Finite plasticity $\xrightarrow{\Gamma}$ Linearized plasticity

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Outline

Aim of the result:

Rigorously justify small-strain linearization in plasticity

1. Linearization in elasticity
2. Finite vs. linearized plasticity
3. Γ-convergence for rate-independent systems

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- Equilibrium: $\varphi : \Omega \to \mathbb{R}^d$

$$\min \left(\int_\Omega W_{\rm elast}(\nabla \varphi)\,{\rm d}x - \int_\Omega \boldsymbol{\ell}\cdot\varphi\,{\rm d}x \right)$$

Hyperleastic energy: $W_{\rm elast} : \mathbb{R}^{d\times d} \to [0,\infty]$, $W_{\rm elast} = \infty$ out of ${\rm GL}_+(d)$

Frame indifference: $W_{\rm elast}(\mathbf{RF}) = W_{\rm elast}(\mathbf{F})$ for all $\mathbf{R} \in {\rm SO}(d)$

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- Formal linearization: let $\varphi = \mathbf{id} + \varepsilon\mathbf{u}$ so that $\nabla\varphi = \mathbf{I} + \varepsilon\nabla\mathbf{u}$

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$$W_{\rm elast}(\mathbf{I} + \varepsilon\nabla\mathbf{u}) \overset{\text{Taylor}}{\approx} \underbrace{W_{\rm elast}(\mathbf{I})}_{=0} + \varepsilon\,\underbrace{\partial_{\mathbf{F}} W_{\rm elast}(\mathbf{I})}_{=\mathbf{0}} : \nabla\mathbf{u} + \frac{\varepsilon^2}{2}\nabla\mathbf{u} : \underbrace{\partial^2_{\mathbf{FF}} W_{\rm elast}(\mathbf{I})}_{=:\mathbb{C}}\,\nabla\mathbf{u}$$

$$\overset{\text{frame ind.}}{=} \frac{\varepsilon^2}{2}\nabla^{\rm s}\mathbf{u}:\mathbb{C}\nabla^{\rm s}\mathbf{u}$$

Linearization in elasticity

- Rigorous justification: [Dal Maso, Negri, Percivale 2002]

$$\frac{1}{\varepsilon^2}\int_\Omega W_{\rm elast}(\mathbf{I}+\varepsilon\nabla\mathbf{u})\,{\rm d}x \ \stackrel{\Gamma}{\to}\ \frac{1}{2}\int_\Omega \nabla^{\rm s}\mathbf{u}{:}\mathbb{C}\nabla^{\rm s}\mathbf{u}\,{\rm d}x$$

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- Γ-convergence:

$$f_\varepsilon \stackrel{\Gamma}{\to} f \iff \begin{cases} f(x) \leq \liminf_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \ \text{ for all } \ x_\varepsilon \to x & \text{(lim inf inequality)} \\ \forall y \ \exists y_\varepsilon \to y \ \ s.t. \ \lim_{\varepsilon\to 0} f_\varepsilon(y_\varepsilon) = f(y) & \text{(recovery sequence)} \end{cases}$$

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Fundamental Theorem of Γ-convergence

Let $f_\varepsilon \stackrel{\Gamma}{\to} f$, x_ε minimize f_ε and $x_\varepsilon \to x$

$\implies x$ minimizes f and $\min f_\varepsilon \to \min f$

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Let $f_\varepsilon \stackrel{\Gamma}{\to} f$, x_ε minimize f_ε and $x_\varepsilon \to x$

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One line proof by contradiction:

$$\limsup_{\varepsilon\to 0} f_\varepsilon(y_\varepsilon) \stackrel{recovery}{=} f(y) \stackrel{contrad.}{<} f(x) \stackrel{\liminf}{\leq} \liminf_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \stackrel{minimality}{\leq} \liminf_{\varepsilon\to 0} f_\varepsilon(y_\varepsilon)$$

Linearization in elasticity

- Application:

$$\mathbf{u}_\varepsilon \quad \text{minimize} \quad \mathbf{u} \mapsto \frac{1}{\varepsilon^2}\int_\Omega W_{\rm elast}(\mathbf{I} + \varepsilon \nabla \mathbf{u})\,{\rm d}x - \int_\Omega \boldsymbol{\ell}\cdot\mathbf{u}$$

