\infty), \\ v_{tt} - \Delta v + \alpha u = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u = 0 \quad \text{in } \Gamma_0 \times (0, +\infty), \ v = 0 \quad \text{in } \Gamma \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} + \eta(x) u = 0 \quad \text{in } \Gamma_1 \times (0, +\infty), \\ (u, u_t)(0) = (u^0, u^1), \ (v, v_t)(0) = (v^0, v^1) \quad \text{in } \Omega, \end{cases}$$

such that

$$b(x) u_t = 0 \quad \text{in } \Gamma_1 \times (0, +\infty),$$

we have $u \equiv v \equiv 0$ in $\Omega \times (0, +\infty)$.

Second model example: the boundary stabilization of two Kirchhoff plates:

$$\begin{cases} u_{tt} + \Delta^2 u + \alpha v = 0 \quad \text{in } \Omega \times (0, +\infty), \\ v_{tt} + \Delta^2 v + \alpha u = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u = \dfrac{\partial u}{\partial \nu} = 0 \quad \text{in } \Gamma_0 \times (0, +\infty), \; v = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{in } \Gamma \times (0, +\infty), \\ \Delta u + (1-\mu) B_1 u = -k \dfrac{\partial u_t}{\partial \nu}, \quad \text{in } \Gamma_1 \times (0, +\infty), \\ \dfrac{\partial \Delta u}{\partial \nu} + (1-\mu) \dfrac{\partial B_2 u}{\partial \tau} = l \nu u_t \quad \text{in } \Gamma_1 \times (0, +\infty), \\ (u, u_t)(0) = (u^0, u^1), \; (v, v_t)(0) = (v^0, v^1) \quad \text{in } \Omega\,. \end{cases}$$

where $\Omega \subset \mathbb{R}^2$, $k \geqslant k_- > 0$ and $l \geqslant l_- > 0$ p.p. in Γ_1, $\mu \in]0, 1/2[$ is the Poisson coefficient, B_1, B_2 are the boundary operators defined by

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx}, \; B_2 u = (\nu_1^2 - \nu_2^2) u_{xy} + \nu_1\nu_2(u_{xx} - u_{yy})$$

The total energy of a solution U is given by

$$E_U(t) = \frac{1}{2}\left(\int_\Omega (u_t^2 + a(u,u) + v_t^2 + a(v,v)dxdy\right) + \alpha \int_\Omega uvdx$$

où

$$a(u,z) =: \int_\Omega (u_{xx}z_{xx} + u_{yy}z_{yy} + \mu(u_{xx}z_{yy} + u_{yy}z_{xx}) + 2(1-\mu)u_{xy}z_{xy})\,dxdy$$

For strong solutions, we have the dissipation relation

$$\frac{dE_U}{dt}(t) = -\int_{\Gamma_1}\left(lu_t^2 + k\left|\frac{\partial u}{\partial \nu}\right|^2\right)d\sigma \leqslant 0.$$

Goal: to be able to treat these model problems, but also other ones, by a fundamental approach. That is with a unified approach to be determined. Meaning that looking for the mathematical tools is part of the game.

And that we look for a robust methodology that can be adapted to treat various classes of coupled PDEs $\rightsquigarrow$ through a direct approach to the evolution PDE.

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Theorem (A. CRAS 1999)

For these two model examples but also for the system of two wave equations with different speed of propagation (under conditions) and for $\alpha > 0$ sufficiently small, one can prove the strong stabilization:

$$E_U(t) \to 0 \quad t \to \infty$$

for every initial data in the energy space.
One can also prove that for smoother initial data, the energy is polynomially stabilized, that is for any integer $n \geqslant 1$ the initial data are in $D(\mathcal{A}_\alpha^{2n})$ then we have

$$E_U(t) \leqslant c(n,\alpha) ||(u^0, u^1, v^0, v^1)||^2_{D(\mathcal{A}_\alpha^{2n})} \, t^{-n}, \quad \forall \, t > 0,$$

where $\mathcal{A}_\alpha$ is the infinitesimal generator of the underlying semigroup associated to the coupled system.

