Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Compressible Phase Change Flows and the Existence of Transition Profiles

Gabriele Witterstein

Mohrenstrasse 39 · 10117 Berlin · Germany · Tel. +49 30 20372 0 · www.wias-berlin.de

Content

1 General Model

- Mass and Momentum Conservation
- Evolution of the Phase Parameter
- Thermodynamical Correctness, Entropy Principle

2 Compressible / Incompressible Model

Two mass densities ϱ_δ^1, ϱ_δ^2,

$$\partial_t \varrho_\delta^1 + \text{div}\,(\varrho_\delta^1 v) = +\boldsymbol{\tau}_\delta$$
$$\partial_t \varrho_\delta^2 + \text{div}\,(\varrho_\delta^2 v) = -\boldsymbol{\tau}_\delta$$
$$\partial_t(\varrho v) + \text{div}\,(\varrho v \otimes v + T_\delta) = 0,$$

where the total mass $\varrho := \varrho_\delta^1 + \varrho_\delta^2$

Reaction rate $\boldsymbol{\tau}_\delta$:

$$\varrho_\delta^1 \xrightarrow[+\boldsymbol{\tau}_\delta]{} \varrho_\delta^2 \qquad \varrho_\delta^1 \xleftarrow[-\boldsymbol{\tau}_\delta]{} \varrho_\delta^2$$

Tension tensor T_δ consists of pressure tensor P_δ and stress tensor S

$T_\delta = P_\delta - S, \quad P_\delta$ from the entropy principle,

$$S \equiv S(\varrho, \varphi) = \nu_1(\varrho, \varphi)\text{div}\, v\, \mathbb{I} + \nu_2(\varrho, \varphi)\big(\tfrac{1}{2}(\nabla v + (\nabla v)^{\mathrm{T}}) - \tfrac{1}{n}\text{div}\, v\, \mathbb{I}\big)\,.$$

$\varphi := \varrho_\delta^2 / \varrho$ phase parameter - represents mass fraction

$$-\boldsymbol{\tau}_\delta = \partial_t \varrho_\delta^2 + \text{div}\,(\varrho_\delta^2 v) = \partial_t(\varphi\varrho) + \text{div}\,(\varphi\varrho\, v) = \varrho\big(\partial_t\varphi + v\cdot\nabla\varphi\big)$$

$$\Longrightarrow \qquad \partial_t \varrho + \text{div}\,(\varrho v) = 0$$
$$\varrho\big(\partial_t\varphi + v\cdot\nabla\varphi\big) = -\boldsymbol{\tau}_\delta$$
$$\partial_t(\varrho v) + \text{div}\,(\varrho v \otimes v + T_\delta) = 0$$

Aim

Derive a Diffuse Interface Model which approximates a Sharp Interface Model consisting of Compressible / Incompressible Flow

Free Energy Density $\varepsilon > 0, \delta > 0,$ later ε as function of δ

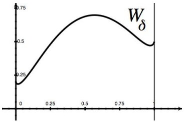
$$f_\delta(\varrho, \varphi, \nabla\varphi) := \frac{1}{\varepsilon} W^\varepsilon(\varrho, \varphi) + \frac{1}{\delta} W^\delta(\varrho, \varphi) + \delta h(\varrho) \frac{|\nabla\varphi|^2}{2} + U(\varrho, \varphi),$$

where W^δ double-well potential, local minima $m_1 = 0$ and $m_2 = 1$

and $h(\varrho) > 0$ is an arbitrary function,

$W^\varepsilon := (\varrho - \varrho^\star)^2 \varphi^2$

$\varrho^\star =$ constant > 0 (Incompressible density)



U Free energy of the bulk phases

$\varphi := \varrho_\delta^2/\varrho$ mass fraction, $\boldsymbol{\tau}_\delta := \eta_\delta(\varrho, \varphi)\mu,$ $\eta_\delta(\varrho, \varphi) > 0,$ μ chemical potential

$$\Longrightarrow \qquad \varrho\big(\partial_t \varphi + v \cdot \nabla\varphi\big) = -\eta_\delta(\varrho, \varphi)\mu$$