Then, we have that $\mathbf{u}_\varepsilon \to \mathbf{u}$ which minimizes

$$\frac{1}{2}\int_\Omega \nabla^{\rm s}\mathbf{u}{:}\mathbb{C}\nabla^{\rm s}\mathbf{u}\,{\rm d}x - \int_\Omega \boldsymbol{\ell}\cdot\mathbf{u}$$

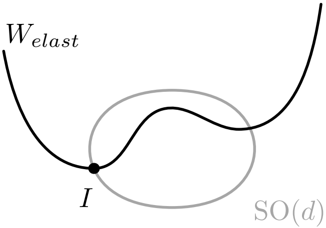
Equivalently, it solves

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{\ell} = \mathbf{0}, \quad \boldsymbol{\sigma} = \mathbb{C}\nabla^{\rm s}\mathbf{u}, \quad \text{(b.c.)}$$

Outline

- Coercivity: by geometric rigidity [Friesecke, James, Müller, 2002]

$$\infty > \int_\Omega W_{\rm elast}(\nabla\varphi)\,{\rm d}x \overset{assume}{\geq} c\int_\Omega d^2(\nabla\varphi, {\rm SO}(d)) \overset{rigidity}{\geq} \tilde{c}\int_\Omega |\nabla\varphi - \tilde{\mathbf{R}}|^2\,{\rm d}x$$



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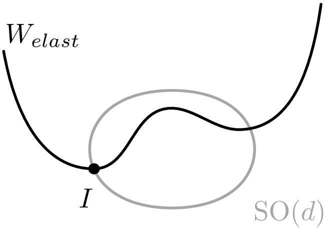

- Literature:
  - [Agostiniani, Dal Maso, DeSimone, 2011] refined assumptions
  - [Schimidt, 2008] multiwell energy
  - [Paroni, Tomassetti, 2009, 2011] non-stress-free reference config.

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Linearized plasticity

Linearized elasticity:

$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{\ell} = \mathbf{0}$

$\nabla^{\rm s} \mathbf{u} = \mathbb{C}^{-1} \boldsymbol{\sigma}$

(b.c.)

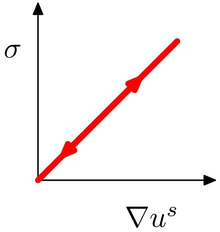
Linearized plasticity:

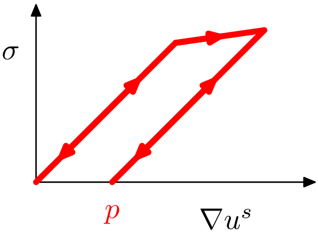
$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{\ell} = \mathbf{0}$

$\nabla^{\rm s} \mathbf{u} = \mathbb{C}^{-1} \boldsymbol{\sigma} + \mathbf{p} \qquad \mathbf{p} \in \mathbb{R}^{d \times d}_{\rm dev}$

$\partial R |\dot{\mathbf{p}}| + \mathbb{H}\mathbf{p} \ni \boldsymbol{\sigma}$

(b.c) + (i.c.)





Linearized plasticity

- Complementary energy:

$$\mathcal{E}(\mathbf{u},\mathbf{p},t) = \frac{1}{2}\int_\Omega (\nabla^{\rm s}\mathbf{u}-\mathbf{p}):\mathbb{C}(\nabla^{\rm s}\mathbf{u}-\mathbf{p})\,{\rm d}x + \frac{1}{2}\int_\Omega \mathbf{p}:\mathbb{H}\mathbf{p}\,{\rm d}x - \int_\Omega \ell(t)\cdot\mathbf{u}\,{\rm d}x$$

- Dissipation:

$$\mathcal{D}(\mathbf{p}_0,\mathbf{p}_1) = R\int_\Omega |\mathbf{p}_1-\mathbf{p}_0|\,{\rm d}x = \mathcal{R}(\mathbf{p}_1-\mathbf{p}_0)$$

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Linearized plasticity:

$$\partial_\mathbf{u}\mathcal{E}(\mathbf{u},\mathbf{p},t) \ni 0 \qquad \text{equilibrium}$$

$$\partial_{\dot{\mathbf{p}}}\mathcal{R}(\dot{\mathbf{p}})+\partial_\mathbf{p}\mathcal{E}(\mathbf{u},\mathbf{p},t) \ni 0 \qquad \text{plastic flow}$$