These results have a broader scope and can be generalized to abstract coupled systems of the form

$$\begin{cases} u_1'' + A_1 u_1 + Bu_1' + \alpha P u_2 = 0, \\ u_2'' + A_2 u_2 + \alpha P^* u_2 = 0, \\ (u_i, u_i')(0) = (u_i^0, u_i^1), i = 1, i = 2 \end{cases}$$

in the following abstract framework

Let V_i, $i = 1, 2$ and H be Hilbert spaces such that the embeddings $V_i \subset H$ are continuous , denses and compacts for $i = 1, 2$, and the embedding $V_2 \subset V_1$ is continuous.

H is identified with its dual space, so that the embeddings $V_i \subset H \subset V_i'$ are continuous, denses and compacts.

The operator A_i, $i = 1, 2$ is the duality operator between V_i and V_i' defined by

$$< A_i w, z >_{V_i', V_i} = (w, z)_{V_i} \ \ \forall w , z \in V_i \,,$$

We suppose that $V_1 \supset V_0$ where V_0 is a closed subset, equipped with the norm and scalar product induced by those of V_1.

We denote by i the canonical injection of V_0 in V_1 and by P_0 the projection operator of V_1 on V_0. Hence for every $u_1 \in V_1$, $P_0 u_1$ is characterized by

$$\begin{cases} < A_1 i(P_0 u_1), i(\phi) >_{V_1', V_1} = < A_1 u_1, i(\phi) >_{V_1', V_1} , \\ \forall\, \phi \in V_0 \,, \forall\, u_1 \in V_1 \,, \\ P_0 u_1 \in V_0 \,. \end{cases}$$

We further define an operator $A_0 : V_0 \mapsto V_0'$ by

$$< A_0 \phi, \psi >_{V_0', V_0} = < A_1 i(\phi), i(\psi) >_{V_1', V_1} \quad \forall\, \phi \,, \psi \in V_0 \,.$$

A_0 is the duality operator between V_0 and V_0'.

We assume that B is a bounded linear from V_1 in V_1' satisfying

$$< Bu, u >_{V_1', V_1} \geqslant 0 \,, \; < Bu, z >_{V_1', V_1} = < Bz, u >_{V_1', V_1} \quad \forall\, u \,, z \in V_1 \,,$$

We further assume that P is a bounded linear operator in H, and $\alpha > 0$ a coupling parameter.

We denote by the same symbol the unbounded operator $A_i \;: D(A_i) \subset H \to H$ where $D(A_i) = \{\varphi \in V_i, A_i\varphi \in H\}$ for $i = 0, 1, 2$.

We consider the operator $\mathcal{A}_1$ defined by
$D(\mathcal{A}_1) = \{(u, q) \in V_1 \times V_1, A_1 u + Bq \in H\}$, and $\mathcal{A}_1(u, q) = (-q, A_1 u + Bq)$.

For $i = 0$ and $i = 2$, we denote by $\mathcal{A}_i$, the operators defined by
$D(\mathcal{A}_i) = D(A_i) \times V_i$, and $\mathcal{A}_i(u, q) = (-q, A_i u)$ and we define the partial energies by

$$e_i(t) = \frac{1}{2}\left(|u_i'|_H^2 + |u_i|_{V_i}^2\right)\,, \text{ for } i = 1, 2.$$

We assume the following hypotheses

$$(H1)\begin{cases}\exists \gamma_i > 0 \; i=1,2,3 \text{ such that for all } f_1 \in \mathcal{C}^1([0,+\infty);H) \\ \text{ and all } 0 \leqslant S \leqslant T \text{, the solution } (u_1,v_1) \text{ of} \\ (u_1,v_1)' + \mathcal{A}_1(u_1,v_1) = (0,f_1) \\ (u_1,v_1) = (u_1^0,v_1^0) \in D(\mathcal{A}_1)\,, \\ \text{satisfies} \\ \int_S^T e_1(t)\,dt \leqslant \gamma_1(e_1(S)+e_1(T)) + \gamma_2 \int_S^T ||f_1(t)||_H^2\,dt + \\ \gamma_3 \int_S^T < Bv_1, v_1 >_{V_1',V_1} \; dt\,. \end{cases}$$

and

$$(H2)\begin{cases} < Bu_1, i(\phi) >_{V_1',V_1} = 0 \quad \forall\, \phi \in V_0,\ \forall\, u_1 \in V_1\,, \\ \text{et} \\ \exists \beta > 0\,,\ ||u_1 - P_0 u_1||_H^2 \leqslant \beta\, \langle Bu_1, u_1 \rangle_{V_1',V_1} \quad \forall\, u_1 \in V_1\,. \end{cases}$$

The first hypothesis implies in particular that $-\mathcal{A}_1$ generates an exponentially stable semigroup.