For example $\mu := \frac{\delta f_\delta}{\delta \varphi}$ or $\mu := -\triangle \frac{\delta f_\delta}{\delta \varphi}$

[Alt 2009], Objectivity

Isothermal case: entropy principle is equivalent to the free energy inequality;

Total free energy density: $f_{\rm tot} = f_\delta + \frac{1}{2}\varrho|v|^2\,,$ $\qquad f_\delta = f_\delta(\varrho,\varphi,\nabla\varphi)$

Free energy flux: $\psi_{\rm tot} = f_{\rm tot}v + T_\delta^{\rm T} v + \psi\,,$ where $\psi = -(\partial_t\varphi + v\cdot\nabla\varphi) f_{\delta|\nabla\varphi}\,,$

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the solution (ϱ, v, φ) of the system satisfy the inequality

$$\partial_t f_{\rm tot} + \mathrm{div}(f_{\rm tot}v + T_\delta^{\rm T} v + \psi) = \partial_t f_\delta + \mathrm{div}(f_\delta v + \psi) + Dv\cdot T_\delta$$
$$= -\frac{1}{\varrho}\eta_\delta(\varrho,\varphi)\Big(\frac{\delta f_\delta}{\delta\varphi}\Big)^2 - Dv\cdot S \qquad \leq \quad 0\,.$$

- as usual, $\nu_1, \nu_2 > 0$, so that $Dv\cdot S \geq 0\,;\, \eta_\delta > 0\,;$
- and where Pressure tensor P_δ:

$$P_\delta \equiv P_\delta(\varrho,\varphi,\nabla\varphi) = (-f_\delta + \varrho f_{\delta|\varrho})\,\mathbb{I} + \nabla\varphi\otimes f_{\delta|\nabla\varphi}$$
$$= \Big(\frac{1}{\varepsilon}p_{W^\varepsilon}(\varrho,\varphi) + \frac{1}{\delta}p_{W^\delta}(\varrho,\varphi) + p_U(\varrho,\varphi) + \delta p_h(\varrho)\frac{|\nabla\varphi|^2}{2}\Big)\,\mathbb{I} \;+\; \delta h(\varrho)\,\nabla\varphi\otimes\nabla\varphi$$

with $\quad p_U(\varrho,\varphi) := -U(\varrho,\varphi) + \varrho U_{|\varrho}(\varrho,\varphi), \quad p_{W^\delta},\;\; p_h$ dito

the last two summands (terms with $\nabla\varphi$) stand for the **'surface tension'** between the different phases;

1 General Model

2 Compressible / Incompressible Model

- Diffusive Interface Model
- Sharp Interface Model - Bulk Region
- Sharp Interface Model - Interface
- Interfacial Profile

Assume: $W^\delta(\varrho,\varphi) := \varrho W(\varphi), \quad W(m_1) \neq W(m_2)$
$\eta_\delta = 1/\delta, \qquad W^\varepsilon(\varrho,\varphi) := (\varrho - \varrho^\star)^2\varphi^2$

Connection of both parameters

$$\delta \equiv \delta(\varepsilon) = \varepsilon^2$$

The incompressible limes slower than the sharp interface limes

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,, \tag{1}$$

$$\partial_t(\varrho v) + \operatorname{div}(\varrho v \otimes v) + \nabla\Big(\frac{1}{\varepsilon} p_{W^\varepsilon}(\varrho,\varphi) + \overbrace{\frac{1}{\varepsilon^2} p_{\varrho W}(\varrho,\varphi)}^{=0} + p_U(\varrho,\varphi)\Big) \tag{2}$$

$$+\,\varepsilon^2 \operatorname{div}\Big(p_h(\varrho)\,\frac{|\nabla\varphi|^2}{2}\,\mathbb{I} + h(\varrho)\,\nabla\varphi \otimes \nabla\varphi\Big) = \operatorname{div}(S)\,,$$

$$\delta\,\varrho(\partial_t\varphi + v\cdot\nabla\varphi) = -\frac{1}{\varepsilon} W^\varepsilon_{|\varphi}(\varrho,\varphi) - \frac{1}{\varepsilon^2}\varrho W'(\varphi) - U_{|\varphi}(\varrho,\varphi) \tag{3}$$

$$+\,\varepsilon^2 \operatorname{div}(h(\varrho)\nabla\varphi)\,,$$

where

$$p_{W^\varepsilon}(\varrho,\varphi) = (\varrho - \varrho^\star)(\varrho + \varrho^\star)\varphi^2$$