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- Energetic formulation: [Mielke-Theil, 2004]

Stability:

$$\mathcal{E}(\mathbf{u}(t),\mathbf{p}(t),t) \leq \mathcal{E}(\widehat{\mathbf{u}},\widehat{\mathbf{p}},t) + \mathcal{R}(\widehat{\mathbf{p}}-\mathbf{p}) \quad \forall(\widehat{\mathbf{u}},\widehat{\mathbf{p}})$$

Energy conservation:

$$\underbrace{\mathcal{E}(\mathbf{u}(t),\mathbf{p}(t),t)}_{\text{actual energy}} + \underbrace{\int_0^T \mathcal{R}(\dot{\mathbf{p}})\,\mathrm{d}t}_{\text{dissipated energy}} = \underbrace{\mathcal{E}(\mathbf{u}^0,\mathbf{p}^0,0)}_{\text{initial energy}} - \underbrace{\int_0^T\int_\Omega \dot{\ell}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t}_{\text{work external forces}}$$

Finite plasticity

Multiplicative decomposition: $\nabla\varphi = \mathbf{F}_{\rm elast}\mathbf{P}, \quad \mathbf{P} \in {\rm SL}(d) = \{\det \mathbf{P} = 1\}$

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- Dissipation:

$$D(\mathbf{P}_0,\mathbf{P}_1) = \inf\left\{ R\int_0^1\int_\Omega |\dot{\mathbf{P}}\mathbf{P}^{-1}|\,\mathrm{d}x\,\mathrm{d}t \ : \ \mathbf{P}_0 \overset{\mathbf{P}(t)}{\rightsquigarrow} \mathbf{P}_1 \right\}$$

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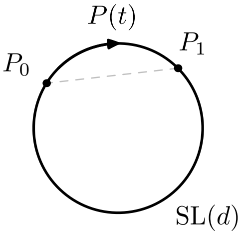

Linearization in plasticity

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- Nonlinear to linear settings
  - multiplicative to additive splits:

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- Evolutive situation
  - Need for an evolutive notion of variational convergence

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Energetic solution

Stability: $\mathcal{E}_\varepsilon(q(t),t) \leq \mathcal{E}_\varepsilon(\widehat{q},t) + \mathcal{D}_\varepsilon(q,\widehat{q}) \quad \forall \widehat{q}$

Energy: $\mathcal{E}_\varepsilon(q(t),t) + \int_0^T \mathcal{R}_\varepsilon(\dot{q})\,\mathrm{d}t = \mathcal{E}_\varepsilon(q^0,0) - \int_0^T\!\!\int_\Omega \dot{\boldsymbol{\ell}}\cdot q\,\mathrm{d}x\,\mathrm{d}t$

Question:

Which $(\mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon) \rightsquigarrow (\mathcal{E}_0, \mathcal{D}_0)$ entails convergence of energetic solutions?

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Which $(\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon) \rightsquigarrow (\mathcal{E}_0,\mathcal{D}_0)$ entails convergence of energetic solutions?

First (but wrong) guess: $\mathcal{E}_\varepsilon \stackrel{\Gamma}{\to} \mathcal{E}_0$ and $\mathcal{D}_\varepsilon \stackrel{\Gamma}{\to} \mathcal{D}_0$

Linearization in plasticity

Sufficient condition: [Mielke, Roubicek, S., 2008]

two separate $\Gamma - \liminf$ conditions ...

$$\mathcal{E}_0 \le \Gamma\text{–}\liminf \mathcal{E}_\varepsilon \quad \text{and} \quad \mathcal{D}_0 \le \Gamma\text{–}\liminf \mathcal{D}_\varepsilon$$

... plus the existence of a Mutual Recovery Sequence

$\forall q_\varepsilon \to q_0, \ \forall \widehat{q}_0$ there exists $\widehat{q}_\varepsilon$ such that

$$\limsup \Big(\mathcal{E}_\varepsilon(\widehat{q}_\varepsilon) - \mathcal{E}_\varepsilon(q_\varepsilon) + \mathcal{D}_\varepsilon(q_\varepsilon, \widehat{q}_\varepsilon)\Big) \le \mathcal{E}_0(\widehat{q}_0) - \mathcal{E}_0(q_0) + \mathcal{D}_0(q_0, \widehat{q}_0)$$