This hypothesis concern a scalar equation. One can prove that the hypothesis is satisfied for most of the PDE's (e.g. waves, Kirchhoff plates,...).

The second hypothesis implies that B satisfies a "weak type coercivity" property.

One can also prove that this hypothesis is satisfied for most of the usual examples (e.g. waves, Kirchhoff . . .) with a boundary feedback. It is based on elliptic estimates.

We further assume the following compatibility hypothesis

$$\begin{cases} V_2 \subset V_0 \text{ with continuous embedding, and,} \\ \text{there exists an operator } C \text{ in } H \text{ such that} \\ CV_2 \subset V_0\,, CD(A_2) \subset D(A_0) \text{ et } A_0 C u_2 = C A_2 u_2 \quad \forall u_2 \in D(A_2)\,. \end{cases}$$

We further assume that

$$\exists \gamma > 0 \text{ such that } (Pu_2, Cu_2)_H \geqslant \gamma ||u_2||_H^2 \quad \forall\, u_2 \in H\,.$$

Remark

In the case of identical dynamics for the two equations of the coupled system, i.e. if $A_0 \equiv A_2$, these hypotheses are satisfied with $C = Id$ and P being a multiplication operator by a function which is bounded from below by a strictly positive constant. It is more restrictive if $A_0 \neq A_2$, it requires then, in particalar, the existence of a common basis of eigenfunctions for A_0 and A_2.

Then under these hypotheses, we prove

Theorem (A. SICON 2002)

$\exists\ \alpha_1 > 0$ such that $\forall\ 0 < |\alpha| < \alpha_1$, the energy $E(U)$ of the solutions U of the above abstract system satisfies for any $n \in$ mathbbN*

$$E(U(t)) \leqslant \frac{c(n,\alpha)}{t^n} \sum_{p=0}^{2n} E(U^{(p)}(0)) \quad \forall\ t > 0,\ \forall\ U^0 \in D(\mathcal{A}_\alpha^{2n})\ .$$

Moreover, if $U^0 \in \mathcal{H}$, then $E(U(t))$ converges to zero when t goes to ∞. Moreover this system is not exponentially stable.

The coercivity hypothesis on the coupling operator and the compatibility hypotheses on A_1 and A_2 allows us to prove

$$\int_S^T E(U(t))\,dt \leqslant c\Big(E(U(S)) + E(U'(S)) + E(U''(S))\Big), \quad \forall\ 0 \leqslant S \leqslant T.$$

where U' , U'', et $U^{(p)}$ denote respectively the time derivatives of order 1 , 2 and p of U.

This inspired the terminology **"method of higher order energy".**

- We show that this system is never exponentially stable by applying a result of A. & Cannarsa & Komornik JEE 2002.
- Note that the loss of exponential stability occurs for very different reasons than the ones where the Geometric Control Condition (GCC) of Bardos Lebeau Rauch (1992) is violated (trapped rays), see Lebeau 1996.
- What is really interesting here is that the indirect stabilization phenomenon studied in 1999 and 2002, the loss of exponential stability occurs also when the feedbacks are globally distributed and the coupling operator is constant (A. Cannarsa Komornik JEE 2002).
- So this is an important framework,of mathematical models for which the loss of exponential stability is not due to the fact that the support of the feedback does not satisfy (GCC).
- The mathematical tool developed in the "method of higher order energy can be formalized in an abstract setting as follows, provided that there is an underlying semigroup structure and a decaying property, related to dissipation of an energy.

Let A be the infimitesimal generator $\mathcal{C}^0$ semigroup $\exp(tA)$ in a Hilbert space $\mathcal{H}$, and $D(A)$ its domain. We set $U(t) = \exp(tA)U^0$, for $U^0 \in \mathcal{H}$.