Formal Matched Asymptotic Expansion in ε

$$\varepsilon \searrow 0$$

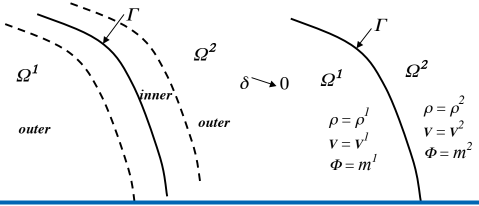
'Low Mach Number Limit' as $\varepsilon \searrow 0$, then $\delta = \varepsilon^2 \searrow 0$,

Outer Expansion

$$\varrho(t,x) = \varrho_0(t,x) + \varepsilon\varrho_1(t,x) + \delta\varrho_2(t,x) + \mathcal{O}(\varepsilon^2)$$
$$v(t,x) = v_0(t,x) + \varepsilon v_1(t,x) + \delta v_2(t,x) + \mathcal{O}(\varepsilon^2)$$
$$\varphi(t,x) = \varphi_0(t,x) + \varepsilon\varphi_1(t,x) + \delta\varphi_2(t,x) + \mathcal{O}(\varepsilon^2)$$

Interfacial Expansion $\varrho(t, y + \delta r \vec{n}_\Gamma(t,y)) = R(t,y,r)$

$$\varrho(t,x) = R(t,y,r) = R_0(t,y,r) + \varepsilon R_1(t,y,r) + \delta R_2(t,y,r) + \mathcal{O}(\varepsilon^2)$$
$$v(t,x) = V(t,y,r) = V_0(t,y,r) + \varepsilon V_1(t,y,r) + \delta V_2(t,y,r) + \mathcal{O}(\varepsilon^2)$$
$$\varphi(t,x) = \Phi(t,y,r) = \Phi_0(t,y,r) + \varepsilon\Phi_1(t,y,r) + \delta\Phi_2(t,y,r) + \mathcal{O}(\varepsilon^2)$$



ϱ_1, v_1, and φ_1
are perturbations in the bulk regions

R_1, V_1, and Φ_1
are perturbations in the interfacial regions

Sharp Interface Model - Bulk Region

For $t > 0$, there is an unknown free interface Γ_t, $\Omega = \Omega_t^1 \cup \Gamma_t \cup \Omega_t^2$,

Bulk Region: Compressible / Incompressible Navier-Stokes system

$$\partial_t \varrho^1 + \mathrm{div}(\varrho^1 v^1) = 0\,, \qquad \text{in } \Omega_t^1$$

$$\partial_t(\varrho^1 v^1) + \mathrm{div}(\varrho^1 v^1 \otimes v^1 + T^1(\varrho^1, m_1)) = 0\,,$$

$$\mathrm{div}\, v^2 = 0\,, \qquad \text{in } \Omega_t^2$$

$$\varrho^\star \left(\partial_t v^2 + \mathrm{div}(v^2 \otimes v^2)\right) + \mathrm{div}\, T^2(\varrho^\star, m_2) = 0\,,$$

$T^k = p^k\,\mathbb{I} - s^k$ tension tensor, p^k scalar pressure, s^k stress tensor,

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in Ω_t^1 : $\varphi = 0$

$$p^1 \equiv p_U(\varrho^1, m_1)\,,$$

$$s^1 \equiv s(\varrho^1, m_1) = \nu_1(\varrho^1, m_1)\mathrm{div}\, v^1\,\mathbb{I} + \nu_2(\varrho^1, m_1)\left(\frac{1}{2}(\nabla v^1 + (\nabla v^1)^{\mathrm{T}}) - \frac{1}{n}\mathrm{div}\, v^1\,\mathbb{I}\right)$$