Linearization in plasticity

- Formal linearization: $\nabla\varphi = \mathbf{I} + \varepsilon\nabla\mathbf{u}$, $\mathbf{P} = \mathbf{I} + \varepsilon\mathbf{p}$

$$W_{\rm elast}(\nabla\varphi_\varepsilon \mathbf{P}_\varepsilon^{-1}) = W_{\rm elast}\big((\mathbf{I}+\varepsilon\nabla\mathbf{u})(\mathbf{I}+\varepsilon\mathbf{p})^{-1}\big) \approx W_{\rm elast}\big((\mathbf{I}+\varepsilon\nabla\mathbf{u})(\mathbf{I} \overset{!}{-} \varepsilon\mathbf{p})\big)$$

$$\approx \frac{\varepsilon^2}{2}(\nabla^{\rm s}\mathbf{u}-\mathbf{p}^{\rm s}):\mathbb{C}(\nabla^{\rm s}\mathbf{u}-\mathbf{p}^{\rm s})$$

$$W_{\rm hard}(\mathbf{P}_\varepsilon) \approx \frac{\varepsilon^2}{2}\mathbf{p}:\underbrace{\partial^2_{\mathbf{PP}}W_{\rm hard}(\mathbf{I})}_{\mathbb{H}}\,\mathbf{p}$$

$$D(\mathbf{P}_{0\varepsilon},\mathbf{P}_{1\varepsilon}) = D(\mathbf{I},\mathbf{P}_{1\varepsilon}\mathbf{P}_{0\varepsilon}^{-1}) = D\big(\mathbf{I},(\mathbf{I}+\varepsilon\mathbf{p}_1)(\mathbf{I}+\varepsilon\mathbf{p}_0)^{-1}\big) \overset{\Gamma}{\approx} \varepsilon R|\mathbf{p}_1 \overset{!}{-} \mathbf{p}|$$

- $(\mathbf{I}+\varepsilon\mathbf{p}_1)(\mathbf{I}+\varepsilon\mathbf{p}_0)^{-1} \approx \mathbf{I}+\varepsilon(\mathbf{p}_1-\mathbf{p}_0) \approx {\rm e}^{\varepsilon(\mathbf{p}_1-\mathbf{p}_0)}$

$$D\big(\mathbf{I},(\mathbf{I}+\varepsilon\mathbf{p}_1)(\mathbf{I}+\varepsilon\mathbf{p}_0)^{-1}\big) \approx D\big(\mathbf{I},{\rm e}^{\varepsilon(\mathbf{p}_1-\mathbf{p}_0)}\big) \overset{\mathbf{P}(t)={\rm e}^{t\varepsilon(\mathbf{p}_1-\mathbf{p}_0)}}{\approx} \varepsilon R|\mathbf{p}_1-\mathbf{p}_0|$$

Linearization in plasticity

- Coercivity of the energy

  Rigidity for $W_{\rm elast}$ and coercivity for $W_{\rm hard}$:

$$\|\nabla \mathbf{u}_\varepsilon(t)\|^2_{\mathrm{L}^2} + \|\mathbf{p}_\varepsilon(t)\|^2_{\mathrm{L}^2} \leq c\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon(t), \mathbf{p}_\varepsilon(t), t) < \infty$$

- Two Γ–liminf inequalities ...
  - Poinwise inequalities for the integrands

$$\frac{1}{2}(\nabla^{\rm s}\mathbf{u}-\mathbf{p}):\mathbb{C}(\nabla^{\rm s}\mathbf{u}-\mathbf{p}) \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} W_{\rm elast}\big((\mathbf{I}+\varepsilon\nabla\mathbf{u}_\varepsilon)(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon)^{-1}\big)$$

$$\frac{1}{2}\mathbf{p}:\mathbb{H}\mathbf{p} \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} W_{\rm hard}(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon)$$

$$R|\mathbf{p}_1-\mathbf{p}_0| \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} D\big(\mathbf{I}, (\mathbf{I}+\varepsilon\mathbf{p}_1)(\mathbf{I}+\varepsilon\mathbf{p}_0)^{-1}\big)$$