Theorem (A. CRAS 1999 The method of higher order energy)

Assume that E is a functional defined on $\mathcal{C}([0, +\infty), \mathcal{H})$ such that for all U^0 in $\mathcal{H}$, $t \mapsto E(\exp(tA))$ is a nonincreasing function, locally a.c from $[0, +\infty)$ to $[0, +\infty)$. We, moreover, assume that there exists an integer $k \in \mathbb{N}^*$ and $c_p > 0$ for $p = 0, \dots k$ such that $E(U)$ satisfies the following integral inequality:

$$\int_S^T E(U(t))\, dt \leqslant \sum_{p=0}^{k} c_p E(U^{(p)}(S)) \quad \forall\, 0 \leqslant S \leqslant T\,, \forall\, U^0 \in D(A^k).$$

Then for all $n \in \mathbb{N}^*$ there exists $c = c_n > 0$ such that for all U^0 in $D(A^{kn})$ we have

$$E(U(t)) \leqslant c \sum_{p=0}^{kn} E(U^{(p)}(0)) t^{-n} \quad \forall t > 0 \quad , \forall\, U^0 \in D(A^{kn}).$$

The proof is simple, once one identifies the key intermediate inequality (to be proven by induction on n):

for all $U^0 \in D(A^{kn})$ et tout $n \in \mathbb{N}^*$ we have

$$\int_S^T E(U(\tau)) \frac{(\tau - S)^{n-1}}{(n-1)!}\, d\tau \leqslant c \sum_{p=0}^{kn} E(U^{(p)}(S)),$$

$$\forall\, 0 \leqslant S \leqslant T, \forall\, U^0 \in D(A^{kn}),$$

This result is related to interpolation properties in the domains of fractional powers of $\mathcal{A}$.

The proof of the inequality by induction is easy.

We saw, on one of the model example, that one of the crucial step is to prove the integral inequality involing higher order energies:

$$\int_S^T E(U(t))\,dt \leqslant c\Big(E(U(S)) + E(U'(S)) + E(U''(S))\Big), \quad \forall\, 0 \leqslant S \leqslant T.$$

For this, one has to understand how the "missing feedback" in the second equation, i.e. the fact to not act directly through a feedback on the second equation

is compensated by the fact that the action of the direct feedback of the first equation is transmitted to the second equation through the coupling

This is also at this stage that the restrictions on the dynamics, the fact to have globally distributed feedbacks or not, bounded or undbounded feedbacks, and properties of the coupling operator are extremly important to understand and to determine.

These are the questions and goals studied, in several works.

This work has inspired developments in functional analysis, more precisely to give resolvent characterizations on infinitesimal generators of semigroups which are not exponentially stables, but are polynomially stables, after a talk I gave in 2004, in Freudenstadt, Germany, at the 4th Euromaghrebian conference where Schnaubelt was very much interested in such behaviors from the functional analysis point of view:

- Batkai, Engel, Nagel & Schnaubelt Math. Nachr 2006, which has inspired the next works on the functional analysis approach:
- Batty & Duyckaerts JEE 2008, Borichev & Tomilov Math. Ann. 2010, ...Batty & Chill & Tomilov JEMS 2013...
- see also Liu & Rao JMAA 2007, Rao & Loreti SICON 2006

Framework:

Let A be the i.g of a $\mathcal{C}^0$ semigroup e^{tA} in the Hilbert space H such that $i\mathbb{R} \subset \rho(A)$ where $\rho(A)$ is the resolvent set of A.

We denote by $R(\lambda, A) = (\lambda I - A)^{-1}$, the resolvant of A where $\lambda \in \rho(A)$.

* "i.g." stands here for the infinitesimal generator.

Borichev and Tomilov (see also other papers):

Let $\beta > 0$. Then

$$||R(is, A)|| = O(|s|^{\beta}) \text{ when } s \to \infty \longleftrightarrow ||e^{tA}A^{-1}|| = O(t^{-1/\beta}), \quad t \to \infty.$$

- This type of results is important. It allows to give an upper estimate of the asymptotic behavior of the semigroup thanks to the analysis of the asymptotic beavior of the resolvent on the imaginary axis, and the exact behavior for applications if the excat behavior is proved:
  evolution problem $\longleftrightarrow$ elliptic probleme
- Note that it concerns only one step in the global work described in the previous slides.
- Comparable, in the method of higher order energy , to the step which allows to deduce form the integral inequality , the polynomial decay of the energy.
- However, the way people use the resolvent approach do not treat ***upstream***, and in a general way the PDE problem, and ***downstream*** the question to characterize for a large class of coupled systems , or PDE's the asymptotic behavior of the resolvent on the imaginary axis..