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in Ω_t^2 : $\varphi = 1$

p^2 unknown ,

$$s^2 \equiv s(\varrho^\star, m_2) = \tilde{\nu}(\varrho^\star, m_2)\triangle v^2$$

Energy-Inequality

$$\int_0^t \int_\Omega \frac{1}{\varepsilon}(\varrho-\varrho^\star)^2\varphi^2 dxdt \le C_{\mathrm{data}} \quad \Rightarrow \quad \varrho = \varrho^\star \quad \text{in} \quad \Omega_t^2 \qquad \text{for} \quad \varepsilon \searrow 0$$

Then in Ω_t^2 $\qquad \varrho(t,x) \approx \varrho_0(t,x) + \varepsilon \varrho_1(t,x)$

Comparison order ε^0

$$\partial_t \varrho_0 + \mathrm{div}(\varrho_0 v_0) = 0\,,$$

$$\partial_t(\varrho_0 v_0) + \mathrm{div}(\varrho_0 v_0 \otimes v_0) + \nabla\left(p_U(\varrho_0,\varphi_0) + 2\varrho_0|\varphi_0|^2\varrho_1 + 2(\varrho_0-\varrho^\star)(\varrho_0+\varrho^\star)\varphi_0\varphi_1\right)$$
$$= \mathrm{div}\Big(\nu_1(\varrho_0,\varphi_0)\mathrm{div}\, v_0\, \mathbb{I} + \nu_2(\varrho_0,\varphi_0)\Big(\tfrac{1}{2}(\nabla v_0 + (\nabla v_0)^{\mathrm{T}}) - \frac{1}{n}\mathrm{div}\, v_0\, \mathbb{I}\Big)\Big)\,.$$

and $\quad \varrho_0(t,x) = \varrho^\star,\ \varphi_0 = 1$ and $\varphi_1 = 0 \quad$, then

$$\Rightarrow \quad \varrho^\star \mathrm{div}(v_0) = 0\,, \qquad \varrho^\star\big(\partial_t v_0 + \mathrm{div}(v_0\otimes v_0)\big) + \nabla\left(2\varrho^\star\varrho_1\right) = \tilde{\nu}\triangle v_0$$

$$\Rightarrow \quad p(t,x) = 2\varrho^\star\varrho_1(t,x)\,, \qquad \gamma := \frac{1}{2\varrho^\star}$$

that means in Ω_t^2 $\qquad \varrho(t,x) \approx \varrho^\star + \varepsilon\gamma p(t,x)$

Different Jump conditions on $\Gamma := \{(t,x) : x \in \Gamma_t\}$: $[v]_\Gamma \neq 0$ and $[\varrho]_\Gamma \neq 0$

Relative interface velocity in normal direction $\lambda := (v_\Gamma - v) \cdot \vec{n}_\Gamma$

$v_\Gamma \cdot \vec{n}_\Gamma$ normal interface velocity of Γ, κ curvature

Interface Conditions

$$
\left.\begin{aligned}
&[\varrho\lambda]_\Gamma = 0\,, && \text{on } \Gamma\,,\\
&-\varrho\lambda[v]_\Gamma + [T^k \vec{n}_\Gamma]_\Gamma = \mathrm{div}_y^\Gamma(\boldsymbol{\gamma}(\mathbb{I} - \vec{n}_\Gamma \otimes \vec{n}_\Gamma)) = \boldsymbol{\gamma}\kappa \cdot \vec{n}_\Gamma\, \vec{n}_\Gamma + \nabla^\Gamma \boldsymbol{\gamma}\,, && \text{on } \Gamma\,,\\
&[v]_\Gamma = \boldsymbol{\omega}\, \vec{n}_\Gamma\,, && \text{on } \Gamma\,,\\
&G(\varrho^1, \varrho^\star) = 0\,, && \text{on } \Gamma\,,
\end{aligned}\right\}
$$