  - Balder-Ioffe lower semicontinuity theorem

Linearization in plasticity

- ...and a mutual recovery sequence

Let $(\widehat{\mathbf{u}}_0, \widehat{\mathbf{p}}_0) := (\mathbf{u}_0, \mathbf{p}_0) + (\tilde{\mathbf{u}}, \tilde{\mathbf{p}})$ with $(\tilde{\mathbf{u}}, \tilde{\mathbf{p}}) \in \mathrm{C}^\infty_\mathrm{c}(\Omega; \mathbb{R}^d \times \mathbb{R}^{d\times d}_\mathrm{dev})$
Define $\psi_\varepsilon := \mathbf{id} + \varepsilon\tilde{\mathbf{u}}$ and $\varphi_\varepsilon := \mathbf{id} + \varepsilon \mathbf{u}_\varepsilon$ and let

$$\widehat{\mathbf{u}}_\varepsilon := \frac{1}{\varepsilon}\left(\psi_\varepsilon \circ \varphi_\varepsilon - \mathbf{id}\right)$$

Moreover, define the *big set*

$$\Omega_\varepsilon := \left\{x \in \Omega \mid \mathrm{e}^{\varepsilon\tilde{\mathbf{p}}(x)}(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon(x)) \in \{W_\mathrm{hard} < \infty\} \subset\subset \mathrm{SL}(d)\right\}$$

and choose

$$\widehat{\mathbf{p}}_\varepsilon := \begin{cases} \frac{1}{\varepsilon}\left(\mathrm{e}^{\varepsilon\tilde{\mathbf{p}}}(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon) - \mathbf{I}\right) & \text{on } \Omega_\varepsilon \\ \mathbf{p}_\varepsilon & \text{else} \end{cases}$$

Linearization in plasticity

- Complicated definitions to simplify complicated expressions
  - $\widehat{\mathbf{u}}_\varepsilon := \frac{1}{\varepsilon}\big(\boldsymbol{\psi}_\varepsilon \circ \boldsymbol{\varphi}_\varepsilon - \mathbf{id}\big)$

$$\Longrightarrow \ (\mathbf{I}+\varepsilon\nabla\widehat{\mathbf{u}}_\varepsilon) = \nabla\boldsymbol{\psi}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\nabla\boldsymbol{\varphi}_\varepsilon = \underbrace{(\mathbf{I}+\varepsilon\nabla\tilde{u}(\boldsymbol{\varphi}_\varepsilon))}_{\to 1}\nabla\boldsymbol{\varphi}_\varepsilon \to \nabla\boldsymbol{\varphi}_\varepsilon$$

[multiplicative decomposition]

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[multiplicative decomposition]

  - $\widehat{\mathbf{p}}_\varepsilon = \dfrac{1}{\varepsilon}\big(\mathrm{e}^{\varepsilon\tilde{\mathbf{p}}}(\mathbf{I} + \varepsilon\mathbf{p}_\varepsilon) - \mathbf{I}\big) \;\; \Longrightarrow \;\; (\mathbf{I} + \varepsilon\widehat{\mathbf{p}}_\varepsilon) = \mathrm{e}^{\varepsilon\tilde{\mathbf{p}}}(\mathbf{I} + \varepsilon\mathbf{p}_\varepsilon)$

$$\frac{1}{\varepsilon}\mathcal{D}\big(\mathbf{I}, (\mathbf{I} + \varepsilon\widehat{\mathbf{p}}_\varepsilon)(\mathbf{I} + \varepsilon\mathbf{p}_\varepsilon)^{-1}\big) = \frac{1}{\varepsilon}\mathcal{D}\big(\mathbf{I}, \mathrm{e}^{\varepsilon\tilde{\mathbf{p}}}\big) \to \mathbf{R}(\tilde{\mathbf{p}})$$