- Proving the required estimates on the resolvent for classes of coupled PDE's in a general framework is a challenging question. And in general people using this approach do it only on ponctual examples. **Note also that the approach does not apply for nonlinear equations. I will come back later to this.**

- Note that In Batkai et al. 2006, an almost characterization by the growth of the resolvent on the imaginary axis is obtained (in a Banach setting), but with a loss ε, and applications given on two examples of A. Cannarsa Komornik 2002 with globally distributed dampings, none on boundary dampings.

- The general approch, presented here, is a global one: upstream and downstream. It aims at identifying classes of systems, hypotheses on the dynamics A_1, A_2, the feedbacks operators B and the coupling operators P ...

  $\rightsquigarrow$ with into the bargain: general results with applications targeted in a "global way": same dynamics or not, same boundary conditions or not, localized couplings or not, locally distributed or boundary feebacks, non linear feedbacks or not ...

Let us come back to our indirect stabilization problem.

We proved that $E(U)$ satisfies a (nondifferential) integral inequality with $k = 2$ by the "method of higher order energy"(A.1999).

We thus can apply our above abstract result and prove the claimed polynomial energy decay.

This abstract framework applies to the cases $A_0 = A_2$, i.e. the two coupled wave equations with the same speed of propagation, the coupled Kirchhoff equations and in the case where $A_0 \neq A_2$, for the wave equations with distinct speeds of propagation but under restrictive geometric hypotheses on Ω on the ratio of the speeds.It applies also to wave coupled with a Petrowsky (bilaplacien) equation.

Thus, we could treat cases with different dynamics

Can we treat, in a global way, examples allowing more general disctinct dynamics?

Yes, we can, in the case of globally distributed feedbacks

Sommaire

We consider the coupled system with $P = Id$ (can be extended)

$$\begin{cases} u_1'' + A_1 u_1 + Bu_1' + \alpha u_2 = 0, \\ u_2'' + A_2 u_2 + \alpha u_2 = 0, \\ (u_i, u_i')(0) = (u_i^0, u_i^1), i = 1, i = 2 \end{cases}$$

assuming:

- For $i = 1, 2$, $A_i \; : D(A_i) \subset H \to H$ is a closed operator with dense domain with

$$A_i = A_i^* \;, \qquad \langle A_i x, x \rangle \geqslant \omega_i |x|^2 \quad \forall x \in D(A_i) \text{ where } \omega_i > 0.$$

- $B \in \mathcal{L}(H)$ $B = B^*$, $\qquad \langle Bx, x \rangle \geqslant \beta |x|^2 \quad \forall x \in H$ where $\beta > 0$.
- $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$.

B is now assumed to be a bounded and coercive operator in H (globally distribued feedback operator).

Model example

Let Ω be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary Γ and consider the coupled system

$$\begin{cases} u_{tt} - \Delta u + \kappa u - \kappa v + \beta u_t = 0 \quad \text{in } \ \Omega \times (0, +\infty), \\ v_{tt} - \Delta v + \kappa v - \kappa u = 0 \quad \text{in } \ \Omega \times (0, +\infty), \\ u = v = 0 \quad \text{in } \Gamma \times (0, +\infty), \\ (u, u_t)(0) = (u^0, u^1), \ (v, v_t)(0) = (v^0, v^1) \quad \text{in } \Omega \, . \end{cases}$$

where $\kappa > 0$ and $\beta > 0$.

But, we can consider as well wave-wave systems with any positive speeds of propagation (which can also depend on space), wave-plate, or plate-wave, elasticity . . .