where $\Phi = m_1$ in Ω_t^1 and $\Phi = m_2$ in Ω_t^2, and

$\boldsymbol{\gamma}$ surface tension, $\boldsymbol{\omega}$ capillarity force,

the two mass densities of the bulk phases are connected by G

$$
\boldsymbol{\omega} = \boldsymbol{\omega}[\varrho\lambda] := \int_{m_1}^{m_2} \frac{e_h(R)}{\tilde{\mu}(R,s)} \left\{ \frac{1}{2}\frac{1}{h(R)} \sqrt{2\int_{m_1}^{s} h(R)\, R\, W' d\bar{s}} \right\} ds\,,
$$

$$
\boldsymbol{\gamma} = \boldsymbol{\gamma}[\varrho\lambda] := \int_{m_1}^{m_2} \left(1 - \frac{1}{2}\frac{\mu_2(R,s)}{\tilde{\mu}(R,s)}\frac{e_h(R)}{h(R)}\right) \sqrt{\int_{m_1}^{s} 2\, h(R)\, R W' d\bar{s}}\, ds\,.
$$

Mass flux $M := \lambda^1 \varrho^1 = \lambda^2 \varrho^2$. The function R is the solution of the

Inner Interface Problem

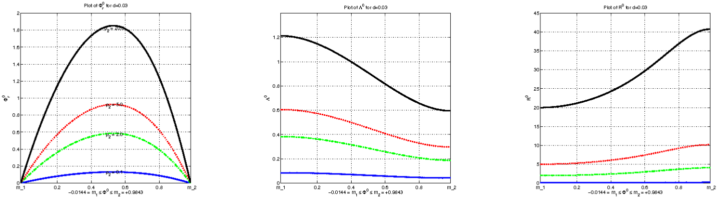
$$M^2 \partial_s \left(\left| h(R)\, g(R,.) \partial_s \left(\frac{1}{R} \right) \right|^2 \right) = 2\, h(R)\, R\, \partial_s W$$
$$\partial_s R(m_1) = 0\,, \qquad \partial_s R(m_2) = 0\,.$$

where $g = \widetilde{\nu}/e_h$, $e_h(z) = \left[z\, h(z)\right]'$ energetic part, and $e_h \gtrsim 0$.

Theorem

We consider $$W(\varphi) := W_0(\varphi) + \varepsilon W_1(\varphi)\,.$$

W_0 *and* W_1 *double-well potentials, twice continuously differentiable functions in* φ*, local minima on* m_1 *and* m_2*, and local maximum on* $s_a = (m_1 + m_2)/2$.
$W_0(m_1) = W_0(m_2)$ *and symmetric function on* $\varphi = s_a$, $W_1(m_1) \neq W_1(m_2)$, $h \equiv 1$.
$\Longrightarrow$
Then, there is a $\varepsilon_0 \in \mathbb{R}_+$ *and for each* $M \in \mathbb{R} \setminus \{0\}$ *has the Inner Interface Problem a solution* $R \in C^1([m_1, m_2]) \cap C^2((m_1, m_2))$ *as* $0 < \varepsilon < \varepsilon_0$.



Let $h \equiv 1$ and $\widetilde{\mu} \equiv 1$, then $g(R,.) = 1$. Then

$$M^2 \partial_s \left(\left| \partial_s \left(\frac{1}{R} \right) \right|^2 \right) = 2\, R\, \partial_s W \qquad \partial_s R(m_1) = 0\,, \qquad \partial_s R(m_2) = 0\,.$$

Then
$$\partial_s \left(\frac{1}{R} \right) = \frac{1}{|M|} \sqrt{\int_{m_1}^{s} 2RW' d\bar{s}} \quad \text{and}$$
$$0 = \partial_s \left(\frac{1}{R} \right)(m_2) = \frac{1}{|M|} \sqrt{\int_{m_1}^{m_2} 2RW' d\bar{s}} \quad ,$$

that is

$$\partial_s \left(\frac{1}{R} \right) = \frac{1}{|M|} \sqrt{\int_{m_1}^{s} 2RW' d\bar{s}}\,, \qquad \int_{m_1}^{m_2} RW' d\bar{s} = 0$$