Linearization in plasticity

- Constraints fulfilled
  - Orientation $[\widehat{\mathbf{u}}_\varepsilon := (\boldsymbol{\psi}_\varepsilon \circ \boldsymbol{\varphi}_\varepsilon - \mathbf{id})/\varepsilon]$

$$
\begin{aligned}
\det(\mathbf{I}+\varepsilon\nabla\widehat{\mathbf{u}}) &= \det(\nabla\boldsymbol{\psi}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\nabla\boldsymbol{\varphi}_\varepsilon) \\
&= \underbrace{\det(\mathbf{I}+\varepsilon\nabla\tilde{\mathbf{u}}(\boldsymbol{\varphi}_\varepsilon))}_{\approx 1}\underbrace{\det(\mathbf{I}+\varepsilon\nabla\mathbf{u}_\varepsilon)}_{>0} > 0
\end{aligned}
$$

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  - Orientation $[\widehat{\mathbf{u}}_\varepsilon := (\boldsymbol{\psi}_\varepsilon \circ \boldsymbol{\varphi}_\varepsilon - \mathbf{id})/\varepsilon]$

$$\det(\mathbf{I}+\varepsilon\nabla\widehat{\mathbf{u}}) = \det(\nabla\boldsymbol{\psi}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\nabla\boldsymbol{\varphi}_\varepsilon) = \underbrace{\det(\mathbf{I}+\varepsilon\nabla\tilde{\mathbf{u}}(\boldsymbol{\varphi}_\varepsilon))}_{\approx 1}\underbrace{\det(\mathbf{I}+\varepsilon\nabla\mathbf{u}_\varepsilon)}_{>0} > 0$$

  - Volume $[\widehat{\mathbf{p}}_\varepsilon = (e^{\varepsilon\tilde{\mathbf{p}}}(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon) - \mathbf{I})/\varepsilon]$

$$\det(\mathbf{I}+\varepsilon\widehat{\mathbf{p}}) = \det\left(e^{\varepsilon\tilde{\mathbf{p}}}(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon)\right) = \det e^{\varepsilon\tilde{\mathbf{p}}}\underbrace{\det(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon)}_{=1} = e^{\mathrm{tr}(\varepsilon\tilde{\mathbf{p}})} = e^0 = 1$$

Linearization in plasticity

- Constraints fulfilled
  - Orientation $[\widehat{\mathbf{u}}_\varepsilon := (\boldsymbol{\psi}_\varepsilon \circ \boldsymbol{\varphi}_\varepsilon - \mathbf{id})/\varepsilon]$

$$\det(\mathbf{I}+\varepsilon\nabla\widehat{\mathbf{u}}) = \det(\nabla\boldsymbol{\psi}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\nabla\boldsymbol{\varphi}_\varepsilon) = \underbrace{\det(\mathbf{I}+\varepsilon\nabla\tilde{\mathbf{u}}(\boldsymbol{\varphi}_\varepsilon))}_{\approx 1}\underbrace{\det(\mathbf{I}+\varepsilon\nabla\mathbf{u}_\varepsilon)}_{>0} > 0$$

  - Volume $[\widehat{\mathbf{p}}_\varepsilon = \big(\mathrm{e}^{\varepsilon\tilde{\mathbf{p}}}(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon) - \mathbf{I}\big)/\varepsilon]$

$$\det(\mathbf{I}+\varepsilon\widehat{\mathbf{p}}) = \det\big(\mathrm{e}^{\varepsilon\tilde{\mathbf{p}}}(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon)\big) = \det \mathrm{e}^{\varepsilon\tilde{\mathbf{p}}}\underbrace{\det(\mathbf{I}+\varepsilon\mathbf{p}_\varepsilon)}_{=1} = \mathrm{e}^{\mathrm{tr}(\varepsilon\tilde{\mathbf{p}})} = \mathrm{e}^0 = 1$$

- Strategy
  - Exploit cancellations (a very refined version of the quadratic trick)

Linearization in plasticity

Convergence

Let $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$ be energetic solutions of finite plasticity.

Then, $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \to (\mathbf{u}_0, \mathbf{p}_0)$ pointwise in $\mathrm{H}^1 \times \mathrm{L}^2$-weak where $(\mathbf{u}_0, \mathbf{p}_0)$ is an energetic solution of linearized plasticity

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- [A. Mielke, U.S. Linearized plasticity is the evolutionary Γ-limit of finite plasticity, *J. Eur. Math. Soc. (JEMS)*, to appear 2012]

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