We assume the following compatibiity condition between the two operators A_1 and A_2, i.e. we assume that there exists an integer $j \geqslant 2$ such that

$$|A_1 u| \leqslant c|A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2}) .$$

Then we prove

Theorem (Alabau Cannarsa Komornik JEE 2002)

• *If $U_0 \in D(\mathcal{A}^{nj})$ for an integer $n \geqslant 1$, then there exists a constant $c_n > 0$ such that the solution U satisfies*

$$E(U(t)) \leqslant \frac{c_n}{t^n} \sum_{k=0}^{nj} E(U^{(k)}(0)) \qquad \forall t > 0$$

• *For all $U_0 \in \mathcal{H}$ we have*

$$E(U(t)) \to 0 \quad quand \quad t \to \infty.$$

• *Moreover the system is never exponentially stable.*

This shows that the loss of exponential stability is not due to a violation of (GCC) as in the scalar case, but is due here to the quality of theinformation transmitted through the coupling operator from the directly stabilized equation to the one which is not directly stabilized.

The proof is based on the proof of the following (nondifferential) integral inequality

$$\int_0^T E(U(t))dt \leqslant c \sum_{k=0}^{j} E(U^{(k)}(0)) \qquad \forall\, T \geqslant 0\,,$$

where j is as in our compatibility hypothesis.

We then apply our previous result and deduce our claimed polynomial decay.

Sommaire

Next classes to be explored: can we extend these results to hybrid boundary conditions?

If the answer is yes, these type of classes of problems will exhibit another type of example of operators A_1 et A_2 with distinct dynamics.

An example :

Let Ω be a bounded open subset of $\mathbb{R}^n$ with a smoothboundary Γ. We consider the coupled system

$$\begin{cases} u_{tt} - \Delta u + \kappa u - \kappa v + \beta u_t = 0 \quad \text{in } \ \Omega \times (0, +\infty), \\ v_{tt} - \Delta v + \kappa v - \kappa u = 0 \quad \text{in } \ \Omega \times (0, +\infty), \\ u = 0, \dfrac{\partial v}{\partial \nu} + v = 0 \quad \text{in } \Gamma \times (0, +\infty), \\ (u, u_t)(0) = (u^0, u^1), \ (v, v_t)(0) = (v^0, v^1) \quad \text{in } \Omega\,. \end{cases}$$

where $\kappa > 0$ and $\beta > 0$.

$$D(A_1) = H^2(\Omega) \cap H_0^1(\Omega)\,,\; A_1 u = -\Delta u + \kappa u\,,\; u \in D(A_1)\,,$$

and

$$D(A_2) = \{u \in H^2(\Omega)\,, \frac{\partial u}{\partial \nu} + u = 0 \text{ in } \partial\Omega\}\,,$$

$$A_2 u = -\Delta u + \kappa u\,,\; u \in D(A_2)\,.$$

One can prove that there does not exist an integer $j \geq 2$ such that:

$$D(A_2^{j/2}) \subset D(A_1)\,.$$

Thus,the operators A_1 and A_2 do not satisfy the compatibility condition of A. & Cannarsa & Komornik 2002.

In A.-B. & Cannarsa & Guglielmi MCRF 2011, we prove the polynomial stability under a different compatibility condition, that includes as a particular case the above example.

We recall that the compatibility condition of A. & Cannarsa & Komornik 2002 is:

$$D(A_2^{j/2}) \subset D(A_1)\,,\, j \in \mathbb{N}\,,\, j \geqslant 2\,.$$

The new one is:

$$D(A_2) \subset D(A_1^{1/2}) \qquad \text{et} \qquad |A_1^{1/2}u| \leqslant c|A_2 u| \quad \forall u \in D(A_2),$$

which is satisfied for various systems with hybrid boundary conditions, in particular for our example of with Dirichlet Robin boundary conditions.

Under this new compatibility condition, we prove that the solutions U satisfy the integral inequality

$$\int_0^T E(U(t))dt \leqslant c_1 \sum_{k=0}^{4} E(U^{(k)}(0)) \qquad \forall\, T > 0\,,\; U_0 \in D(\mathcal{A}^4)\,.$$

We deduce from our abstract Lemma (A. 1999) that

$$E(U(t)) \leqslant \frac{c_n}{t^n} \sum_{k=0}^{4n} E(U^{(k)}(0)) \qquad \forall t > 0$$

for all $n \geqslant 1$ and $U_0 \in D(\mathcal{A}^{4n})$.