Fixed point operator

Let $B \subset C^0([m_1, m_2])$ closed with

$$B := \left\{ u \in C^0([m_1, m_2]) : 0 < c_k \leq u \leq C_g, \int_{m_1}^{m_2} \frac{W'}{u} ds = 0, u \text{ monotone increasing} \right\}.$$

We introduce the operator

$$F : B \to C^0([m_1, m_2]) , \quad u_1 \in B \mapsto u_3 = F(u_1) \in C^0([m_1, m_2]) ,$$

where

$$u_2(s) = \frac{1}{|M|} \int_{m_1}^{s} \sqrt{\int_{m_1}^{\bar{s}} \frac{2W'}{u_1} d\bar{\bar{s}}} \; d\bar{s}$$

$$u_3(s) = u_2(s) + C(u_2) , \qquad \text{with} \quad \int_{m_1}^{m_2} \frac{W'}{u_2 + C(u_2)} ds = 0 .$$

1. Existence of $C(u_2)$ - intermediate value theorem
2. F is a non-expanding map - by choosing c_k and C_g
3. F is a contraction - for a weighted norm, Banach fixed point theorem

$$f_\delta(\varrho,\varphi,\nabla\varphi) := \tfrac{1}{\varepsilon}W^\varepsilon(\varrho,\varphi) + \tfrac{1}{\delta}W^\delta(\varrho,\varphi) + \delta h(\varrho)\frac{\left|\nabla\varphi\right|^2}{2} + U(\varrho,\varphi)\,,$$

Generalization to $h \not\equiv 1$

1. The function h fulfills a special growth condition

$$h(z_2){z_2}^{\frac{2-n}{n}} \le h(z_1){z_1}^{\frac{2-n}{n}} \qquad \text{for all} \quad z_1 \le z_2\,, \;\; z_1, z_2 \in \mathbb{R}_+\,,$$

   Then, $e_h(z) \neq 0$ for all $z \in \mathbb{R}$ and surface tension $\boldsymbol{\gamma} > 0$

2. $h > 0$ is an arbitrary continuously differentiable function and bounded from below, that is $h(z) \ge h_s > 0$ for all $z \in \mathbb{R}$

Theorem

Let $W(m_1) \neq W(m_2)$ *. Then, it exists* G *with* $\mathbf{G}(\boldsymbol{\varrho}^{\mathbf{1}}, \boldsymbol{\varrho}^\star) = \mathbf{0}$ *.*

Definition: *The function* G *is defined by solution* $R = R[M]$ *with*
$R[M(t,y)](m_1) = \varrho^1(t,y)\,, \quad R[M(t,y)](m_2) = \varrho^\star(t,y)\,.$

Perturbations

- Incompressible limes is a singular limit,

On Ω^1: compressible

$$\partial_t \varrho_1' + \operatorname{div}(\varrho_1' v_1 + \varrho_1 v_1') = 0\,,$$

$$\partial_t(\varrho_1' v_1 + \varrho_1 v_1') + \operatorname{div}(\varrho_1' v_1 \otimes v_1 + \varrho_1 v_1' \otimes v_1 + \varrho_1 v_1 \otimes v_1') + \nabla\big(p_{U_1|\varrho}(\varrho_1, m_1)\varrho_1'\big)$$

$$= \operatorname{div}(\mathrm{Lin}S)$$

On Ω^2: incompressible

$$\partial_t \varrho_2' + \operatorname{div}(\varrho_2' v_2 + \varrho^\star v_2') = 0\,,$$

$$\partial_t(\varrho_2' v_2 + \varrho^\star v_2') + \operatorname{div}(\varrho_2' v_2 \otimes v_2) + \varrho^\star \operatorname{div}(v_2' \otimes v_2 + v_2 \otimes v_2') + \nabla p' = \operatorname{div}(\mathrm{Lin}S)$$

- In compressible flows the sound velocity is finite, there are wave solutions for small perturbations
- In incompressible flows the sound velocity is infinite, but $\varepsilon > 0$ consider $\varrho = \varrho^\star + \varepsilon\gamma p$

Thank you for your attention !