Using interpolation theory, we further prove that

$$E(U(t)) \leqslant \frac{c_n}{t^{n/4}} \sum_{k=0}^{n} E(U^{(k)}(0)) \qquad \forall t > 0$$

for all $n \geqslant 1$ and $U_0 \in D(\mathcal{A}^n)$.

In particular, for $n = 1$, we have $U_0 \in D(\mathcal{A})$,

$$e_1(t) + e_2(t) \leqslant \frac{c}{t^{1/4}} \|U_0\|^2_{D(\mathcal{A})} \qquad \forall t > 0\,,$$

and there exists $c_1 > 0$ such that

$$\|U_0\|^2_{D(\mathcal{A})} \leqslant c_1 \left(|A_1 u^0|^2 + |A_1^{1/2} u^1|^2 + |A_2 v^0|^2 + |A_2^{1/2} v^1|^2 \right)\,.$$

• In A. & Léautaud 2011 we generalize A. 1999 et 2002 and A. & Cannarsa & Komornik 2002 using the method of higher order energy to the cases with same dynamics, i.e. $A_1 = A_2$, bounded as well as unbounded feedback operators,

$\rightsquigarrow$ to the case of partially coercive coupling operators

i.e. assuming now for our classes of abstract upstream coupled systems that

$$\rightsquigarrow \ \exists\, \Pi_P \in \mathcal{L}(H) \text{ and } p_- > 0 \text{ such that } \langle Pu, u\rangle_H \geqslant p_- \, |\Pi_P u|_H^2 \quad \forall\, u \in H$$

Applications to wave coupled systems with same speed of propagation, localized feedback and

$$P = p(x)Id, B = b(x)Id$$

assuming a condition of type (MGC) on ω_p and ω_b where $p \geqslant p_- > 0$ on ω_p and $b \geqslant b_- > 0$ on ω_b.
In particular, in dimension 1 the result is valid for arbitrary non empty open subsets ω_p, ω_b, and thus for cases for which $\omega_b \cap \omega_p = \emptyset$.

Case of wave-wave system with same speed and localized feedback

In A. & Léautaud 2011, we prove that if $\omega_b \cap \omega_p \neq \emptyset$, then

$$E(U(t)) \leqslant \frac{c_n}{(\ln(2+t))^n} \sum_{i=0}^{n} E(U^i(0)), \quad \forall\, t > 0,$$

for all $U^0 \in (D(-\Delta_D)^{(n+1)/2})^2 \times (D(-\Delta_D)^{n/2})^2$, where $-\Delta_D$ denotes the operator $-\Delta$ with Dirichlet boundary conditions.

The proof relies on resolvent estimates due to Lebeau 1996 and Lebeau & Robbiano 1998, and an interpolation inequality for an associated elliptic system due to Léautaud 2010.

This interpolation inequality is not known in the case $\omega_b \cap \omega_p = \emptyset$, for which even the strong stablity is not proved.

In A. & Léautaud 2013, we extend these results by indirect arguments – i.e. via an observability result for the free dual system– to geometric situations in multi-D when $\omega_b \cap \omega_p = \emptyset$.

Further extensions:

Nonlinear indirect stabilization results:

• in A. NoDEA 2007 for the Timoshenko coupled system with nonlinear damping
• A. & Wang & Yu ESAIM COCV 2014 for wave and Petrowsky coupled systems with nonlinear locally distributed feedbacks of the form $Bu_t = b(x)\rho(u_t)$ with $\rho(s)s \geqslant 0$ for all $s \in \mathbb{R}$ and ρ with arbitrary growth close to 0.

The higher order energy method introduced in A. 1999 and further developed with co-authors in several directions allow us to treat in a unified way indirect damping mechanisms for coupled systems with either same dynamics or not, bounded as well as unbounded feedback operators, different types of couplings: coercive, or partially coercive, of higher order, linear or nonlinear.

Several open problems are still open on these questions:

- larger classes of coupled systems, more unstructured
- semilinear extensions
- time dependent coefficients
- localization of stabilization and coupling regions
- numerical analysis

⋮

Thanks for your attention.